How long does it take until a quantum system reemerges after a gravitational collapse?

Inauguraldissertation
der Philosophisch-naturwissenschaftlichen Fakultät
der Universität Bern

vorgelegt von

Marcel Ambrus

von Wohlen/BE

Leiter der Arbeit:

Prof. Dr. P. Hájíček

Institut für theoretische Physik

Universität Bern

How long does it take until a quantum system reemerges after a gravitational collapse?

Inauguraldissertation
der Philosophisch-naturwissenschaftlichen Fakultät
der Universität Bern

vorgelegt von

Marcel Ambrus

von Wohlen/BE

Leiter der Arbeit:

Prof. Dr. P. Hájíček

Institut für theoretische Physik

Universität Bern

Von der Philosophisch-naturwissenschaftlichen Fakultät angenommen.

Der Dekan:

Bern, den 11. November 2004 Prof. Dr. P. Messerli

Contents

Chapter 1

Introduction and summary

Two notorious problems have been plaguing the theory of gravitational collapse for decades. The first one is the occurrence of *singularities*. The classical collapse of a gravitating system leads inevitably to a singularity [Pen65, Haw67, HP70], cf. also [HE73]. The consequence is that the classical theory breaks down because the structure of the singularity contradicts the basic principles of the theory such as the equivalence principle. But there is some hope that a future quantum theory of gravity (or at least of gravitational collapse) will avoid the singularities of its classical counterpart.

The second problem is the so-called *black hole information paradox*. The semi-classical analysis of fields near a black hole by Hawking [Haw75] has led to the surprising conclusion that black holes evaporate by emitting radiation that is thermal. This implies the violation of basic principles of quantum mechanics. As an idealization, the initial quantum state of the collapsing system can be viewed as a pure state. But since it is thermal, the radiation does not depend on the detailed structure of the body that collapsed to form the black hole. Thus, the pure initial state evolves into a mixed state, by this violating the unitarity and time reversal symmetry of quantum theory. In other words, it is, in principle, impossible to reconstruct the initial state from the final state. Also, even if the initial state were precisely known, it would be impossible to predict with certainty what the final quantum state will be. Moreover, the pure initial state has zero entropy, but at the end, according to Hawking, its entropy is of the order of $\frac{M^2}{m_{\mathrm{Pl}}^2}$, where M is the total mass of the system and m_{Pl} is the Planck mass. The entropy can be interpreted as a measure for the information lost during the evolution. If unitarity does not hold anymore, then also the energy seems to fail to be conserved [BSP84]. For these reasons Hawking conjectured [Haw76, Haw82] that quantum theory breaks down in the case of black holes and that the information is lost in principle in quantum gravity, summarized in the quotation: *"God not only plays dice, He sometimes throws the dice where they cannot be seen."*

This conjecture has led to a great controversy, mainly between the bulk of the relativists defending Hawking's position on one side and particle physicists that adhered to a unitary time evolution on the other. Particle physicists abhorred the idea to reject one of the basic

principles of quantum mechanics and came up with various ideas to explain the apparent paradox or to modify Hawking's calculations, e.g. [BSP84, tH90, CGHS92, Gid92]. But all of these approaches are either highly improbable or suffer from conceptual defects. There is an abundance of reviews on the controversy and its tentative solutions, e.g. [Pre92, Gid95]. Most recently, at the GR17 conference in Dublin, Hawking surprised the community by announcing that he has disproved his own conjecture and that unitarity is preserved after all. But since his calculations have not been published yet at the time this thesis is written, we do not know how he obtained his results. So the controversy still remains unsettled.

Hawking's results on evaporating black holes are only plausible as long as the semi-classical approach is valid. This seems to be not the case near the singularity, where a more fundamental modification of the classical theory due to quantum effects is expected to occur. If the semi-classical approximation near the singularity is rejected, then one is forced to look for a different kind of approximative scheme, since there seems to be no hope for a satisfactory full quantum theory of gravity in the near future. The attempts of the Berne group led by P. Hájíček are based on the use of *simplified models*, where a non-perturbative quantum theory can be explicitly constructed. The main reason to use simplified models is that they strongly simplify technicalities: such systems are easier to reduce and to quantize and, nevertheless, still possess some of the important qualitative features of the full theory. Simplified models lead to a kind of effective theory of gravity. An infinite number of degrees of freedom that are unimportant for the problem can be safely frozen such that one obtains a quantum theory rather than a quantum *field* theory after quantization. Of course, the thus simplified theory must still have the right (semi-) classical limit.

The use of simplified models has led to some promising results. The quantum theory of a single spherically symmetric self-gravitating null thin shell constructed by Hájíček and Kiefer in [HK01, Háj01] seems to describe some important features of the collapse sufficiently well and yields the surprising conclusion that, in contrast to the classical collapse, the thus defined quantum system does not form a singularity but always bounces at the centre and returns to the same asymptotic region it originated from! So it seems that *unitarity is preserved* and that no black hole forms. Some quantum systems can even be squeezed such that a significant part of them lies under their Schwarzschild horizon and still they reappear. The Schwarzschild radius of such quantum systems is hence a mixture of white and black hole states, and the colour of the horizon gradually changes from black to white in the course of the evolution of the system.

The construction of the quantum theory required the solution of several conceptual problems commonly associated with the quantization of gravity. The exclusive use of gauge-invariant quantities as observables in the quantum theory successfully addresses the issue of diffeomorphism invariance. Asymptotically flat models, which are suitable to describe the collapse, allow the definition of a complete set of gauge-invariant quantities that are also integrals of motion, the so-called *Dirac observables*. The physical problem can thus be viewed at as a scattering system where every relevant question can be answered by a

measurement in the asymptotically flat region of spacetime. In the case of the spherically symmetric shells the Dirac observables can, however, also be found without relying on its asymptotic properties. The computation of the algebra of Dirac observables required the specification of a gauge. This has been obtained by the so-called *Kuchař-decomposition* [Kuc71], where the canonical variables describing the system are neatly split into physical degrees of freedom (the Dirac observables), embeddings (gauge degrees of freedom) and their momenta. The existence of such a decomposition has been shown to exist for a wide class of models by Hájíček and Kijowski in [HK00]. In the course of their work the important notions of background manifold and covariant gauge fixing have been introduced, cf. [Háj00a]. The issues and results of this approach are collected and discussed in detail in [Háj03].

The question what is the metric outside of the shell in the quantum theory remains unanswered. It is an intricate question because the metric components are not gauge invariant and must be expressed solely in terms of invariant quantities by some means, which may be not feasible. It thus seems to be a more promising idea to obtain the information about the outside geometry by introducing a second (test) shell that probes the field of the first one. The classical analysis of the system containing two or more shells has been done by Hájíček and Kouletsis [HK02a, HK02b, KH02]. Finding the corresponding quantum theory seems to be a non-trivial task. Work is being done on this topic by Hájíček and Minassian.

The second important question is the motivation for the present work. Consider the following hypothetical scenario: A quantum system with large (i.e. astrophysical) energy — of course a single null shell is hardly an adequate description of such a system — collapses and re-expands after a huge scattering time. The black hole phase is so long that Hawking radiation becomes important and therefore influences the scattering time. Only when the black hole becomes very small the change of horizon colour takes place such that only a tiny remnant of the system reemerges. Such a process does not necessarily violate unitarity and may be compatible with astronomical observations. So it is interesting to know when the shell does reappear after the collapse.

The model we are using to compute the *scattering times* is again a single spherically symmetric null thin shell surrounded by its own gravitational field. In order to incorporate the reflection a spherically symmetric, perfectly reflecting mirror with some positive radius is placed at the centre.

The main findings of this work are summarized in the following paragraph: We found that the time delay, a quantity used in the quantum theory of scattering processes, cannot be defined for our model because the infrared divergence at infinity common to all Coulomb-like potentials (to which also gravity belongs) cannot be removed by any suitable regularization. The question was then, what time we wanted to consider as the scattering time of our quantum system and how it could be defined in a gauge-invariant way. The *sojourn time* seems to be a good candidate. It can be interpreted as the mean value of the operator associated with the proper scattering time measured by an ideal clock carried along by an observer at a constant radius between his two encounters with the shell. We

found quantities that can be associated with the scattering time that are gauge-invariant and thus can be represented by self-adjoint operators. In order to obtain the true degrees of freedom of the model we adapted the canonical reduction program by Hájíček, Kiefer and Kouletsis [HK01, HK02b, KH02] to our model. This required the construction of a specific gauge. This gauge has the beautiful property that it also can be used for the perturbative approach by DeWitt [DeW63]. We thus have proven an existence theorem for a suitable gauge that is compatible with DeWitt's methods for our model. We have constructed the quantum theory in which the shells are represented by wave packets and the scattering time by a self-adjoint operator the expectation value of which is the sojourn time. Explicit and numeric calculations of the sojourn time have shown that it displays a rather surprising behaviour: It is *finite* for wave packets of finite extension even if the corresponding classical scattering time diverges! And it, furthermore, remains well-defined and finite when a significant part of the packet falls under its Schwarzschild horizon. A minor flaw in the calculation is, however, that one of the designated classical observables is not represented by a self-adjoint operator in our quantum theory. But the quantum theory itself is nevertheless well-defined, and the main results do not seem to depend significantly on the corrections that would have to be done to incorporate a self-adjoint operator representing this observable.

The program of this thesis is given by the following summary of the chapters.

Chapter 2: In this chapter we summarize the most important results in our context of the quantum theory of scattering processes. A precise definition of the sojourn time is given, and several ways to define the time delay are presented. The problems concerning the time delay appearing in the case of the Coulomb potential are pointed out.

Chapter 3: The definition of massive and null self-gravitating thin shells is given. Then, a spherically symmetric mirror centered at $R = 0$ is introduced, such that the in- and outgoing shell trajectories are glued together in a unique way. The (proper) scattering time of the shell measured by an observer at a fixed radius is computed.

Chapter 4: In this chapter it is discussed why the sojourn time is chosen as the measured quantity instead of the time delay and why the mean value of the operator representation of the scattering time can be interpreted as the sojourn time. The scattering time is newly constructed such that it is a gauge invariant.

Chapter 5: The space of solutions of our system is determined and the distinction between singular and regular gauges is pointed out.

Chapter 6: The central-regular (CR) gauge is constructed using an interpolation. By this the difficulties with the canonical theory at the inner boundary of space-time induced by the mirror are removed.

Chapter 7: The Hamiltonian action for a spherical null thin shell and its gravitational field (the LWF action, cf. [LWF98]) is adapted to include the spherical mirror. The canonical reduction of the action is performed using the methods developed by Hájíček, Kiefer and Kouletsis and the CR gauge that has been constructed in the preceding chapter.

The true degrees of freedom are identified, they are the shell energy E and its asymptotic advanced time v.

Chapter 8: In this chapter a well-defined quantum theory and a self-adjoint operator representing the scattering time are constructed. Analytical and numerical computations of the sojourn time using a special type of wave packets are made. The rather surprising results are discussed and illustrated by diagrams.

Chapter 9: A brief introduction into DeWitt's [DeW63] method of small disturbances is given. Then, an existence theorem for a suitable gauge (the CR gauge) for DeWitt's method is stated. The dynamics of the shell and the metric disturbance on the background is determined and the effect of infinitesimal gauge transformations on it is computed.

Chapter 10: The main results and problems found in this thesis are discussed, followed by some suggestions for further research.

A series of appendices containing material that is either too lengthy and technical for the main text or not about the main topics of this thesis completes this work.

Unless otherwise defined, we use units for which $c = G = \hbar = 1$ throughout this thesis. The metrics have signature -2, the Minkowski metric being given by $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$.

Chapter 2

Time delay in scattering theory

2.1 Quantum scattering theory

We summarize the most important features of the quantum theory of non-relativistic potential scattering of a structure-less particle. We first write down the relevant ideas of the time-dependent approach and state the basic definitions. In the second subsection we introduce the main aspects of the time-independent setting, an approach more familiar to most physicists. This short introduction is, of course, in no way complete, but the appropriate references are given in the text.

2.1.1 Time-dependent setting

A mathematically precise framework of this approach to scattering theory is given in [AJS77] and [RS79]. This introduction follows in some parts roughly [Mar81]. We first define the basic notions of non-relativistic time-dependent potential scattering theory.

Scattering states

Let $\mathcal{H}$ be a Hilbert space of quantum states generated by the self-adjoint operators $\hat{H}$ and $\hat{H}_0$, being, respectively, the *free* (or unperturbed) and the *total Hamiltonian*, where $\hat{H} = \hat{H}_0 + \hat{V}$. The operator $\hat{V}$ is defined as the multiplication by the function $V(\vec{x})$, which is the *potential*. The spectrum of $\hat{H}_0$ is clearly absolutely continuous: $\mathcal{M}_\infty(\hat{H}_0) = \mathcal{H}$. In the domain of $\hat{H}$ are *scattering states* $\psi \in \mathcal{M}_\infty(\hat{H})$ as well as *bound states* $\phi_b \in \mathcal{M}_0(\hat{H})$. We suppose that $\mathcal{M}_\infty(\hat{H}) = \mathcal{H}_{ac}$ and $\mathcal{M}_0(\hat{H}) = \mathcal{H}_p$, i.e. the set of scattering states of $\hat{H}$ is exactly the subspace of absolute continuity $\mathcal{H}_{ac}$ of $\mathcal{H}$ and, moreover, $\mathcal{M}_0(\hat{H})$, the set of the bound states, is equal to the subspace $\mathcal{H}_p \subset \mathcal{H}$ belonging to the pure point spectrum of $\hat{H}$. The two spaces $\mathcal{H}_{ac}$ and $\mathcal{H}_p$ are orthogonal. These assumptions are satisfied if the potential verifies $V = V_1 + V_2$, $V_1 \in \mathcal{L}^2(\mathbb{R}^3)$ and V_2 essentially bounded (i.e. $\exists M < \infty$,

$|V_2(\vec{x})| \leq M$ almost everywhere $\forall \vec{x} \in \mathbb{R}^3$). This applies in particular to the important case of the Coulomb potentials $V_c = \frac{\gamma}{r}$, $r = |\vec{x}|$, $\gamma \in \mathbb{R}$. One simply has to put $V_1 = V_c \chi_r$, $V_2 = V_c - V_1$, where $\chi_r(\vec{x}) = 1$ as $|\vec{x}| \leq r$ and 0 otherwise, $r \in (0, \infty)$.

The asymptotic condition and the scattering operator

The *free* and *total time evolutions*, two continuous unitary one-parameter groups on $\mathcal{H}$ corresponding to the two Hamiltonians, are given by $U_t = e^{-i\hat{H}_0 t}$ and $V_t = e^{-i\hat{H}t}$. The *wave* or *Møller operators* $\hat{\Omega}_\pm$ are defined by the *asymptotic condition*

$$\begin{aligned} \lim_{t\to-\infty} \left\| (V_t^* U_t - \hat{\Omega}_-)\phi \right\| &= 0, \\ \lim_{t\to\infty} \left\| (V_t^* U_t - \hat{\Omega}_+)\phi \right\| &= 0, \end{aligned} \tag{2.1}$$

for the states $\phi \in \mathcal{H}$. We suppose that they are complete, i.e. $\operatorname{Range}\hat{\Omega}_- = \operatorname{Range}\hat{\Omega}_+ = \mathcal{H}_{ac}$. An equivalent definition is

$$\hat{\Omega}_\pm \doteq \mathrm{s}\text{–}\lim_{t\to\pm\infty} V_t^* U_t, \tag{2.2}$$

where the strong limit s–lim is defined implicitly by eq. (2.1). $\hat{\Omega}_-$ transforms an incoming state ϕ_{in} into a scattering state $\psi = \hat{\Omega}_- \phi_{in}$. The same happens if one operates with $\hat{\Omega}_+$ on an outgoing state ϕ_{out}. ψ is unique for a given incoming state. *Asymptotic completeness* holds if for each ψ there is a (free) outgoing state ϕ_{out} to which the scattering state converges in the far future. The so-called *intertwining relation* $\hat{H}\hat{\Omega}_\pm = \hat{\Omega}_\pm \hat{H}_0$ is satisfied. This property is transmitted to the corresponding evolution: $V_t \hat{\Omega}_\pm = \hat{\Omega}_\pm U_t$. The wave operators are *isometries*: $\hat{\Omega}_\pm \hat{\Omega}_\pm^* = \hat{\Omega}_\pm^* \hat{\Omega}_\pm = \hat{I}$, where $\hat{I}$ is the unit operator. The time evolution of a scattering state ψ_t corresponding to an incoming state ϕ_{in} reads hence $\psi_t = V_t \hat{\Omega}_- \phi_{in}$, whereas the time evolution of the free incoming state is given by $\phi_t = U_t \phi_{in}$. The Møller operators $\hat{\Omega}_\pm$ exist and are complete for potentials falling off at infinity as $\frac{1}{r^{1+\epsilon}}, \epsilon > 0$, so called *short-range potentials*. There exists a generalization to a broader class of potentials, including the Coulomb case, that belongs to the class of the *long-range potentials*, which we will treat in the next section.

The *scattering operator* $\hat{S}$ is defined by $\hat{S} \doteq \hat{\Omega}_+^* \hat{\Omega}_-$. It is unitary and conserves the energy, i.e. $[\hat{S}, \hat{H}_0] = 0$; it therefore also commutes with the free evolution. $\hat{S}$ transforms an incoming free state ϕ_{in} into an outgoing one: $\phi_{out} = \hat{S}\phi_{in}$.

It follows from the asymptotic condition and the completeness, that the evolving scattering state behaves asymptotically in time as

$$\begin{aligned} \lim_{t\to-\infty} \|\psi_t - \phi_t\| &= 0, \\ \lim_{t\to\infty} \|\psi_t - \hat{S}\phi_t\| &= 0. \end{aligned} \tag{2.3}$$

We observe that in the remote past the total evolution is approximated by the unperturbed one and that in the far future the difference between the scattering and the outgoing state is arbitrarily small.

Spectral representations

Some important operators can be brought in a special form, the so-called diagonal form in the spectral representation of a particular operator. A *spectral representation* is a particular representation of the Hilbert space in which the operators in question appear as multiplication operators [JM65, JM66, AJS77]. We denote the spectral variable belonging to $\hat{H}_0$ by λ, the energy of a particular state. The spectrum Λ of the free Hamiltonian $\hat{H}_0$, where $\lambda \in \Lambda$, is continuous. Let $\hat{C}$ be a set of operators commuting with $\hat{H}_0$ and c the corresponding spectral variables. We then can denote a state in this particular representation by $|\lambda, c\rangle$. Let $\hat{B}$ be another bounded linear operator with $[\hat{B}, \hat{H}_0] = 0$ which possesses an integral representation

$$\hat{B} = \int_{-\infty}^{\infty} dt\, U_t^* \hat{A} U_t. \tag{2.4}$$

Then the matrix elements of $\hat{B}$ in the spectral representation with respect to $\hat{H}_0$ read [JM67].

$$\langle \lambda, c|\, \hat{B} |\lambda', c'\rangle = \delta(\lambda - \lambda') \langle c|\, B(\lambda) |c'\rangle \tag{2.5}$$

with

$$\langle c|\, B(\lambda) |c'\rangle = 2\pi \langle \lambda, c|\, \hat{A} |\lambda, c'\rangle. \tag{2.6}$$

We call the operator $B(\lambda)$ *on the energy shell* or *on-shell*. It will be shown in the next section that the scattering and the time delay operators are examples of operators which can be brought on-shell.

2.1.2 Time-independent setting

We very briefly introduce the basic ideas of the stationary formalism of scattering theory. We omit proofs as well as the discussion of cross sections and the perturbation theory. For a more accurate treatment we refer to the abundant literature (e.g. [Far73, New66, GW64]).

The Asymptotic Condition

We consider the scattering by a potential $V(\vec{x})$. Let $\hat{H}_0 = -\frac{1}{2m}\triangle$ and $\hat{H} = \hat{H}_0 + \hat{V}(\vec{x})$ on the Hilbert space $\mathcal{H} = \mathcal{L}^2(\mathbb{R}^3)$, where $\triangle$ is the Laplacian in $\mathbb{R}^3$ and $\mathcal{L}^2(\mathbb{R}^3)$ is the space of square integrable functions on $\mathbb{R}^3$. The potential is assumed to be such that eigenfunctions $\psi(\vec{k},\vec{x})$ given by the stationary solutions of the Schrödinger equation

$$\left(-\frac{1}{2m}\triangle + \hat{V}(\vec{x}) - \lambda\right)\psi(\vec{k},\vec{x}) = 0, \qquad \lambda = \frac{k^2}{2m}, \quad k = |\vec{k}| \tag{2.7}$$

exist. These eigenfunctions are required to be scattering states with the following asymptotic spatial behaviour as $|\vec{x}| = r \to \infty$,

$$\psi(\vec{k},\vec{x}) \overset{r\to\infty}{\simeq} e^{i\vec{k}\cdot\vec{x}} + f(\lambda,\vec{\omega}_{\vec{x}},\vec{\omega}_{\vec{k}})\frac{e^{ikr}}{r}, \tag{2.8}$$

where $\vec{\omega}_{\vec{x}} \doteq \frac{\vec{x}}{r}, \vec{\omega}_{\vec{k}} \doteq \frac{\vec{k}}{k}$ are the angles of $\vec{x}$ and $\vec{k}$, respectively, and $f(\lambda,\vec{\omega}_{\vec{x}},\vec{\omega}_{\vec{k}})$ is the *scattering amplitude*. This is the *asymptotic condition* in the time-independent setting. According to eq. (2.8), $\psi(\vec{k},\vec{x})$ is an outgoing plane wave modified by a spherical wave. The correction to the plane wave depends on the energy of the ingoing wave and on the scattering angle. These stationary states with infinite extension are of course by no means a physically realistic description of a particle scattered by a potential. We will therefore, later in this section, also discuss spatially localized wave packets and their asymptotic behaviour.

The Lippmann-Schwinger equation

We introduce the Lippmann-Schwinger equation and the definition of the Møller operators in the time-independent setting. For scattering by a short-range potential we can write the scattering states as follows ($\eta > 0$):

$$\psi^{\pm} \doteq \phi_{in} + (\lambda - \hat{H} \mp i\eta)^{-1}\,\hat{V}\,\phi_{in}. \tag{2.9}$$

After some algebraic manipulations we obtain the so-called *Lippmann-Schwinger equation*

$$\psi^{\pm} = \phi_{in} + (\lambda - \hat{H}_0 \mp i\eta)^{-1}\,\hat{V}\,\psi^{\pm}. \tag{2.10}$$

This equation is equivalent to an integral equation whose solution behaves asymptotically as a plane wave plus an ingoing ($-$ sign) and outgoing ($+$ sign) wave (cf. eq. (2.8)), respectively. By using the relation $\psi^{\pm} = \hat{\Omega}_{\pm}\phi_{in}$ we can define the Møller operators

$$\hat{\Omega}_{\pm} \doteq \hat{I} + (\lambda - \hat{H} \mp i\eta)^{-1}\,\hat{V}. \tag{2.11}$$

Notice that in the preceding section about the time-dependent formalism we always denoted the ingoing scattering state ψ^- simply by ψ. We also define the *transition operator* $\hat{T}$ given by $\hat{T} \doteq \hat{V}\hat{\Omega}_-$. The scattering amplitude can be hence written as follows:

$$f(\phi_{in} \to \phi_{out}) = -\frac{m}{2\pi}\langle\phi_{out}|\,\hat{T}|\phi_{in}\rangle. \tag{2.12}$$

We assume that the ingoing state has the wave vector $\vec{k}$ and the outgoing one $\vec{\ell}$:

$$\phi_{in} = \phi_{\vec{k}}, \quad \phi_{out} = \phi_{\vec{\ell}}, \tag{2.13}$$

with $|\vec{k}| \doteq k = \ell \doteq |\vec{\ell}|$. Then we can rewrite the scattering amplitude,

$$f(\vec{k} \to \vec{\ell}) = -4\pi^2 m\langle\vec{\ell}|\,\hat{T}|\vec{k}\rangle = -\frac{4\pi^2}{k}\langle\lambda_\ell, \vec{\omega}_{\vec{\ell}}|\,\hat{T}|\lambda_k, \vec{\omega}_{\vec{k}}\rangle. \tag{2.14}$$

The *scattering matrix element* between these two states reads

$$S_{\vec{\ell}\vec{k}} \doteq \langle\phi_{\vec{\ell}}|\,\hat{S}|\phi_{\vec{k}}\rangle = \langle\psi^+_{\vec{\ell}}|\psi^-_{\vec{k}}\rangle = \delta(\vec{\ell} - \vec{k}) - 2\pi i\delta(\lambda_\ell - \lambda_k)\langle\phi_{\vec{\ell}}|\,\hat{T}|\phi_{\vec{k}}\rangle. \tag{2.15}$$

This result is valid at least for short-range potentials.

Wave packets

This paragraph follows closely the treatment in [Far73]. Since a plane wave with sharply defined energy is not a realistic description of a particle we now introduce wave packets. The *wave packet for a free particle* shall be given by

$$\bar{\varphi}(\vec{x}, t) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int d\vec{k}\, C(\vec{k}, t)e^{i\vec{k}\cdot\vec{x}}, \tag{2.16}$$

where

$$C(\vec{k}, t) = C(\vec{k})e^{-i\lambda_k t} \tag{2.17}$$

is an energy- and time-dependent contour function. The free wave packet can be also written as

$$\bar{\varphi}(\vec{x}, t) = \int d\vec{k}\, C(\vec{k})\, \varphi_k(\vec{x})\, e^{-i\lambda_k t}, \tag{2.18}$$

where we have introduced the normalized eigenstates

$$\varphi_k(\vec{x}) \doteq \frac{e^{i\vec{k}\cdot\vec{x}}}{(2\pi)^{\frac{3}{2}}} \tag{2.19}$$

of the momentum operator $\hat{\vec{P}}$. The probability to find the particle with the wave vector $\vec{k}$ in the volume element $d\vec{k}$ of $\vec{k}$-space is $P(\vec{k}) = |C(\vec{k})|^2$ and is constant in time. It follows that the contour functions satisfies $\int d\vec{k}\,|C(\vec{k})|^2 = 1$. As $t \to \pm\infty$, the wave packet tends to zero: $\bar{\varphi}(\vec{x}, t) \to 0$. Hence, the probability to find the particle in any finite region tends to zero in the remote past and far future, respectively.

The *scattering* wave function of a single non-relativistic structure-less particle can be described by the *wave packet*

$$\bar{\psi}(t) \doteq \int d\vec{k}\, C(\vec{k})\, \psi^-_{\vec{k}}\, e^{-i\lambda_k t}. \tag{2.20}$$

It is free in the remote past and in the far future:

$$\begin{aligned} \bar{\psi}(t) &\overset{t\to-\infty}{\simeq} \bar{\varphi}_{in}(t) = \int d\vec{k}\, C(\vec{k})\, \varphi_{\vec{k}}\, e^{-i\lambda_k t}, \\ \bar{\psi}(t) &\overset{t\to\infty}{\simeq} \bar{\varphi}_{out}(t) = \int d\vec{\ell}\, D(\vec{\ell})\, \varphi_{\vec{\ell}} e^{-i\lambda_\ell t}, \end{aligned} \tag{2.21}$$

where $\bar{\varphi}_{in}(t)$ and $\bar{\varphi}_{out}(t)$ are the initial and the final wave packets, respectively. $C(\vec{k})$ is the initial momentum amplitude; the final momentum amplitude is given by

$$D(\vec{\ell}) = \int d\vec{k}\, S_{\vec{\ell}\vec{k}}\, C(\vec{k}). \tag{2.22}$$

Inserting the formula for the scattering matrix elements, eq. (2.15), we obtain after some manipulations, that

$$D(\vec{\ell}) = C(\vec{\ell}) + \Delta(\vec{\ell}), \tag{2.23}$$

where

$$\Delta(\vec{\ell}) \doteq \frac{i\ell}{2\pi} \int d\vec{\omega}_{\vec{k}}\, C(\ell\vec{\omega}_{\vec{k}})\, f(\ell\vec{\omega}_{\vec{k}} \to \vec{\ell}) \tag{2.24}$$

is the change produced in the momentum amplitude by the collision. We observe that the amount of scattering increases with the scattering amplitude. For a well collimated ingoing beam, the shape of the final wave packet is approximately

$$\bar{\varphi}_{out} = \bar{\varphi}_{in} + \bar{\psi}_s, \tag{2.25}$$

where the outgoing scattered wave $\bar{\psi}_s$ is given by

$$\bar{\psi}_s \approx \frac{1}{\sqrt{2\pi}ir} \int_0^\infty \ell d\ell \, |\Delta(\ell\vec{\omega}_{\vec{x}})| \, e^{i\ell r - i\lambda_\ell t + iarg\Delta(\ell\vec{\omega}_{\vec{x}})}. \tag{2.26}$$

Partial wave analysis

After this short deviation we again consider a wave function with fixed energy for the sake of simplicity. The results can be taken over to wave packets rather straightforwardly.

First we have to record how the stationary solutions behave in time. The time evolution of the ingoing scattering state (here we again omit the $-$ sign) reads explicitly

$$\psi_t(\vec{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k \, \psi(\vec{k}, \vec{x}) \, e^{-i\frac{k^2}{2m}t} \, \phi(\vec{k}). \tag{2.27}$$

$\psi(\vec{k}, \vec{x}) \doteq (2\pi)^{\frac{3}{2}} \langle \vec{x}| \, \hat{\Omega}_- |\vec{k}\rangle$ is the kernel of the Møller operator. If the potential is spherically symmetric, $V(\vec{x}) = V(r)$, then the kernel has a development in *radial partial waves*

$$\psi(\vec{k}, \vec{x}) = 4\pi \sum_{l,m} i^l \, \frac{u_l(k,r)}{kr} \langle \omega_{\vec{x}}|l,m\rangle \, \langle l,m|\omega_{\vec{k}}\rangle. \tag{2.28}$$

Here, $\langle \omega_{\vec{x}}|l,m\rangle = Y_l^m(\omega_{\vec{x}})$ is the normalized spherical harmonic and $u_l(k,r)$ is the regular solution of the radial equation

$$\left(-\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} + 2mV(r) - 2mk^2 \right) \, u_l(k,r) = 0 \tag{2.29}$$

that has the asymptotic behaviour

$$u_l(k,r) \overset{r\to\infty}{\simeq} \frac{1}{2i} \left(S_l(\lambda) \, e^{i(kr - l\frac{\pi}{2})} - e^{-i(kr - l\frac{\pi}{2})} \right) \tag{2.30}$$

as r goes to infinity.

$$S_l(\lambda) = e^{i\delta_l(\lambda)} \tag{2.31}$$

is the *scattering matrix element* associated with the angular momentum l and the corresponding *phase shift* $\delta_l(\lambda)$. We observe that the effect of the potential is a phase shift of the scattering wave. This phase shift is dependent on the angular momentum and the energy of the ingoing state, as well as on the strength and the shape of the potential.

2.1.3 Short- and long-range potentials

It turns out to be useful to divide the potentials into two distinct groups: *short-range* and *long-range potentials*. We will call a potential $V(\vec{x})$ short-range, if it falls off faster than $r^{-1-\epsilon}$ as r tends to infinity, where $\epsilon > 0$. Otherwise it will be called long-range. The hard sphere and square well potentials belong clearly to the first group, whereas the Coulomb and the gravitational potential belong to the second. For a long range potential it is not possible to define the wave operators in the usual way, and the time delay in the standard definition becomes infinite. Possible ways to solve these problems will be proposed in section 3 of this chapter.

2.2 The time delay

The time delay is an important theoretical concept, although it is usually too small to be directly measurable in experiments. In this section we give various definitions of the quantity *time delay* and the associated operator. A clear and comprehensive review of various aspects of the time delay is [Mar81]. It also includes an extensive list of references.

2.2.1 A heuristic definition

The time delay can be considered as the difference between the time spent by a particle within the region of the interaction and the time that it would have spent there without the interaction. What follows in this subsection is a heuristic definition of the time delay based on the time-independent approach to quantum scattering theory that has been introduced in the previous chapter.

The Eisenbud-Wigner formula

We analyse the scattered wave far away from the scattering centre for large positive times. From eq. (2.3) it follows that

$$\psi_t(\vec{x}) \overset{t\to\infty}{\simeq} \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k \, e^{i(\vec{k}\cdot\vec{x}-\frac{k^2}{2m}t)} \, (\hat{S}\phi)(\vec{k}). \tag{2.32}$$

Its radial components with angular momentum l behave asymptotically as

$$\psi_t^l(r) \overset{t\to\infty}{\simeq} \frac{1}{i\sqrt{2\pi}} \frac{1}{r} \int k dk \, \left(e^{i(kr-\frac{k^2}{2m}t+\delta_l(\lambda))} - (-1)^l e^{-i(kr+\frac{k^2}{2m}t)} \right) \phi^l(k), \tag{2.33}$$

where we have used the asymptotic form (2.30) of the free radial function $u_l^0(k,r)$. $\psi_t^l(r)$ differs perceptibly from zero only for those values of (r,t) for which the arguments of

the exponentials in eq. (2.33) are stationary with respect to energy variations, where we assume that $\phi^l(k)$ is real. As r and t are both positive in our considerations, the second term does not contribute to such variations. The first term produces a contribution if the energy derivative of the exponential vanishes (this is in fact the stationary phase argument)

$$\frac{d}{dk}\left(kr - \frac{k^2}{2m}t + \delta_l(\lambda)\right) = 0. \tag{2.34}$$

This is equivalent to

$$r = v(t - \tau_l(\lambda)), \quad \tau_l(\lambda) \doteq \frac{d}{d\lambda}\delta_l(\lambda), \tag{2.35}$$

where $v = \frac{k}{m}$ is the free radial velocity. Since the phase shift $\delta_l(\lambda)$ vanishes in the case of free motion, we can interpret $\tau_l(\lambda)$ as the *time delay suffered by the radial wave in the presence of the potential*. eq. (2.35) is called the *Eisenbud-Wigner formula* [Eis48], [Wig55].

Resonances and the time delay

There is a noteworthy relation between the time delay and the width Γ of the resonance belonging to a particular angular momentum l. If the S-matrix $S_l(\lambda)$ has a pole just below the real axis which corresponds to a resonance at the energy λ_0 and of width $\Gamma > 0$, then one can approximate the S-matrix by

$$S_l(\lambda) \approx \frac{\alpha(\lambda)}{\lambda_0 - i\Gamma - \lambda} \tag{2.36}$$

for $\lambda \approx \lambda_0$. Here, $\alpha(\lambda)$ is some slowly varying function of the energy. The time delay hence obtains a Breit-Wigner-like form

$$\tau_l(\lambda) = \Im\left(\frac{d}{d\lambda}\ln S_l(\lambda)\right) \approx \frac{\Gamma}{(\lambda - \lambda_0)^2 + \Gamma^2}, \quad \lambda \approx \lambda_0. \tag{2.37}$$

Therefore, $\tau_l(\lambda_0) = \frac{1}{\Gamma}$ is the *lifetime of the resonance*.

Remarks

In principle, the quantity 'time delay' is measurable. The measurement is although in the most cases not feasible in experiments because the delay is usually very small. Suppose that in an experiment one can measure the time delay of a scattering process. Then it will give a complementary piece of information to the determination of the scattering

amplitude. Namely, by the Eisenbud-Wigner formula one then can calculate the phase shift which is closely related to the scattering amplitude. In fact, the scattering operator and the time delay contain the same information, although from very complementary points of view.

2.2.2 The time delay of Jauch-Marchand

The definition due to Jauch and Marchand is based on the time-dependent setting of scattering theory. It goes back to an idea of Goldberger and Watson [GW64]. The concept of this approach to time delay has been written down in [JM67] and refined involving advanced mathematical techniques in [JSM72], see also [BO75] for a slight modification. A proof using somewhat different methods is given in [Mar76].

Jauch and Marchand associate a self-adjoint operator with the observable time delay. Their mathematically precise, abstract formalism uses only simple operator identities such as the intertwining property or the relation between the wave operators and the scattering matrix. Moreover, it avoids the use of eigenfunction expansions. It is therefore also valid for more general problems for which such an expansion does not exist. Another advantage of their method is that it works for all types of wave packets. A drawback is that the restriction on the potentials for which their approach is valid is stronger than for other definitions.

The basic concept entering in the definition of the time delay is that of the *sojourn time*. The following treatment loosely follows the article by Martin [Mar81].

The sojourn time

The introduction of the quantity sojourn time in relation with time delay first appeared in [Smi60] and in [GW64]. Jauch and Marchand formulated a precise formulation of the concept in the article [JM67].

First, we define the projection operator $\hat{P}$, that projects into a subspace $\hat{P}\mathcal{H}$ of states of $\mathcal{H}$. The probability of finding the state $\psi_t = U_t\psi$, where $\psi \in \mathcal{H}_{ac}$ in $\hat{P}\mathcal{H}$ at the time t is

$$\langle\psi_t|\,\hat{P}|\psi_t\rangle = |\hat{P}\psi_t|^2 \tag{2.38}$$

and the total mean time spent there is given by

$$\int_{-\infty}^{\infty} dt\,|\hat{P}\psi_t|^2. \tag{2.39}$$

This quantity is called *sojourn (or residence, or transit) time* of the state ψ_t in the subspace $\hat{P}\mathcal{H}$. It can be finite or infinite depending on the state and the projection. In

potential scattering we take $\hat{P} = \hat{P}_\Sigma$, where Σ is a bounded subset of $\mathbb{R}^3$ and the projection acts on a state as follows:

$$(\hat{P}_\Sigma\phi)(\vec{x}) = \chi_\Sigma(\vec{x})\phi(\vec{x}), \qquad \chi_\Sigma \doteq \begin{cases} 1, & \vec{x} \in \Sigma \\ 0, & \vec{x} \notin \Sigma \end{cases} . \tag{2.40}$$

The *sojourn time of a free state* ϕ is therefore given by

$$T^0_\Sigma(\phi) = \int_{-\infty}^{\infty} dt\, |\hat{P}\phi_t|^2 = \int_{-\infty}^{\infty} \langle\phi|\, U_t^* \hat{P}_\Sigma U_t|\phi\rangle dt, \tag{2.41}$$

whereas that of a *scattering state* $\psi = \hat{\Omega}_-\phi$ corresponding to the same incoming state ϕ reads

$$T_\Sigma(\phi) = \int_{-\infty}^{\infty} dt\, |\hat{P}\psi_t|^2 = \int_{-\infty}^{\infty} \langle\phi|\, \hat{\Omega}_-^* V_t^* \hat{P}_\Sigma V_t \hat{\Omega}_-|\phi\rangle dt. \tag{2.42}$$

There is a mathematical issue: $T^0_\Sigma(\phi)$ and $T_\Sigma(\phi)$ shall be finite for a significant amount of state vectors $\phi \in \mathcal{H}$, i.e. a dense subset. This imposes conditions on the interaction. The finiteness of the free sojourn time $T^0_\Sigma(\phi)$ can be expected for finite regions Σ in the case of scattering states. For bound states, of course, the quantity is infinite. If the explicit form of the time development of the free kernel (for $t \neq 0$),

$$(U_t\phi)(\vec{x}) = \left(\frac{m}{2\pi i|t|}\right)^{\frac{3}{2}} \int d^3y\, e^{im\frac{|\vec{x}-\vec{y}|^2}{2t}}\, \phi(\vec{y}), \tag{2.43}$$

and its behaviour at large times, cf. [AJS77], are used, then the integrand in (2.41) can be estimated:

$$|\hat{P}\phi_t|^2 = \left(\frac{m}{2\pi|t|}\right)^3 \int_\Sigma d^3x \left| \int d^3y\, e^{im\frac{|\vec{x}-\vec{y}|^2}{2t}}\, \phi(\vec{y}) \right|^2 \leq |\Sigma| \left(\frac{m}{2\pi|t|}\right)^3 \|\phi\|_1^2, \tag{2.44}$$

where $\|\phi\|_1 = \int d^3x\, |\phi(\vec{x})|$ is the norm in $\mathcal{L}^1(\mathbb{R}^3)$. We observe that $|\hat{P}\phi_t|^2$ falls off as $|t|^{-3}$ at large t if $\phi \in \mathcal{L}^1(\mathbb{R}^3) \cap \mathcal{L}^2(\mathbb{R}^3)$. Thus, the finiteness of $T^0_\Sigma(\phi)$ holds at least for the dense set of $\phi \in \mathcal{L}^1(\mathbb{R}^3) \cap \mathcal{L}^2(\mathbb{R}^3) \subset \mathcal{L}^2(\mathbb{R}^3)$.

The finiteness of the sojourn time $T_\Sigma(\phi)$ of scattering states is not so obvious. It is based on the asymptotic behaviour of the scattering states. One can show that the quantity $T_\Sigma(\phi)$ is finite if

$$\begin{aligned} &1. \quad |\psi_t - \phi_t| \in \mathcal{L}^1(-\infty, 0), \\ &2. \quad |\hat{P}_\Sigma \hat{S}\phi_t| \text{ and } |\psi_t - \hat{S}\phi_t| \in \mathcal{L}^1(0, \infty). \end{aligned} \tag{2.45}$$

The two conditions are fulfilled if the asymptotic condition is sufficiently well satisfied and if $\hat{S}$ has some smoothness in its energy dependence (see the article by Martin [Mar81] and the references stated therein). There are mathematically more advanced methods to show the finiteness of sojourn time for a broad class of potentials: the operator $\hat{V}$ has to be $\hat{H}_0$-bounded (i.e. $|\hat{V}\phi| \leq \alpha|\phi| + \beta|\hat{H}_0\phi|$ for some $\alpha \geq 0, \beta \geq 0, \forall\phi \in \mathcal{H}$), the wave operators $\hat{\Omega}_\pm$ have to exist and the state vectors are to be members of a dense subset $\mathcal{D}$ of $\mathcal{H}_{ac}$, with $\mathcal{D} = \{\phi \in \mathcal{H}_{ac} \,|\, \phi \text{ has compact support on } \Lambda\}$ (Λ is the spectrum of $\mathcal{H}_0$). This holds in particular for potentials decaying faster than the Coulomb potential. For details cf. the references mentioned above.

The time delay

The difference between the sojourn times of the scattering and the free states in Σ

$$\tau_\Sigma(\phi) \doteq T_\Sigma(\phi) - T^0_\Sigma(\phi) \tag{2.46}$$

is the delay in Σ due to the interaction. $\tau_\Sigma(\phi)$ is therefore the time delay for the region Σ and the incoming state ϕ. As the limit $\Sigma \to \mathbb{R}^3$ is taken, both sojourn times $T_\Sigma(\phi)$ and $T^0_\Sigma(\phi)$ diverge, but it is expected that the difference (2.46) remains finite. This is the case if the interaction is sufficiently short-ranged. Thus, if the limit exists, then the *time delay* of the scattering process with incoming state ϕ reads

$$\tau(\phi) \doteq \lim_{\Sigma\to\mathbb{R}^3} \tau_\Sigma(\phi). \tag{2.47}$$

In terms of operators one can also write

$$\hat{\tau} \doteq \lim_{\Sigma\to\mathbb{R}^3} \int_{-\infty}^{\infty} dt\, U_t^* \left[\hat{\Omega}_-^* \hat{P}_\Sigma \hat{\Omega}_- - \hat{P}_\Sigma\right] U_t, \tag{2.48}$$

where the operator $\hat{\tau}$ is related to the quadratic form $\tau(\phi)$ by the equation $\tau(\phi) = \langle\phi|\,\hat{\tau}|\phi\rangle$. $\hat{\tau}$ is a uniquely defined, bounded and self-adjoint linear operator. In the passage to the operator definition the intertwining relation $V_t\hat{\Omega}_- = \hat{\Omega}_- U_t$ has been used. There are various methods to show the existence of the limit (2.47). In this section we will briefly show the results based on the time-dependent setting. For the other methods we refer to the literature [Mar81].

We use the asymptotic condition (2.1) to show that the time delay $\tau(\phi)$, if it exists, depends on the $\hat{S}$-operator alone. We will state two assertions which connect the time delay with the scattering operator. The proof of them is given in [Mar81]. Assume that the conditions (2.45) hold. Then, whenever the limit exists, it holds that

$$\tau(\phi) = \lim_{\Sigma\to\mathbb{R}^3} \int_0^\infty dt\, \langle\phi_t|\,(\hat{S}^*\hat{P}_\Sigma\hat{S} - \hat{P}_\Sigma)|\phi_t\rangle. \tag{2.49}$$

The limit (2.47) exists if (2.45) are satisfied and ϕ has compact support on Λ. Moreover, in this case, the time delay is related to the energy derivative of the $\hat{S}$-operator by

$$\tau(\phi) = -i\langle\phi|\,\hat{S}^* \frac{d\hat{S}}{d\hat{H}_0}\,|\phi\rangle. \tag{2.50}$$

This is the *Eisenbud-Wigner formula in terms of operators*. $\hat{\tau}$ evidently commutes with $\hat{H}_0$. Eq. (2.50) is independent of the representation; it is in principle a general functional relation between $\hat{\tau}$ and $\hat{S}$. A proof of eq. (2.50) is given in [AJS77], Prop. 7. 14.

Both the scattering and the time delay operator can be brought to the form of operators on the energy shell [JM66]:

$$\begin{aligned}
\langle\lambda, l, m|\,\hat{S}|\lambda', l', m'\rangle &= \delta(\lambda-\lambda')\langle l, m|\,S(\lambda)|l', m'\rangle, \\
\langle\lambda, l, m|\,\hat{\tau}|\lambda', l', m'\rangle &= \delta(\lambda-\lambda')\langle l, m|\,\tau(\lambda)|l', m'\rangle, \\
\langle l, m|\,\tau(\lambda)|l', m'\rangle &= 2\pi \lim_{\Sigma\to\mathbb{R}^3} \langle\lambda, l, m|\,(\hat{\Omega}_-^* \hat{P}_\Sigma \hat{\Omega}_- - \hat{P}_\Sigma)|\lambda, l', m'\rangle.
\end{aligned} \tag{2.51}$$

Here l is the spectral variable corresponding to $\hat{\vec{L}}^2$ and m that to $\hat{L}_3$. If the potential is spherically symmetric and $\{\Sigma\}$ is a sequence of spheres centered at the origin, and if the limit exists, then the energy shell components of the time delay operator are given by the Eisenbud-Wigner formula (2.35):

$$\tau(\lambda) = -iS^*(\lambda)\frac{d}{d\lambda}S(\lambda) = \frac{d}{d\lambda}\delta(\lambda). \tag{2.52}$$

It has been shown [JSM72] that the eq. (2.50) holds for potentials decaying faster than $\frac{1}{r^{3+\epsilon}}$ as r tends to infinity. Weaker conditions are discussed in [Mar81]. For spherical potentials the weakest condition is $V(r) \overset{r\to\infty}{\simeq} \frac{1}{r^{\frac{5}{2}+\epsilon}}$.

2.2.3 Other formulations of the time delay

We write down the most important features of some other definitions of the time delay.

The time-independent formulation of time delay

This method is due to Smith [Smi60]. It is based on the time-independent description of scattering processes. One considers the stationary state solution of the time-independent Schrödinger equation for a fixed energy λ. Then one calculates the excess number of particles near the scattering centre after subtracting the number that would have been there without the interaction. The time delay is then given by this excess or defect

number divided by the incoming (or outgoing) flux of particles at a large distance R from the scattering centre. The time delay for a specific angular momentum and energy is given by

$$\tau_l(\lambda) = \lim_{R\to\infty} \frac{1}{R} \int_R^{2R} dR' \int_0^{R'} r^2 dr \left[\frac{4\pi}{F_{in}} |\psi_l^-(r,k)|^2 - \frac{2}{vr^2} \right], \tag{2.53}$$

where the incoming flux is $F_{in} = \frac{2}{km}$ and the velocity of the incoming particles is given by $v = \frac{k}{m}$. The average limit in the first integral is in order to get rid of oscillatory terms in R. A way to remove the rather unphysical averaging is given in a paper by Bollé and Osborn [BO75]. In the same paper they carefully show that the method of Smith and the time-dependent one are in fact equivalent. A brief summary of the main results is also given in the article by Martin [Mar81]. This definition of the time delay holds for potentials fulfilling the condition (2.8).

Lavine's formula for the time delay

Lavine's formula [Lav73] gives the connection between the time delay τ and the potential $V(\vec{x})$ in scattering theory. The method is closely related to the geometric approach to scattering which will be briefly introduced in the next paragraph.

We first give the definition of an important operator in this context: the *generator of dilations*

$$\hat{D} = \frac{1}{2i} \left(\hat{\vec{X}} \cdot \vec{\nabla} + \vec{\nabla} \cdot \hat{\vec{X}} \right). \tag{2.54}$$

$\hat{D}$ is the generator of the dilation group in $\mathcal{H} = \mathcal{L}^2(\mathbb{R}^3)$. On the operators of position and momentum its action is defined by

$$\begin{aligned} e^{i\alpha\hat{D}} \hat{\vec{P}} e^{-i\alpha\hat{D}} &= e^{-\alpha} \hat{\vec{P}}, \\ e^{i\alpha\hat{D}} \hat{\vec{X}} e^{-i\alpha\hat{D}} &= e^{\alpha} \hat{\vec{X}}, \end{aligned} \tag{2.55}$$

where α is some real number.

The following results are true for potentials that satisfy

$$\begin{aligned} V &= V_1 + V_2, \\ V_2 &\simeq \frac{1}{r^{2+\epsilon}}, \\ V_1 &\simeq \frac{1}{r^{1+\epsilon}}, \quad \vec{x} \cdot \vec{\nabla} V_1 \simeq \frac{1}{r^{1+\epsilon}} \quad \text{as} \quad r \to \infty. \end{aligned} \tag{2.56}$$

The self-adjoint dilation operator $\hat{D}$ satisfies the two important equalities

$$\begin{aligned} \hat{H} - \frac{i}{2}[\hat{H}, \hat{D}] &= \hat{V} + \frac{1}{2}\hat{\vec{X}} \cdot \vec{\nabla}\hat{V}, \\ \hat{H}_0\hat{\tau} &= \frac{1}{2}\hat{S}^*[\hat{S}, \hat{D}], \end{aligned} \tag{2.57}$$

where the Eisenbud-Wigner time delay τ is defined by the decomposition in the spectral representation of $\hat{H}_0$:

$$\tau(\lambda) = -iS(\lambda)^* \frac{dS(\lambda)}{d\lambda}. \tag{2.58}$$

See the papers by Jensen [Jen82] and Narnhofer [Nar84] for proofs and details. Note that the decay properties on the potential (2.56) are very weak compared to those necessary in the formalism by Jauch-Marchand, but still the important class of Coulomb-like potentials is not included.

For incoming states $\phi \in \mathcal{H}_{ac}$ Lavine's formula reads

$$\langle\phi|\, \hat{H}_0\hat{\tau}|\phi\rangle = \int_{-\infty}^{\infty} dt\, \langle\psi_t|\, (\hat{H} - \frac{i}{2}[\hat{H}, \hat{D}])|\psi_t\rangle. \tag{2.59}$$

The right hand side of this equation can be rewritten using eqs. (2.57), such that the dependence on the potential becomes visible

$$\langle\phi|\, \hat{H}_0\hat{\tau}|\phi\rangle = \int_{-\infty}^{\infty} dt\, \langle\psi_t|\, (2\hat{V} + \hat{\vec{X}} \cdot \vec{\nabla}\hat{V})|\psi_t\rangle. \tag{2.60}$$

This formula is valid for potentials that satisfy the conditions (2.56). We observe that the time delay operator is closely related to the virial $V(\vec{x}) + \frac{1}{2}\vec{x} \cdot \vec{\nabla}V(\vec{x})$.

For a spherically symmetric potential the energy shell components of the time delay operator in the corresponding representation by Lavine reads

$$\tau_l(\lambda) = \frac{4m}{k\lambda} \int_0^{\infty} dr\, |u_l(k, r)|^2 \left(V(r) + \frac{r}{2}\frac{dV(r)}{dr} \right). \tag{2.61}$$

This formula gives some qualitative information on the behaviour of the time delay, especially on the sign of $\tau_l(\lambda)$: it is the same as that of the expression $V(r) + \frac{r}{2}\frac{dV(r)}{dr}$. $\tau_l(\lambda)$ is small if $\frac{dV(r)}{dr} < 0$, i.e. if the potential decreases monotonously so that there are no potential barriers which capture the particle near the origin. The time delay is negative for all energies if $V(r) + \frac{r}{2}\frac{dV(r)}{dr} < 0$. This condition is satisfied if the potential is strongly repulsive and has no barriers. $\tau_l(\lambda)$ is of course not only governed by the term $V(r) + \frac{r}{2}\frac{dV(r)}{dr}$, but also by the radial wave $u_l(\lambda)$.

Jensen [Jen81] has proven the Eisenbud-Wigner formula for not necessarily spherically symmetric potentials which fall off at infinity faster than $\frac{1}{r^4}$.

Geometric time delay

The notion of time delay already appears in classical scattering theory. There it can be derived with purely geometrical considerations, see, for a short introduction [Nar80, Bol81, Mar81] and the references stated in these papers. We write down only the most important results.

Consider a particle of mass $m = 1$ with trajectory $(\vec{x}(t), \vec{p}(t))$ scattered by a potential $V(\vec{x})$. The time delay is then given by

$$\tau = \frac{\vec{x}_- \cdot \vec{p}_- - \vec{x}_+ \cdot \vec{p}_+}{|\vec{p}_+|^2}, \tag{2.62}$$

where $\vec{p}_\pm \doteq \lim_{t\to\pm\infty} \vec{p}(t)$ is the momentum of the particle as $t \to \pm\infty$ and $\vec{x}_\pm \doteq \lim_{t\to\pm\infty} (\vec{x}(t) - \vec{p}(t)t)$. The scattering matrix S transforms $\vec{x}_-$ into $\vec{x}_+ = S\vec{x}_-$. It depends on the energy and the angular momentum of the particle. As $r \doteq |\vec{x}|$ tends to infinity one can write $S = e^{i\delta}$ (δ is the usual phase shift), and the fundamental relation

$$\tau = -i\frac{d}{d\lambda} \ln S \tag{2.63}$$

holds.

2.2.4 Remarks

The quantity $\tau(\phi)$ introduced in this section has to be viewed as a global time delay in the sense that its evaluation involves the scattering in all possible directions. One can also define a so-called angular time delay which depends on the direction of the incoming beam and that of the observer of the scattered particles. The global time delay is then a kind of average over all directions of the angular one (cf. [AJS77]).

We have observed that the time delay has been defined for a class of potentials which does not include important cases as the Coulomb one, that is long-range potentials. The situation is even worse — long-range potentials do not even fulfill the asymptotic condition (2.1), and thus the wave operators do not exist. Possible ways to solve these problems are discussed in the next section.

2.3 Time delay in the Coulomb potential

2.3.1 Introduction

Since there are several problems associated with it, the special case of long-range potentials needs a deeper investigation. Most of the definitions and formulae from the preceding sections break down in the case of potentials decaying as slowly as, or more slowly than $\frac{1}{r}$ as $r \to \infty$. In this section we discuss the problems and how some of them can be removed. At the end we present a method of how one can define a finite time delay for the Coulomb potential, but its physical interpretation seems not obvious.

2.3.2 Time-independent approach

Surprisingly there exists an explicit solution of the time-independent Schrödinger equation for the Coulomb potential despite the various difficulties associated with it. We give a short summary of the results which is based on the treatment in [Far73].

We consider the scattering of a particle with mass m and charge $Z_1 e$ by a fixed charge $Z_2 e$ placed at the origin. The corresponding Schrödinger equation is

$$\left[-\frac{\triangle}{2m} + \frac{Z_1 Z_2 e^2}{4\pi\epsilon_0 r}\right] \Psi(\vec{x}) = \lambda \Psi(\vec{x}). \tag{2.64}$$

In an abbreviated form this equation reads

$$\left[\triangle + k^2 - \frac{2\gamma k}{r}\right] \Psi(\vec{x}) = 0, \quad \gamma \doteq \frac{m Z_1 Z_2 e^2}{4\pi\epsilon_0 k}. \tag{2.65}$$

We put

$$\Psi(\vec{x}) = e^{ikz} f(\vec{x}), \tag{2.66}$$

where the z-axis has been set to lie in the direction of the incoming particle. eq. (2.65) then simplifies to

$$\triangle f + 2ik\frac{\partial f}{\partial z} - \frac{2\gamma k}{r} f = 0. \tag{2.67}$$

With another change of variable, $u = ik(r - z) = 2ikr\sin^2(\frac{\theta}{2})$, where θ is the polar angle given by $z = r\cos\theta$, we find that

$$u\frac{d^2 f}{du^2} + (1 - u)\frac{df}{du} + i\gamma f = 0. \tag{2.68}$$

This is a special case of the confluent hypergeometric equation; solutions regular at $u = 0$ must be proportional to the confluent hypergeometric function

$$ {}_1F_1(-i\gamma, 1; u) = \sum_{n=0}^{\infty} \frac{\Gamma(-i\gamma + n)}{\Gamma(-i\gamma)} \frac{u^n}{(n!)^2}, \tag{2.69} $$

where Γ is the gamma function. The solution of (2.65) reads hence

$$ \Psi(\vec{x}) = A e^{ikz} {}_1F_1[-i\gamma, 1; ik(r - z)], \tag{2.70} $$

where A is some constant.

For $\Im(u) > 0$ the asymptotic behaviour of ${}_1F_1(u)$ as $|u| \to \infty$ is given by

$$ \begin{aligned} {}_1F_1(-i\gamma, 1; u) \;\simeq\; & \frac{e^{\pi\gamma} u^{i\gamma}}{\Gamma(1 + i\gamma)} \left[1 + \frac{\gamma^2}{u} + \mathcal{O}\left(\frac{1}{u^2}\right)\right] \\ & + \frac{1}{\Gamma(-i\gamma)} \frac{e^u}{u^{1+i\gamma}} \left[1 + \frac{(1 + i\gamma)^2}{u} + \mathcal{O}\left(\frac{1}{u^2}\right)\right]. \end{aligned} \tag{2.71} $$

Therefore, as $|r - z| \to \infty$, the scattering wave function behaves like

$$ \begin{aligned} \Psi(\vec{x}) \;\simeq\; & \frac{A e^{\frac{\pi}{2}\gamma}}{\Gamma(1 + i\gamma)} \left\{ e^{ikz + i\gamma \ln(k|r-z|)} \left[1 + \frac{\gamma^2}{ik(r - z)} + \ldots\right] \right. \\ & \left. + \frac{\Gamma(1 + i\gamma)}{\Gamma(-i\gamma)} \frac{e^{ikr - i\gamma \ln(k|r-z|)}}{2ikr \sin^2 \frac{\theta}{2}} \left[1 + \frac{(1 + i\gamma)^2}{ik(r - z)} + \ldots\right] \right\}. \end{aligned} \tag{2.72} $$

The first term in the curly brackets of eq. (2.72) represents a plane wave modified by the phase factor $i\gamma \ln(k|r - z|)$. The second term can be rewritten in the form

$$ f_c(\theta) \frac{e^{ikr - i\gamma \ln(2kr)}}{r}, $$

where

$$ f_c(\theta) \doteq \frac{\Gamma(1 + i\gamma)}{i\Gamma(-i\gamma)} \frac{e^{-i\gamma \ln(\sin^2 \frac{\theta}{2})}}{2k \sin^2 \frac{\theta}{2}} \tag{2.73} $$

is the scattering amplitude. The expression

$$ \frac{e^{ikr - i\gamma \ln(2kr)}}{r} $$

can be interpreted as an outgoing spherical wave modified by a term due to the long-range nature of the Coulomb potential. We observe that both the incident and the scattered waves are distorted by a term logarithmic in r right out to infinity. This is a characteristic of the Coulomb potential and other potentials with the same fall-off. The meaning of this result is that there exist no free asymptotic states in the usual sense in a long-range potential. The notion of asymptotic freedom and therefore also the definition of the asymptotic condition have to be modified in order to be able to give a description of the theory analogous to that for the short-range case.

2.3.3 The time delay in long-range potentials is infinite

The time delay as it has been defined in the previous section is infinite for long-range potentials. The following heuristic arguments shall elucidate this fact. The Gedanken-experiment used in this paragraph has been adopted from [Gas74], where the problem is discussed in more detail.

We consider a fictious experiment in classical particle scattering: a source at $\vec{x}_0 = (x_0, 0, z_0)$ injects test particles into a spherically symmetric force field $\vec{F} = \alpha_\beta \frac{\vec{x}}{|\vec{x}|^{\beta+2}}$; $\beta \geq 1$, $\alpha_\beta > 0$, such that the motion of the particles takes place in the x-z-plane. The particles have initial velocity $\dot{\vec{x}} = (\dot{x}_0, 0, 0)$. At $\vec{x}_1 = (x_1, 0, z_1)$ an experimentalist measures:

$$\begin{aligned} \dot{\vec{s}}(\vec{s}) : &\quad \text{the velocity of the particles at } \vec{s} \doteq \vec{x} - \vec{x}_1, \\ T(\vec{s}) : &\quad \text{the time } T \text{ taken by them to travel from } \vec{x}_1 \text{ to } \vec{x}_1 + \vec{s}, \end{aligned} \tag{2.74}$$

where we have set $T(0) = 0$. We will give the qualitative behaviour of these two quantities for large $|\vec{x}_0|$ and $|\vec{x}_1|$ (both the source and the observer are far away from the scatterer). Introducing polar coordinates $x = r\cos\varphi$, $z = r\sin\varphi$, assuming that $\frac{z_0}{x_0} \ll 1$ and denoting by r_{min} the minimal distance which a particle of given energy $E \simeq \frac{m}{2}\dot{x}_0^2$ can reach from the centre of force we obtain, approximately,

$$\begin{aligned} \dot{\vec{s}}(\vec{s}) &\simeq \sqrt{\frac{2E}{m}}\,(\cos\varphi_{as}, 0, \sin\varphi_{as}) \doteq \frac{\vec{p}_{out}}{m}, \\ T(\vec{s}) &\simeq |\vec{s}|\frac{m}{|\vec{p}_{out}|}, \quad \text{for} \quad \beta > 1 \end{aligned} \tag{2.75}$$

and

$$\begin{aligned} \dot{\vec{s}}(\vec{s}) &\simeq \sqrt{\frac{2E}{m}}\,(\cos\varphi_{as}, 0, \sin\varphi_{as}) \doteq \frac{\vec{p}_{out}}{m}, \\ T(\vec{s}) &\simeq |\vec{s}|\frac{m}{|\vec{p}_{out}|} + \frac{\alpha_1}{2E}\frac{m}{|\vec{p}_{out}|}\ln\left(1 + \frac{|\vec{s}|}{|\vec{x}_1|}\right), \quad \text{for} \quad \beta = 1, \end{aligned} \tag{2.76}$$

where

$$\varphi_{as} \doteq \pi - 2\int_0^{\frac{1}{r_{min}}} \frac{du}{\sqrt{\frac{2mE}{l^2} - \frac{2m}{l}V(\frac{1}{u}) - u^2}}, \quad l \doteq m\dot{x}_0 z_0, \tag{2.77}$$

is the asymptotic angle between the x-axis and the trajectory of the scattered particle.

In the case $\beta > 1$, particles far from the origin behave as free ones within the applied accuracy. They are called asymptotically free particles. In the case $\beta = 1$, however, within the experimental uncertainties, we find that the time taken by the particle has an additional logarithmic contribution; for $\alpha_1 > 0$ it takes more time to travel the same distance than the free one. The distance it covers in the time T is given by the absolute value of

$$\vec{s}(T) \simeq \frac{\vec{p}_{out}}{m}T - \alpha_1 \frac{m}{|\vec{p}_{out}|^3}\vec{p}_{out} \ln\left(1 + \frac{|\vec{p}_{out}|}{|\vec{x}_1|}\frac{T}{m}\right). \tag{2.78}$$

We observe that even in the far future this particle is not asymptotically free.

The time delay suffered by the particle in the potential with $\beta = 1$ can be considered as the time T_s spent by it within the distance d from the centre of force subtracted by the time T_{free} it would have spent there without the interaction:

$$\tau(d) \doteq T_s(d) - T_{free}(d). \tag{2.79}$$

As we have pointed out before, the trajectories in potentials decaying faster than the Coulomb one are free in the far future. We can therefore put $T_{free} \simeq d\frac{m}{|\vec{p}_{out}|}$ for large d. Hence, the time delay in potentials with $\beta = 1$ reads

$$\tau(d) \simeq \frac{\alpha_1}{2E}\frac{m}{|\vec{p}_{out}|} \ln\left(1 + \frac{d}{|\vec{x}_1|}\right). \tag{2.80}$$

It obviously tends to infinity as $d \to \infty$. One, therefore, needs to redefine the notion of asymptotically free particles by some means in order to get a way to define a finite time delay.

The preceding arguments can be adopted to the case of quantum mechanics. One can show that the asymptotic condition (2.1) does not hold for scattering in long-range potentials and that the Møller operators $\hat{\Omega}_\pm = \mathrm{s}\text{–}\lim_{t\to\pm\infty} V_t^* U_t$ do not exist (cf. [AJS77], prop. 13.6).

2.3.4 A new asymptotic condition

Dollard [Dol64] has provided a generalization of the asymptotic condition for the Coulomb potential and has constructed the corresponding wave operators. In this subsection we

shall give a short summary of his work and of improvements by other authors. See also [AJS77] for details and proofs.

We know from the previous subsection that the asymptotic condition (2.1) fails to hold for Coulomb scattering and that the wave operators do not exist in the usual sense. We now briefly introduce Dollard's construction of the Coulomb wave operators. Consider the Coulomb Hamiltonian

$$\hat{H}_c \doteq -\frac{\triangle}{2m} + \frac{\xi}{r}, \quad \xi \doteq \frac{Z_1 Z_2 e^2}{4\pi\epsilon_0}. \tag{2.81}$$

The corresponding evolution is given by

$$W_t \doteq e^{-i\hat{H}_c t}, \tag{2.82}$$

and the *distorted free evolution* reads

$$Y_t \doteq e^{-i\hat{H}_{0c}(t)}, \tag{2.83}$$

where

$$\hat{H}_{0c}(t) \doteq \hat{H}_0 t + \frac{m\xi \operatorname{sgn} t}{\sqrt{-\triangle}} \ln\left(-\frac{2|t|\triangle}{m}\right) \doteq \hat{H}_0 t + \hat{H}'_{0c}(t). \tag{2.84}$$

The operator $\hat{H}'_{0c}(t)$ acts on a state $\phi(\vec{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d\vec{k} e^{i\vec{k}\cdot\vec{x}} \tilde{\phi}(\vec{k})$ as follows:

$$(\hat{H}'_{0c}(t)\phi)(\vec{x}) = \frac{m\xi \operatorname{sgn} t}{(2\pi)^{\frac{3}{2}}} \int d\vec{k} e^{i\vec{k}\cdot\vec{x}} \ln\left(\frac{2k^2|t|}{m}\right) \frac{\tilde{\phi}(\vec{k})}{k}. \tag{2.85}$$

$W_t^* Y_t$ is unitary, and the strong limits

$$\lim_{t\to\pm\infty} \|W_t^* Y_t \phi - \hat{\Omega}^c_\pm \phi\| = 0 \tag{2.86}$$

define the *Coulomb wave operators* $\hat{\Omega}^c_\pm$. They exist on all of $\mathcal{L}^2$. Eqs. (2.86) represent the modified *asymptotic condition*. In the remote past and in the far future respectively, the scattering states behave as free ones with respect to the modified free evolution Y_t:

$$\lim_{t\to\pm\infty} \|W_t \psi - Y_t \phi\| = 0, \quad \psi = \hat{\Omega}^c_\pm \phi. \tag{2.87}$$

$Y_t\phi$ is the only possible form for a wave packet in the Coulomb field. Again, we can define the *scattering operator* by

$$\hat{S}_c \doteq \hat{\Omega}^{c*}_+ \hat{\Omega}^c_-, \quad \phi_{out} = \hat{S}_c \phi_{in}. \tag{2.88}$$

Mulherin and Zinnes [MZ70] have given a somewhat different construction, but their wave operators coincide with that of Dollard. They define their *free Coulomb evolution* Z_t by

$$(Z_t^{\pm}\phi)(\vec{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d\vec{k}\, \Phi_c^{\pm}(\vec{k},\vec{x}) e^{-i\frac{k^2}{2m}t}\, \tilde{\phi}(\vec{k}), \tag{2.89}$$

where $\phi \in \mathcal{L}^2$ is a free state. $\Phi_c^{\pm}(\vec{k},\vec{x})$ is the kernel of the Coulomb transform, which converts a free state into an asymptotic state in the Coulomb potential:

$$\phi_c^{\pm}(\vec{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d\vec{k}\, \Phi_c^{\pm}(\vec{k},\vec{x})\, \tilde{\phi}(\vec{k}); \tag{2.90}$$

the kernel is given by

$$\Phi_c^{\pm}(\vec{k},\vec{x}) \doteq e^{i\left(\vec{k}\cdot\vec{x} \mp \gamma \ln(kr \pm \vec{k}\cdot\vec{x})\right)}. \tag{2.91}$$

The wave operators according to this construction are defined by the limits

$$\hat{\Omega}_{\pm}^{c} \doteq \lim_{t\to\pm\infty} W_t^* Z_t. \tag{2.92}$$

Their action is well-defined on states $\in \mathcal{L}^2$, and they are identical to the wave operators obtained by Dollard.

The evolution $Z_t^{\pm}$ can be written as the product of two operators

$$Z_t^{\pm} = \hat{K}^{\pm}(\xi) U_t. \tag{2.93}$$

The operators $\hat{K}^{\pm}(\xi)$ are defined implicitly by eq. (2.93) and depend on the charge, which is itself proportional to ξ. For $\xi = 0$, it holds that $\hat{K}^{\pm}(0) = \hat{I}$. Hence, Z_t reduces just to the free evolution U_t for short-range potentials. In the case of long-range potentials, where $\xi \neq 0$, however, the operators $\hat{K}^{\pm}(\xi)$ are no longer trivial. They transform the free state ϕ into the asymptotic states given by:

$$(\hat{K}^{\pm}\phi)(\vec{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d\vec{k}\, \Phi_c^{\pm}(\vec{k},\vec{x})\, \tilde{\phi}(\vec{k}). \tag{2.94}$$

The operators $\hat{K}^{\pm}(\xi)$ are sometimes called *time-dependent dressing transformations*.

As we have pointed out, the scattering state $\psi_t^{\pm} = \hat{\Omega}_{\pm}^{c} W_t \phi$ is not asymptotically free ($\|U_t\phi - \psi_t\| \not\to 0$), but its probability distributions for both position and momentum approach those of the free wave function ϕ_t as $t \to \pm\infty$ (cf. [RS79] p. 174). In this sense we can say that the motion is *asymptotically free*.

2.3.5 Coulomb time delay made finite

Based on the considerations of the preceding subsection we show how one can define a *finite time delay* for long-range potentials for a sufficiently large class of state vectors. The method has been developed by Bollé, Gesztesy and Grosse in [BGG83]. It uses the time-dependent formulation of scattering theory together with Dollard's results on the Coulomb wave operators. We only state the most important formulae from [BGG83] and refer the reader to that article for proofs and references.

The Coulomb time delay is defined with respect to the modified free evolution of Dollard (cf. eq. (2.83)). It is shown to be finite and to satisfy a generalization of the Eisenbud-Wigner formula.

The Coulomb potential shall be denoted by $V(r) = \frac{\xi}{r}$, $\xi \in \mathbb{R}$, the unperturbed Hamiltonian is given by $\hat{H}_0 = -\triangle$, $m = \frac{1}{2}$. Hence, the modified free evolution reads

$$Y_t^a = e^{-i\left[\hat{H}_0 t + \frac{\xi \operatorname{sgn} t}{2\sqrt{\hat{H}_0}} \ln(4a\hat{H}_0|t|)\right]}, \; t \neq 0, \tag{2.95}$$

where an explicit dependence on a parameter $a > 0$ has been introduced. Dollard's choice for a was $a = 1$, but we will see that the choice $a = e$ seems more natural. According to the previous subsection the modified free evolution Y_t^a defines, now a-dependent, Coulomb wave operators $\hat{\Omega}_{\pm}^{a,c}$.

Let $\mathcal{S}_0(\mathbb{R}^3) \subset \mathcal{H} = \mathcal{L}^2(\mathbb{R}^3)$ be the subset of the Schwarz space $\mathcal{S}(\mathbb{R}^3)$ given by

$$\mathcal{S}_0(\mathbb{R}^3) \doteq \{\phi \in \mathcal{S}(\mathbb{R}^3) \,|\, \tilde{\phi}(\vec{k}) \text{ vanishes in a neighbourhood of } \vec{k} = 0\}. \tag{2.96}$$

Then, for a state $\phi \in \mathcal{S}_0(\mathbb{R}^3)$, both the *modified free sojourn time*

$$T_R^{a,c}(\phi) \doteq \int_{-\infty}^{\infty} dt \, \|\hat{P}_R Y_t^a \phi\|^2 \tag{2.97}$$

and the *Coulomb sojourn time*

$$T_R^{c}(\phi) \doteq \int_{-\infty}^{\infty} dt \, \|\hat{P}_R W_t \hat{\Omega}_{-}^{a,c} \phi\|^2 \tag{2.98}$$

are finite, as well as their difference

$$\tau_R^{a,c}(\phi) \doteq T_R^{c}(\phi) - T_R^{a,c}(\phi), \tag{2.99}$$

which is the Coulomb time delay for the region of $\mathbb{R}^3$ with $|\vec{x}| \leq R$.

Since the potential is spherically symmetric, we introduce the functions $\phi_{lm} \in \mathcal{S}_0(\mathbb{R}^3)$ which satisfy

$$\tilde{\phi}_{lm}(\vec{k}) = \alpha_{lm}(k) Y_l^m(\vec{\omega}), \tag{2.100}$$

where $\alpha_{lm}(k) \in C_0^\infty(0,\infty)$, $l \in \mathbb{N}$, $-l \leq m \leq l$, $\vec{k} = k\vec{\omega}$ and $\vec{\omega} \in \mathbb{S}^2$. Y_l^m denote the spherical harmonics.

As the distance R from the scattering centre tends to infinity, the sojourn times of the states ϕ_{lm} approach

$$T_R^{a,c}(\phi_{lm}) \overset{R\to\infty}{\simeq} \int_0^\infty k^2 dk \int_{\mathbb{S}^2} d\vec{\omega} \left|\tilde{\phi}_{lm}(k\vec{\omega})\right|^2 \left[\frac{R}{k} + \frac{\xi}{2k^3}\ln(2kaR) - \frac{\xi}{k^3}\right] + \mathcal{O}(1), \tag{2.101}$$

$$\begin{aligned} T_R^{c}(\phi_{lm}) \quad \overset{R\to\infty}{\simeq} \quad & \int_0^\infty k^2 dk \int_{\mathbb{S}^2} d\vec{\omega} \left|\tilde{\phi}_{lm}(k\vec{\omega})\right|^2 \left[\frac{R}{k} + \frac{\xi}{2k^3}\ln(2kR)\right. \\ & \left. -\frac{\xi}{2k^3} + \frac{1}{k}\frac{\partial}{\partial k}\delta_l^c(k)\right] + \mathcal{O}(1). \end{aligned} \tag{2.102}$$

Here,

$$\delta_l^c(k) = \arg\left[\Gamma\left(l+1+\frac{i\xi}{2k}\right)\right] \tag{2.103}$$

is the *Coulomb phase shift*. For $\phi_{lm} \in \mathcal{S}_0(\mathbb{R}^3)$ the *time delay* for infinite space region is *finite* and is given by

$$\tau^{a,c}(\phi_{lm}) = \int_0^\infty k^2 dk \int_{\mathbb{S}^2} d\vec{\omega} \left|\tilde{\phi}_{lm}(k\vec{\omega})\right|^2 \frac{1}{k}\frac{\partial}{\partial k}\left[\delta_l^c(k) - \frac{\xi}{2k}(1-\ln a)\right]. \tag{2.104}$$

For functions $\phi \in \mathcal{S}_0(\mathbb{R}^3)$ whose Fourier transforms $\tilde{\phi}$ have compact support, the time delay can be written as an operator

$$\tau^{a,c}(\phi) = \langle\phi|\,\hat{\tau}^{a,c}|\phi\rangle. \tag{2.105}$$

It is essentially self-adjoint and commutes with $\hat{H}_0$. The on-shell components of $\hat{\tau}^{a,c}$ satisfy a modification of the Eisenbud-Wigner formula:

$$\tau^{a,c}(k) = -\frac{i}{2k} S_c^*(k)\frac{\partial}{\partial k} S_c(k) + \frac{\xi}{2k^3}(1-\ln a), \quad k > 0, \tag{2.106}$$

where $S_c(k)$ is the energy-shell decomposition of the Coulomb scattering operator,

$$S_c(k) = e^{2i \arg \Gamma\left(\frac{1}{2}+\sqrt{\hat{\vec{L}}^2(k)\frac{1}{4}}+\frac{i\xi}{2k}\right)}. \tag{2.107}$$

Here, $\hat{\vec{L}}^2$ is the square of the angular momentum operator.

The choice $a = e$ seems the most natural one, because then (2.106) exactly parallels the short-range form of the Eisenbud-Wigner formula.

2.3.6 Remarks

The construction in the preceding paragraphs is consistent, at least mathematically. The deep reason that the regularization of the time delay works seems to be the state independence of the potential, i.e. the states do not depend on the parameter ξ that appears in the potential and the regularization. The modification of the time evolution due to Dollard is also physically acceptable. It seems, however, not very obvious to interpret the quantity $\tau^{a,c}(\phi)$, defined by eq. (2.104), as 'Coulomb time delay', because, as we have shown earlier, the time delay according to the standard definition for the Coulomb potential is infinite, not only in the quantum, but also in the classical case. It seems, therefore, not justifiable to interpret the finite quantity $\tau^{a,c}(\phi)$ as a time delay in the usual sense. The real physical meaning of it is not clear and remains to be found.

In the fourth chapter we will explain that the regularization of the time delay explained in this section does not work in the case of the scattering of a self-gravitating null shell, because the potential is not state-independent. There are thus two arguments that point out that for our model there is no physically acceptable analog to the well-defined time delay in short-range potentials.

Chapter 3

Light-like shells and scattering time

3.1 Introduction

Since a consistent theory of quantum gravity seems to be out of reach in the present one often chooses a model with a finite number of degrees of freedom and constructs an exact quantum theory thereof. The model is chosen such that the main features of the problems one wants to study are still present. For the study of gravitational collapse self-gravitating spherical thin shells have proven to be quite useful. The collapsing shell, although a simple finite-dimensional model, still suffers from the disease of singularities, at least in the classical theory.

A shell consists of a thin layer of some matter and separates two distinct spacetime geometries which are influenced by the matter distribution inside the shell. Spherically symmetric thin shells surrounded by their own gravitational field are widely used in cosmology and the study of quantum black holes. See, for example, [Háj00b] for a list of references.

In this chapter we first briefly explain the basic ideas about thin shells; we especially treat the spherically symmetric dust shells. We then turn our attention to light-like shells made of null-dust and construct a simple scattering system: a spherically symmetric light-like shell scattered off a spherical mirror that lies outside the horizon of the shell. Finally, we compute the scattering time measured by an observer between his two encounters with the shell.

3.2 Massive shells

3.2.1 Israel's equations

The classical paper on the dynamics of thin shells in general relativity is that by W. Israel, [Isr66]. We, however, summarize the treatment of Berezin et al. [BKT87] which is more

intuitive.

A shell is an infinitely thin layer of matter which separates two media with different properties, dividing by this spacetime into three distinct regions. We denote the world sheet of the shell by Σ. It divides the spacetime into inner ($\mathcal{M}_-$) and outer ($\mathcal{M}_+$) parts. In order to establish the formulae for the general-relativistic back reaction of the moving shell in the spacetime, we need to take Einstein's equations into account. The energy-momentum tensor T^μ_ν of the shell will in general have singularities on it, but in our treatment we assume that these are not stronger than those given by δ-functions.

First we have to introduce some notation and state our conventions. In this chapter we use units with $G = c = 1$, and the Minkowski metric is given by $\eta = \mathrm{diag}(1, -1, -1, -1)$. Greek indices are used for quantities on the four-dimensional spacetime manifold. They run from 0 to 3. Quantities on the three-dimensional hypersurfaces carry Latin indices which take the values $0, 2, 3$. Einstein's equations (with vanishing cosmological constant) in these conventions read

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{R} = 8\pi T_{\mu\nu}, \tag{3.1}$$

where $R_{\mu\nu}$ is the Ricci tensor and $\mathcal{R}$ the curvature scalar. Together with a given matter distribution $T_{\mu\nu}$ and the equations of continuity

$$T^\nu_{\mu;\nu} = 0, \tag{3.2}$$

they determine the dynamics of the shell and the surrounding spacetime. Here, the semicolon denotes the four-dimensional covariant derivative. As the hypersurface Σ may be considered as embedded into four-dimensional spacetime, Einstein's equations at the shell can be expressed in terms of the extrinsic curvature (K_{kl}) and the surface energy density (S_{kl}) tensors. K_{kl} is given by

$$K^\pm_{kl} = -n^\pm_\mu e^{\pm\nu}_{(l)} e^{\pm\mu}_{(k);\nu}, \tag{3.3}$$

where

$$e^{\pm\mu}_{(k)} \doteq \frac{\partial x^\mu_\pm}{\partial y^k} \tag{3.4}$$

are vectors tangent to the shell hypersurface, $x^\mu_\pm$ are coordinates on the inner (-) and outer (+) spacetimes $\mathcal{M}_\pm$ and y^k are coordinates at the shell. The terms 'inner' and 'outer' are to be understood as the position with respect to right spatial infinity. $n^\pm_\mu$ is a unit normal vector of Σ in the respective coordinates directed from $\mathcal{M}_-$ to $\mathcal{M}_+$. It is called unit outer normal. We denote the jump of a quantity Q at the shell by

$$[Q] \doteq Q^+ - Q^-. \tag{3.5}$$

Einstein's equations in this formalism hence read

$$\epsilon \left([K^k_l] - \delta^k_l [K^m_m] \right) = 8\pi S^k_l. \tag{3.6}$$

$\epsilon \doteq n_\mu n^\mu$ is $+1$ (-1) for space-like (time-like) shells. The special case of light-like shells $(\epsilon = 0)$ is treated in the next section, because the usual formalism breaks down. The equations (3.6) are often called *Israel's equations*.

3.2.2 Spherically symmetric time-like dust shells

Israel's equations are decisively simplified if we choose the matter of the shell to be a non-relativistic pressure-less ideal fluid, so-called *dust*. The surface energy tensor then obtains the simple form

$$S^{kl} = S u^k u^l, \quad u_k u^k = -1, \tag{3.7}$$

where S is the surface density of the shell and u^k is a unit vector tangential to the streamlines $y^k(\tau)$ of the dust particles: $\frac{dy^k}{d\tau} = u^k$. τ is the proper time measured along the streamlines. The streamlines are time-like geodesics of Σ, so the world sheet of the shell is a time-like hypersurface.

We now turn our attention to the specific case of *spherically symmetric dust shells*. In this case there is an inside $(\mathcal{M}_-)$ and a outside $(\mathcal{M}_+)$ geometry, both spherically symmetric. According to Birkhoff's theorem they must be Schwarzschild spacetimes with different mass parameters E_-, E_+. If we assume that the shell is the only source of gravity, then $\mathcal{M}_-$ is a piece of Minkowski spacetime which does not contain null infinity, and $E_- = 0$. Consequently, $E = E_+$ is the total mass of the system.

The interior metric in the *polar Minkowski coordinates* T_M, R, θ, ϕ reads:

$$ds^2 = dT_M^2 - dR^2 - R^2 d\Omega^2, \tag{3.8}$$

where we have introduced the abbreviation $d\Omega^2 \doteq d\theta^2 + \sin^2\theta d\phi^2$, that will be used throughout this work. For the exterior we choose the *Schwarzschild coordinates* T, R, θ, ϕ, for which the metric is given by the line element

$$ds^2 = f dT^2 - \frac{1}{f} dR^2 - R^2 d\Omega^2, \quad f \doteq 1 - \frac{2E}{R}. \tag{3.9}$$

The induced metric on the world sheet Σ of the shell in the coordinates τ, θ, ϕ on the shell is

$$ds^2|_\Sigma = d\tau^2 - r^2(\tau) d\Omega^2, \tag{3.10}$$

where τ is the proper time along Σ and $R = r(\tau)$ is the radius of the shell at the time τ.

In the case of spherical symmetry we have $K^2_2 = K^3_3$ (This property holds in fact for any tensor of this type). Israel's equations (3.6) thus reduce to

$$\begin{aligned} -4\pi S^0_0 &= [K^2_2], \\ -8\pi S^2_2 &= [K^0_0] + [K^2_2] \end{aligned} \tag{3.11}$$

and

$$\dot{S}^0_0 + 2\frac{\dot{r}}{r}\left(S^0_0 - S^2_2\right) + [T^n_0] = 0, \tag{3.12}$$

where $T^{\pm n}_0$ is the normal flow of the energy-momentum tensor inside $(-)$ and outside $(+)$ of the shell. Since the matter of the shell is dust, it holds that $S^2_2 = S^3_3 = 0$ and hence, $S = -S^0_0$. The two equations (3.11) thus reduce to a single one,

$$[K^2_2] = 4\pi S. \tag{3.13}$$

For a spherically symmetric geometry the non-vanishing components of the extrinsic curvature tensor are given by

$$K^{2\pm}_2 = -\frac{\sigma_\pm}{r}\sqrt{\dot{r}^2 + f_\pm}, \quad f_\pm \doteq 1 - \frac{2E_\pm}{r}. \tag{3.14}$$

Since $f_- = 1$ and $f_+ = 1 - \frac{2E}{r}$, the equation of motion of the massive shell reads

$$\frac{\sigma_-}{r}\sqrt{\dot{r}^2 + 1} - \frac{\sigma_+}{r}\sqrt{\dot{r}^2 + 1 - \frac{2E}{r}} = 4\pi S. \tag{3.15}$$

The meaning of the two signs $\sigma_\pm$ is as follows: $\sigma = +1$ if the radii increase in the direction of the outer normal to the shell, $\sigma = -1$ if they decrease. We will discuss the behaviour of these signs later in this section.

For a shell of dust, equation (3.12) reduces to

$$\dot{S} + 2\frac{\dot{r}}{r}S = 0. \tag{3.16}$$

Hence,

$$S = \frac{M}{4\pi r^2(\tau)} \tag{3.17}$$

is the surface energy density of a sphere of radius $R = r(\tau)$ with rest mass M. Inserting this result into (3.15), we obtain

$$\sigma_- \sqrt{\dot{r}^2 + 1} - \sigma_+ \sqrt{\dot{r}^2 + 1 - \frac{2E}{r}} = \frac{M}{r}, \tag{3.18}$$

which is the final form of the equation of shell motion.

Solving (3.18) for E yields the *total energy of the shell*

$$E = E(r, \dot{r}) = M\sqrt{\dot{r}^2 + 1} - \frac{M^2}{2r}. \tag{3.19}$$

The difference $E - M$ can be viewed at as the 'binding energy' of the shell due to gravity. We can rewrite eq. (3.19) in the form of the radial equation of motion of a particle moving in an effective potential [Háj92]:

$$\dot{r}^2 + V_{\text{eff}}(r) = 0, \quad V_{\text{eff}}(r) \doteq 1 - \left(\frac{E}{M} + \frac{M}{2r}\right)^2. \tag{3.20}$$

There are only three types of solutions:

1) $E \geq M, \dot{r} > 0$: The shell starts at $r = 0$ at a finite time τ_- and approaches $r = \infty$ with $\tau = +\infty$ and $\dot{r} = \sqrt{\frac{E^2}{M^2} - 1} \geq 0$.

2) $E \geq M, \dot{r} < 0$: The time inversion of case 1).

3) $E < M$: The shell starts at $r = 0, \tau = \tau_-, \dot{r} = \infty$, reaches a turning point at $r_0 = \frac{M^2}{2(M-E)}$ and falls back to $r = 0$ at the finite time $\tau_+, \dot{r} = -\infty$.

We observe that ingoing shells must reach the singularity, so nothing can prevent the collapse to a black hole in the classical theory once the shell has passed the horizon at $R = 2E$. Moreover, an outgoing shell can never return behind the white hole horizon, provided that $E \geq M$.

We add a note on the signs $\sigma_\pm$. In flat (Minkowski) space ($\mathcal{M}_-$) we have always $\sigma_- = +1$ for future-oriented shells (we will consider only these). Outside ($\mathcal{M}_+$) the geometry is Schwarzschild. In the region I (see fig. (3.1)) it holds that $\sigma_+ = +1$, since the outer normal points into the direction of growing radial coordinate. Similarly, in the corresponding region I' at left infinity, the sign has the value $\sigma_+ = -1$ for future-oriented shells. In the two regions II, III beyond the horizons the sign can change. Where this happens depends on the actual trajectory of the shell. But since we will not use massive shells in the following, we do not go into details. The signs $\sigma_\pm$ are not defined for the light-like case, because the normal n_ν lies tangential to the shell, and we can not give it a meaning to be directed 'outwards'.

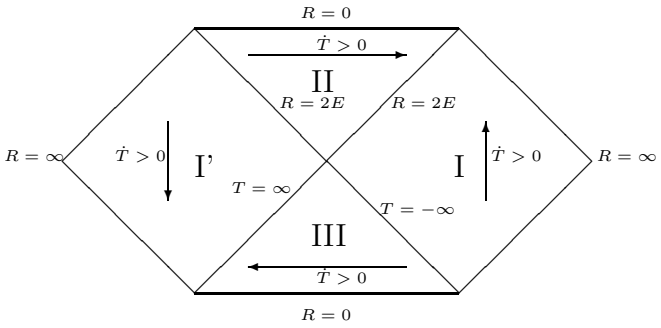

Figure 3.1: *Penrose diagram of the maximal extension of Schwarzschild spacetime showing the directions of growing Schwarzschild time coordinate T in the four regions. The singularities are depicted by thick lines. The horizons lie at $R = 2E$. The regions I and I' lie outside, whereas the regions II and III are inside the horizons.*

3.3 Null shells

3.3.1 The light-like limit

The fundamental paper about light-like thin shells has been written by Barrabès and Israel [BI91]. The description in the preceding paragraphs which works in the subluminal case breaks down for light-like shells because the extrinsic curvature tensor K_{kl} is no longer uniquely definable. Barrabès and Israel have established a beautiful method to circumvent this problem. Their description includes the light-like case as a continuous limit of the time-like one. But we will not take use of their formalism in the next section. We will rather derive the equations of motion of a light-like shell from the equations of null geodesics in Schwarzschild and Minkowski spacetime.

3.3.2 Light-like shells as null geodesics

We determine the equations of motion for light-like (null) spherically symmetric shells of dust. In this section we loosely follow the treatment by Hájíček and Bičák in [HB97].

The spacetime interior of the shell can be described by the Minkowski metric

$$ds^2 = dT_M^2 - dR^2 - R^2 d\Omega^2. \tag{3.21}$$

The outside geometry is given by the Schwarzschild metric

$$ds^2 = f dT^2 - \frac{1}{f} dR^2 - R^2 d\Omega^2, \; f \doteq 1 - \frac{2E}{R}. \tag{3.22}$$

E is the Schwarzschild mass. The metric on the hypersurface Σ defined by the light-like shell is simply

$$ds^2|_\Sigma = -r^2(\lambda)d\Omega^2, \tag{3.23}$$

where λ is some parameter.

Since the motion of the null shell is that of a radial null geodesic in the two-dimensional spacetime spanned by the coordinates T_M and R (or T and R) with the trajectory $R = r(\lambda), T_M = t_M(\lambda)$ (or $T = t(\lambda)$), the formulae for the shell can be obtained from

$$\dot{t}_M^2 - \dot{r}^2 = 0, \tag{3.24}$$

$$f\dot{t}^2 - \frac{1}{f}\dot{r}^2 = 0 \tag{3.25}$$

in the respective interior (exterior) coordinates. We choose the parameter λ of the shell trajectory to be a so-called physical parameter: $\dot{t}_M = \eta E$, $f\dot{t} = \eta E$ (see [HB97] for the exact definition). The overdot represents the derivative with respect to this parameter. Then, the eqs. (3.24) and (3.25) imply the following three equations which govern the shell motion:

$$\begin{aligned} \dot{r}^2 &= E^2, \\ \dot{t}_M^2 &= \dot{r}^2, \\ f^2\dot{t}^2 &= \dot{r}^2. \end{aligned} \tag{3.26}$$

The first equation of (3.26) is the so-called *radial equation* while the two others are the *time equations*. Taking square roots we obtain

$$\begin{aligned} \dot{r} &= \zeta\eta E, \\ \dot{t}_M &= \eta_-|\dot{r}|, \\ f\dot{t} &= \eta_+|\dot{r}|. \end{aligned} \tag{3.27}$$

The four numbers ζ, η, η_- and η_+ can take the values ± 1. Their meaning is explained in the following. ζ just distinguishes which asymptotic region of the maximal extension of Schwarzschild spacetime the geodesic reaches, $\zeta = +1$ (-1) for the right (left) one. The parameter η is connected with the direction of motion of the shell with respect to right infinity. It takes the value $+1$ if the shell is outgoing and -1 if it is ingoing, provided the shell is future-oriented. The meaning of the two remaining signs $\eta_\pm$ is determined

as follows: Consider the case of the outgoing shells. Inside the shell the retarded null coordinate

$$\tilde{U} = T_M - R \tag{3.28}$$

is constant along the trajectory. The same holds for the retarded Eddington-Finkelstein (REF) coordinate

$$\bar{U} = T - R - 2E \ln \left| \frac{R}{2E} - 1 \right| \tag{3.29}$$

in the spacetime exterior of the shell. Hence, along the outgoing shell's trajectory:

$$\dot{\tilde{U}} = 0 = \dot{t}_M - \dot{r} = \eta_- E - E. \tag{3.30}$$

which implies that $\eta_- = +1$ for future-oriented shells. This is consistent with the fact that η_- is the sign of $\dot{t}_M$ which determines the time orientation of the trajectories. For outgoing null geodesics in the exterior spacetime it holds that

$$\dot{\bar{U}} = 0 = \dot{t} - \dot{r} \left(\frac{r}{r - 2E} \right) = \frac{E}{f}(\eta_+ - 1), \tag{3.31}$$

hence also $\eta_+ = +1$ for future-oriented shells. The same results are obtained in the case of the ingoing shells. From the third eq. of (3.27) follows that

$$\eta_+ = \operatorname{sgn}(f\dot{t}). \tag{3.32}$$

For null geodesics the sign of $\dot{t}$ changes as the trajectory crosses a horizon and then does not change anymore, because, in contrary to time-like geodesics, the light-like one never lies tangent to the lines of constant t. Also the sign of f flips behind the horizon and then stays the same, so the sign of $f\dot{t}$ remains constant with value η_+. We thus observe that all the four signs remain constant during the evolution. In the case of a massive (time-like) shell, however, this is not true. The parameter η_+ can change its sign as the shell reaches the region behind the horizon.

Past-orientation is represented by $\eta_\pm = -1$. Since shells with Minkowski spacetime inside cannot reach left infinity, we will consider only future-oriented shells that reach (or come from) right infinity; thus we set

$$\zeta = \eta_\pm = +1. \tag{3.33}$$

Thus, the remaining parameter η distinguishes between ingoing and outgoing shells, and the eqs. (3.27) are reduced to

$$\begin{aligned} \dot{r} &= \eta E, \\ \dot{t}_M &= |\dot{r}| = E, \\ f\dot{t} &= |\dot{r}| = E. \end{aligned} \tag{3.34}$$

We first solve these equations for outgoing shells ($\eta = +1$). In order to fix the parameter λ we have to impose an initial condition. We choose

$$r(0) = 0. \tag{3.35}$$

Hence we obtain the solution of the radial equation,

$$r(\lambda) = E\lambda, \tag{3.36}$$

and that the parameter λ must take values from the interval $[0, \infty)$. We also have to determine the constants appearing in the integration of the two time equations. For an outgoing radial null geodesic in flat spacetime it holds that $\tilde{U} = \tilde{u}$ is constant. $\tilde{U} = T_M - R$ implies that

$$\tilde{u} = t_M(\lambda) - r(\lambda). \tag{3.37}$$

The solution of the Minkowski time equation reads

$$t_M(\lambda) = E\lambda + c, \tag{3.38}$$

where c is an arbitrary constant. From the two last equations and from the fixing of λ by eq. (3.35) follows that $c = \tilde{u}$, hence,

$$t_M(\lambda) = E\lambda + \tilde{u}. \tag{3.39}$$

We thus have determined the first integration constant.

The shell trajectory in the REF spacetime is given by

$$\bar{U} = \bar{u} = t(\lambda) - r(\lambda) - 2E \ln \left| \frac{r(\lambda)}{2E} - 1 \right|, \tag{3.40}$$

whereas the solution of the second time equation reads

$$t(\lambda) = E\lambda + 2E \ln \left| \frac{\lambda}{2} - 1 \right| + c', \tag{3.41}$$

where c' is another arbitrary constant. Hence, the two constants c' and $\bar{u}$ are equal; and we obtain that

$$t(\lambda) = E\lambda + 2E \ln \left| \frac{\lambda}{2} - 1 \right| + \bar{u}. \tag{3.42}$$

The relation between the two time coordinates along the shell is thus given by

$$t(t_M) = t_M - \tilde{u} + \bar{u} + 2E \ln \left| \frac{t_M - \tilde{u}}{2E} - 1 \right|. \tag{3.43}$$

A similar calculation for the ingoing ($\eta = -1$) case yields the following formulae:

$$\begin{aligned} r(\lambda) &= -E\lambda, \quad \lambda \in (-\infty, 0], && (3.44)\\ t_M(\lambda) &= E\lambda + \tilde{v}, && (3.45)\\ t(\lambda) &= E\lambda - 2E \ln \left| -\frac{\lambda}{2} - 1 \right| + \bar{v}, && (3.46)\\ t(t_M) &= t_M - \tilde{v} + \bar{v} - 2E \ln \left| \frac{\tilde{v} - t_M}{2E} - 1 \right|. && (3.47) \end{aligned}$$

Because of the initial condition (3.35), the parameter λ takes negative values for ingoing and positive ones for outgoing shells, in order to preserve the positiveness of r. The horizon lies at the radius $R_H = 2E$. The two values of the parameter of the shell trajectory there are $\lambda_H = 2\eta$. The Schwarzschild time of the shell at the horizon, $t(\lambda_H)$, diverges, but anywhere else it is well-defined.

3.4 Scattering time and the spherical mirror

3.4.1 Glueing in- and outgoing geodesics

In this section we construct a simple scattering system containing a null shell with total energy E and a spherical mirror with the fixed radius $R = R_M$ centered at $R = 0$. We allow the energy of the shell only to have values that are small enough, such that the horizon lies inside the mirror. This leads to a energy cut-off at

$$E_{\max} = \frac{R_M}{2}. \tag{3.48}$$

Hence, only shells with an energy E inside the interval $[0, \frac{R_M}{2})$ are allowed.

We now glue together an ingoing and an outgoing radial null geodesic that meet at the mirror and make a single trajectory out of them. We thus have to find for each ingoing

geodesic $\tilde{V} = \tilde{v}$ ($\bar{V} = \bar{v}$) its corresponding outgoing counterpart $\tilde{U} = \tilde{u}$ ($\bar{U} = \bar{u}$). We moreover require the two functions $t_M(\lambda)$ and $t(\lambda)$ to be continuous in the parameter λ at the mirror. First we fix the affine parameter λ by the condition

$$r(0) = R_M, \tag{3.49}$$

such that the radial shell equation reads

$$r(\lambda) = R_M + \eta E\lambda, \tag{3.50}$$

where

$$\eta = +1, \quad \text{if } \lambda \geq 0, \tag{3.51}$$

$$\eta = -1, \quad \text{if } \lambda < 0, \tag{3.52}$$

For an ingoing trajectory in the interior spacetime given by

$$\tilde{V} = \tilde{v} = t_M(\lambda) + r(\lambda) \tag{3.53}$$

it holds that

$$t_M(\lambda) = E\lambda + \tilde{v} - R_M, \quad \lambda \in (-\infty, 0]. \tag{3.54}$$

Hence, at the mirror, where $\lambda = 0$,

$$t_M(\lambda = 0) = \tilde{v} - R_M \doteq T_0. \tag{3.55}$$

From the outgoing geodesic given by

$$\tilde{U} = \tilde{u} = t_M(\lambda) - r(\lambda) = t_M(\lambda) - E\lambda - R_M \tag{3.56}$$

it follows that

$$t_M(\lambda) = E\lambda + \tilde{u} + R_M, \quad \lambda \in [0, \infty), \tag{3.57}$$

and, therefore,

$$t_M(0) = \tilde{u} + R_M \doteq T_0. \tag{3.58}$$

Since we require $t_M = T_0$ at the mirror for both parts of the trajectory (the Minkowski time of the shell shall be continuous at the mirror), we obtain the junction condition between $\tilde{u}$ and $\tilde{v}$

$$\tilde{u} = \tilde{v} - 2R_M. \tag{3.59}$$

If we put together both parts, we get the function $t_M(\lambda)$ for the entire trajectory

$$t_M(\lambda) = E\lambda - R_M + \tilde{v}, \quad \lambda \in \mathbb{R}. \tag{3.60}$$

A similar calculation in the Schwarzschild spacetime leads to the junction condition

$$\bar{u} = \bar{v} - 2R_M - 4E \ln \left| \frac{R_M}{2E} - 1 \right| \doteq \bar{v} - 2R_M^* \tag{3.61}$$

and to the function $t(\lambda)$ measuring the Schwarzschild time along the shell trajectory,

$$t(\lambda) = E\lambda + 2\eta E \ln \left| \frac{\eta E\lambda + R_M}{2E} - 1 \right| + \bar{v} - R_M - 2(\eta+1)E \ln \left| \frac{R_M}{2E} - 1 \right|. \tag{3.62}$$

Here the sign of λ coincides with η.

Equations (3.50), (3.60) and (3.62) together describe the trajectories of the shells in this simple scattering system. Eqs. (3.59) and (3.61) provide the necessary junction conditions.

3.4.2 Scattering time at the radius R_O

We calculate the scattering time measured by an observer at rest at the radius $R = R_O > R_M$. During the measurement the observer is situated in the Schwarzschild spacetime exterior to the shell. Therefore, we have to compute the two values t_+, t_- of the function $t(\lambda)$ at the radius $R = r(\lambda) = R_O$ and then take the difference. The resulting quantity is the *scattering time* in terms of the Schwarzschild time coordinate, we denote it by

$$\Delta T \doteq t_+(R_O) - t_-(R_O). \tag{3.63}$$

Using the equations (3.50) and (3.62) we obtain the Schwarzschild time along the shell in terms of the radial coordinate r:

$$t(r) = \eta r + 2\eta E \ln \left| \frac{r}{2E} - 1 \right| + \bar{v} - (\eta+1) \left(R_M + 2E \ln \left| \frac{R_M}{2E} - 1 \right| \right). \tag{3.64}$$

Hence,

$$t_-(R_O) = -R_O - 2E \ln\left|\frac{R_O}{2E} - 1\right| + \bar{v}, \tag{3.65}$$

$$t_+(R_O) = R_O + 2E \ln\left|\frac{R_O}{2E} - 1\right| + \bar{v} - 2R_M - 4E \ln\left|\frac{R_M}{2E} - 1\right|. \tag{3.66}$$

Thus, the scattering time (3.63) equals to

$$\Delta T = 2(R_O - R_M) + 4E \ln\left|\frac{R_O - 2E}{R_M - 2E}\right|. \tag{3.67}$$

The proper time measured by an ideal clock carried along by an observer residing in the Schwarzschild spacetime at the constant radius R_O is given by

$$s = \sqrt{1 - \frac{2E}{R_O}}\, T, \tag{3.68}$$

hence, the *proper scattering time* measured by the observer reads

$$\Delta s = \sqrt{1 - \frac{2E}{R_O}}\, \Delta T = 2\sqrt{1 - \frac{2E}{R_O}} \left(R_O - R_M + 2E \ln\left|\frac{R_O - 2E}{R_M - 2E}\right| \right). \tag{3.69}$$

Of course, the scattering time tends to infinity as the observer approaches higher and higher values:

$$\lim_{R_O \to \infty} \Delta s = \infty. \tag{3.70}$$

If the energy of the shell approaches $E_{\max} = \frac{R_M}{2}$ such that the Schwarzschild radius of the shell reaches the mirror, then the scattering time tends to infinity also for observers with finite values of their radial coordinate. So the scattering time becomes infinite if the shell falls under its Schwarzschild horizon.

To summarize the results we state that the scattering time Δs in a model with a fixed spherical mirror at $R = R_M$ can be defined for shells carrying a total energy $E \in [0, \frac{R_M}{2})$ and observers at $R = R_O \geq R_M$. It depends on the energy of the shell E and on the values of the radial coordinate of the mirror (R_M) and the observer (R_O). The scattering time diverges when the shell's energy approaches the cut-off or when the radius of the observer goes to infinity.

Chapter 4

Sojourn time and gauge invariance

4.1 Why the time delay cannot be defined in a finite way in the case of the self-gravitating shell

In chapter 2 it has been pointed out that in the standard definition the time delay (eq. (2.47)) is infinite for long-range potentials, i.e. potentials with a fall-off at infinity as fast as or slower than $\frac{1}{r}$. Both the gravitational and the Coulomb potential belong to this class. It has also been shown that there is a certain way to redefine the time delay in the case of the Coulomb potential, such that it becomes finite (eq. (2.104)). One has to modify the free time evolution operator by adding to the free Hamiltonian a term depending on the central charge of the potential, cf. eq. (2.95). Then the free and Coulomb sojourn times (given by the eqs. (2.97) and (2.98)) are adapted to the new 'free' evolution. The limit as $r \to \infty$ of their difference is the, now finite, modified Coulomb time delay.

In the case of a self-gravitating shell, the system can also be considered having a 'central potential',

$$V(r) \propto \frac{E}{r}, \tag{4.1}$$

that is long-range and depends on the total energy of the shell. This potential has the same fall-off at infinity as the Coulomb one, so the method that has been successful in the Coulomb case could be in principle applied also here. But there is a problem: In contrast to the Coulomb case, where the central charge in the potential does not depend on the scattered particle, the self-gravitating shell produces — so to say — its own potential. When the shell is quantized (i.e., in our approach, the total energy E of the shell becomes an operator), then the quantum states are dependent on the energy E. A wave packet describing a quantum shell is hence a superposition of states with different energies. Thus, also the central potential becomes a superposition of potentials corresponding to different energies. But as the modified free evolution depends on the central charge in the

Coulomb case, the corresponding evolution in the case of the self-gravitating shell would depend on the shell's energy and would be, therefore, state-dependent, thus rendering the regularization itself state-dependent. The definitions of the modified free and Coulomb sojourn times and therewith the modified time delay are hence not applicable in the case of the gravitating shell.

4.2 Scattering time and sojourn time

In the preceding section it has been argued that the time delay can not be made finite in the case of the scattering system consisting of a self-gravitating shell and a spherical mirror. Therefore, we have to find a different quantity that is finite and well-defined and that can be interpreted as a measure for the duration of the scattering. It should also depend on the strength of the potential.

In chapter 3 the *proper scattering time* Δs given by eq. (3.69) has been computed. It is the proper time measured by an ideal clock carried along by an observer at the constant radius R_O between the two instants at which the shell and the observer meet. The thus defined scattering time is a purely classical quantity. It satisfies the requirements stated above.

Although the time delay measured at infinity diverges and cannot be regularized, the *sojourn time* inside a finite region is clearly *finite* in our system. The sojourn time is the mean time a particle resides under a given radius. Its interpretation as the mean residence time of a quantum state in a specific region is plausible, at least in Minkowski spacetime, c.f. section 2 of the appendix A. The sojourn time is, as it has been defined in eq. (2.39), not a classical quantity but the expectation value of a quantum operator.

There is a problem in the definition of the sojourn time when one tries to adopt it to curved spacetimes. In flat spacetime the sojourn time is the mean Minkowski time T_M the particle described by the wave function Φ resides inside a given radius R,

$$T_S(\Phi) = \int_{-\infty}^{\infty} dT_M \, \langle \Phi | \hat{P}_R | \Phi \rangle, \tag{4.2}$$

where $\hat{P}_R$ is the projection operator on the states inside the radius R. In the case of curved spacetimes it is not clear anymore, over which 'time' the integral has to be taken. For the null shell scattered by the mirror the proper time at a fixed radius R seems to be a good choice. But this time coordinate depends on the energy of the shell, E, that is replaced by the corresponding operator $\hat{E}$ in the quantum theory. The (infinitesimal) proper time ds is thus an operator and its mean value in the state Φ has to be computed before the integration can be performed. Besides factor ordering difficulties with the projector $\hat{P}_R$, there emerge several conceptual problems, that are discussed in more detail in the appendix A. We, therefore, have to find a better definition for the quantum scattering time for our model.

In order to achieve this goal we define the quantum scattering time to be the expectation value of the operator $\widehat{\Delta s}$ for the wave packets of a suitable quantum theory. The operator $\widehat{\Delta s}$ results from the replacement of E in the classical scattering time Δs by the corresponding operator $\hat{E}$:

$$\widehat{\Delta s} = \Delta s(E = \hat{E}). \tag{4.3}$$

But one must be careful when replacing classical quantities by operators. The classical observable must be gauge invariant (i.e. a Dirac observable), otherwise severe problems occur in the quantum theories corresponding to different gauges, see [Hájí00a]. The scattering time Δs is, the way it has been defined in the third chapter, not gauge invariant. But there is, fortunately, a method to redefine it, which renders it truly gauge invariant. This is shown in section 5 of this chapter.

Let Δs be the gauge invariant scattering time. From now on we will refer to the *mean value of the scattering time operator* $\widehat{\Delta s}$ as the *sojourn time* of the quantum state under the radius R:

$$t_s(R) \doteq \langle \Delta s(R) \rangle. \tag{4.4}$$

Even though this definition does not coincide with the usual one for the sojourn time, it seems to be the best approximation we are able to do.

4.3 Gauge invariant scattering time

In the preceding section we have argued that the mean value of the scattering time operator can be interpreted as the sojourn time of our quantum system. The main problem with Δs is, that it is not a gauge invariant, as it will be shown in the next section. But only gauge invariant (Dirac) observables can be made into operators that have a well-defined interpretation. It is, however, sometimes difficult to find suitable gauge invariant quantities in general relativistic models. Therefore, one often has to content oneself with a semiclassical approximation of the theory, where it may be less difficult to find the needed gauge invariant. In the case of systems with an infinite-dimensional transformation group the method of small disturbances invented by DeWitt [DeW63] seems to be appropriate. DeWitt's method is discussed in more detail in chapter 9. His approach requires a suitable gauge, however, with respect to which the metric can be written as a sum of a flat background metric and a small perturbation thereupon. Fortunately, such a gauge exists for our model. It is the Cartesian transform of the central-regular gauge that is constructed in chapter 6. The proof that this gauge does really satisfy all of the requirements for DeWitt's method is written down in the appendix C. Since in DeWitt's approximation only infinitesimal gauge transformations remain, it may be easier to find a gauge invariant quantity that can be associated with the sojourn time, than in the exact theory.

We are, however, in the fortunate situation that we have managed to find a satisfactory definition of the scattering time that is also gauge invariant in the exact theory. This is pointed out in section 5 of this chapter. Hence, the approximation due to DeWitt is not needed. But the results related to the DeWitt-approximation we have obtained are interesting also beyond the context of the sojourn time. They hence deserve to be mentioned in this work. Also, the perturbative method could be used to compare the results of the two methods in the low-energy regime.

In the next section we argue why the coordinate-based definition of the scattering time is not gauge invariant. The last section is devoted to a definition that is based on the invariant line element and is, therefore, independent of the choice of coordinates.

4.4 Coordinate-based definition

In the second section of this chapter we have argued that the scattering time operator $\widehat{\Delta s}$ can play the role of the expected value of the sojourn time $t_s(R)$ under the radius R. The scattering time Δs has been constructed by using the trajectories of the shell and the observer in section 4 of chapter 3 and also, for the DeWitt case, in section section 6 of chapter 9. We, therefore, do not write down the construction once more. The thus defined quantity is, unfortunately, not gauge invariant, because the trajectories that have been used depend on the choice of coordinates. This is already true for the corresponding trajectories in the semi-classical approach according to DeWitt and is explicitly shown in subsection 8.3. of chapter 9. There, only infinitesimal gauge transformations act on the coordinates, but they are general enough that the scattering times in two different coordinate systems are not the same. Of course, if a quantity is already not invariant under infinitesimal gauge transformations, it can not be invariant under finite transformations! The quantity Δs is, nevertheless, useful, because it will be demonstrated in the next section, that one can define the scattering time in a gauge invariant way, such that it becomes the difference of two invariant quantities.

4.5 Gauge invariant definition

In this section we present a gauge invariant way to define the scattering time. The *proper time measured by an ideal clock along the trajectory of an observer residing at the constant radius* $R = R_O$ is clearly gauge invariant, because it is the integral over the square root of the invariant line element ds^2, which is coordinate invariant by definition. The thus defined proper time is, however, not unique, because an arbitrary constant turns up, when one integrates over ds. We want to define the proper times measured by the observer at the two instants at which he observes the shell pass by. We denote them by s_- for the

ingoing and by s_+ for the outgoing shell. The two times read

$$s_- = C, \quad s_+ = C + s, \tag{4.5}$$

where C is the integration constant. The term s in s_+ is the proper time an ideal clock carried along by the observer measures along his trajectory between the two instants p_1, p_2 at which he encounters the shell. It is defined as follows:

$$s = \int_\gamma ds, \tag{4.6}$$

where γ is the curve connecting the two points p_1 and p_2 along the trajectory of the observer and where the infinitesimal proper time ds is given by

$$ds = \sqrt{ds^2} = \sqrt{g_{\mu\nu} dx^\mu dx^\nu}. \tag{4.7}$$

The equations (4.6, 4.7) are invariant under coordinate transformations. One can thus compute them in the coordinates that are the most suitable. Since the curve γ lies in the region of spacetime exterior to the shell, the Schwarzschild coordinates (T, R, θ, ϕ) are a good choice. The trajectory of the observer with respect to these coordinates is given by

$$R(\mu) = R_O, \quad T = \mu, \quad \mu \in (\mu_-, \mu_+), \tag{4.8}$$

where μ is the parameter along the curve γ and where $p_1 = (T(\mu_-), R(\mu_-))$ and $p_2 = (T(\mu_+), R(\mu_+))$. The values $\mu_\pm$ of the parameter are hence found by computing the Schwarzschild times $T_\pm$ of the shell meeting the observer. Since $R = R_O$ is constant along γ and since the rate $\frac{ds}{dT}$ between the two time coordinates in the exterior region is given by $\sqrt{1 - \frac{2E}{R}}$, the infinitesimal proper time simply reads

$$ds = \sqrt{1 - \frac{2E}{R_O}} dT. \tag{4.9}$$

The line integral along a curve γ is defined as follows:

$$\int_\gamma f(x) dx = \int_a^b f(\gamma(t)) \dot{\gamma}(t) dt. \tag{4.10}$$

In our case, γ is simply given by $T(\mu) = \mu$. Hence, s reads

$$s = \sqrt{1 - \frac{2E}{R_O}} \int_{\mu_-}^{\mu_+} d\mu = \sqrt{1 - \frac{2E}{R_O}} \int_{T_-}^{T_+} dT = \sqrt{1 - \frac{2E}{R_O}} (T_+ - T_-). \tag{4.11}$$

We thus only have to compute the two Schwarzschild times $T_{\pm}$. For this end we need to know the trajectory of the shell with respect to the coordinates T and R. It can be determined by using the transformation from T, R to the double-null Eddington-Finkelstein (DNEF) coordinates U, V, cf. the next chapter:

$$V = T + R^*, \quad U = T - R^*, \tag{4.12}$$

where

$$R^* \doteq R + 2E \ln \left| \frac{R}{2E} - 1 \right|. \tag{4.13}$$

Let $T(\lambda), R(\lambda)$ be the trajectory of the shell. Then it holds that

$$V = v = T(\lambda) + R^*(\lambda) \tag{4.14}$$

along the ingoing and that

$$U = v - 2R^*_M = T(\lambda) - R^*(\lambda) \tag{4.15}$$

along the outgoing shell. We make the following ansatz for $T(\lambda)$:

$$T(\lambda) = \lambda - \lambda_0 + v - R^*_M. \tag{4.16}$$

At the mirror the in- and the outgoing trajectories meet, hence $T = v - R^*_M$ and, therefore, $\lambda_0 = 0$. It also holds at the mirror that $R(\lambda = 0) = R_M$. Thus, along the ingoing shell ($\lambda < 0$):

$$R^*(\lambda) = v - T(\lambda) = -\lambda + R^*_M, \tag{4.17}$$

whereas along the outgoing shell ($\lambda > 0$):

$$R^*(\lambda) = T(\lambda) - v + 2R^*_M = \lambda + R^*_M. \tag{4.18}$$

Hence, the function $R^*(\lambda)$ reads

$$R^*(\lambda) = R^*_M + |\lambda|. \tag{4.19}$$

Eq. (4.19) can be easily solved for $R(\lambda)$ by using the definition of the Kruskal function, cf. appendix E, yielding the result

$$R(\lambda) = 2E\kappa \left(e^{\frac{|\lambda| + R^*_M}{2E}} \right). \tag{4.20}$$

The trajectory of the shell is thus given by eq. (4.20) and

$$T(\lambda) = \lambda + v - R_M^*. \tag{4.21}$$

The values $T_\pm$ are obtained by solving the equation

$$R(\lambda_\pm) = R_O \tag{4.22}$$

for $\lambda_\pm$, the result being

$$\lambda_\pm = \pm(R_O^* - R_M^*), \tag{4.23}$$

and inserting it into eq. (4.21), yielding

$$T_- = v - R_O^*, \quad T_+ = v + R_O^* - 2R_M^*. \tag{4.24}$$

Hence, according to eq. (4.11) the proper scattering time s reads

$$s = 2\sqrt{1 - \frac{2E}{R_O}}\left(R_O - R_M + 2E \ln\left|\frac{R_O - 2E}{R_M - 2E}\right|\right) = \Delta s. \tag{4.25}$$

Thus, the two definitions Δs and s coincide.

Now, in the definition of the two proper times $s_\pm$, only the constant C remains to be determined. If we require the proper time s_- to be equal to the Minkowski time in the past of the shell, that is given by

$$T_M^- = v - R_O, \tag{4.26}$$

we obtain that

$$C = v - R_O. \tag{4.27}$$

Hence, the proper times $s_\pm$ read

$$s_- = v - R_O, \quad s_+ = v - R_O + s. \tag{4.28}$$

The difference of the two proper times yields of course the scattering time s, as required:

$$s_+ - s_- = s \doteq t_s. \tag{4.29}$$

Here we have introduced the same symbol for t_s for the (classical) gauge invariant scattering time as for the sojourn time to emphasize that it is, so to say, its classical analog.

We thus have found, that the proper scattering time measured by an observer at the fixed radius R_O can be defined in a gauge invariant way and that it is also given by the eq. (4.25). Hence, the sojourn time of a state in the corresponding quantum theory is given by the mean value of the operator $\hat{t}_s$ associated to t_s:

$$t_s(\Phi) = \langle\Phi|\, \hat{t}_s \,|\Phi\rangle. \tag{4.30}$$

In the last section of the chapter 9 on the method by DeWitt we will demonstrate that the second definition of the scattering time is invariant under infinitesimal coordinate transformations, in contrast to the definition based on explicit coordinates. The two definitions of the scattering time, Δs and s, seem to agree only because of the special coordinates chosen in the computation of Δs.

Chapter 5

The space of solutions

5.1 Preliminaries

We construct and discuss the space of all solutions for the system consisting of a single spherically symmetric thin null shell and a spherical, perfectly reflecting mirror at the fixed radius $R = R_M$. The space of solutions for a single shell without the mirror is already well-known (cf. e.g. [HK01]). Also the case of multiple shells has been discussed in the literature, [HK02a]. Our construction closely follows the methods of [HK02a].

There are two kinds of parameters characterizing a solution. Changes in the first kind produce a change of the properties of the solution. These parameters thus have a real physical meaning and can be chosen as coordinates in the physical phase space. On the other hand, changes in the parameters of the second kind do not affect the properties of the solution. Such parameters describe thus just pure gauge.

All spacetime solutions satisfy the following conditions: The metric outside the shell is the Schwarzschild one corresponding to the Schwarzschild energy E. The spacetime inside the shell is flat. It has a regular inner boundary defined by the mirror at the radius $R = R_M > 0$. The Cauchy hypersurfaces have thus the topology of $\mathbb{R}^2 \times \mathbb{S}^1$. The spacetime part outside the shell is denoted by I, the two parts inside the shell by II (for the interior of the outgoing shell) and III (inside the ingoing shell). The shell forms hypersurface boundaries of these three pieces. These hypersurfaces are light-like with respect to the metrics of their adjacent spacetimes. There exist coordinates in the neighbourhood of any shell point such that the metric is continuous in a neighbourhood of this point. The resulting total spacetime is denoted by $\mathcal{M}$. We furthermore assume that the energy density of the shell is positive.

The topology of the Cauchy hypersurfaces is that of $\mathbb{R}^2 \times \mathbb{S}^1$, but we assume that there is only one infinity (that on the right) and thus only one scri, $\mathcal{I} = \mathcal{I}_- \cup \mathcal{I}_+$. It is also assumed that the observers live there and define a reference frame. So each shell starts at $\mathcal{I}_-$, is ingoing until it is reflected by the mirror and ends up as a outgoing shell at $\mathcal{I}_+$.

We are now ready to construct the solutions. The regions inside the shell, II and III, are parts of Minkowski spacetime, $\mathcal{M}_0$, whereas the region outside, I, is a part of the Kruskal spacetime corresponding to the energy E, $\mathcal{M}_E$. The corresponding maximal analytical extensions are denoted by $(\bar{\mathcal{M}}_0, \bar{\mathcal{M}}_E)$ for the Minkowski and the Kruskal spacetimes and by $\bar{\mathcal{M}}$ for the total shell spacetime. The Minkowski spacetime is oriented such that the mirror is on the left and infinity is on the right. From the positivity of the energy density of the shell it follows that then $E > 0$. The diagram fig. (5.1) depicts the regions and the trajectory of the shell.

5.2 Singular DNEF gauge

The so-called Double-Null Eddington-Finkelstein (DNEF) (cf. [HK02a]) coordinates $\bar{U}$, $\bar{V}$ are very suitable to describe the spacetime outside the shell. They are defined by

$$\bar{U} = T - R - 2E \ln \left| \frac{R}{2E} - 1 \right|, \quad \bar{V} = T + R + 2E \ln \left| \frac{R}{2E} - 1 \right|, \tag{5.1}$$

where T and R are the Schwarzschild coordinates. The abbreviation

$$R^* \doteq R + 2E \ln \left| \frac{R}{2E} - 1 \right| \tag{5.2}$$

is very useful. The DNEF coordinates are singular at the horizons of the in- and the outgoing shell. We therefore restrict the allowed values of the Schwarzschild mass such that

$$E < \frac{R_M}{2}. \tag{5.3}$$

Then, the horizon lies inside the mirror and the coordinates (5.1) are regular everywhere in region I. The metric in I has the form

$$ds^2 = \bar{A}(\bar{U}, \bar{V}) d\bar{U} d\bar{V} - \bar{R}^2(\bar{U}, \bar{V}) d\Omega^2, \tag{5.4}$$

where

$$\bar{A} = 1 - \frac{2E}{\bar{R}} \tag{5.5}$$

and

$$\bar{R} = 2E\kappa \left(e^{\frac{\bar{V}-\bar{U}}{4E}} \right). \tag{5.6}$$

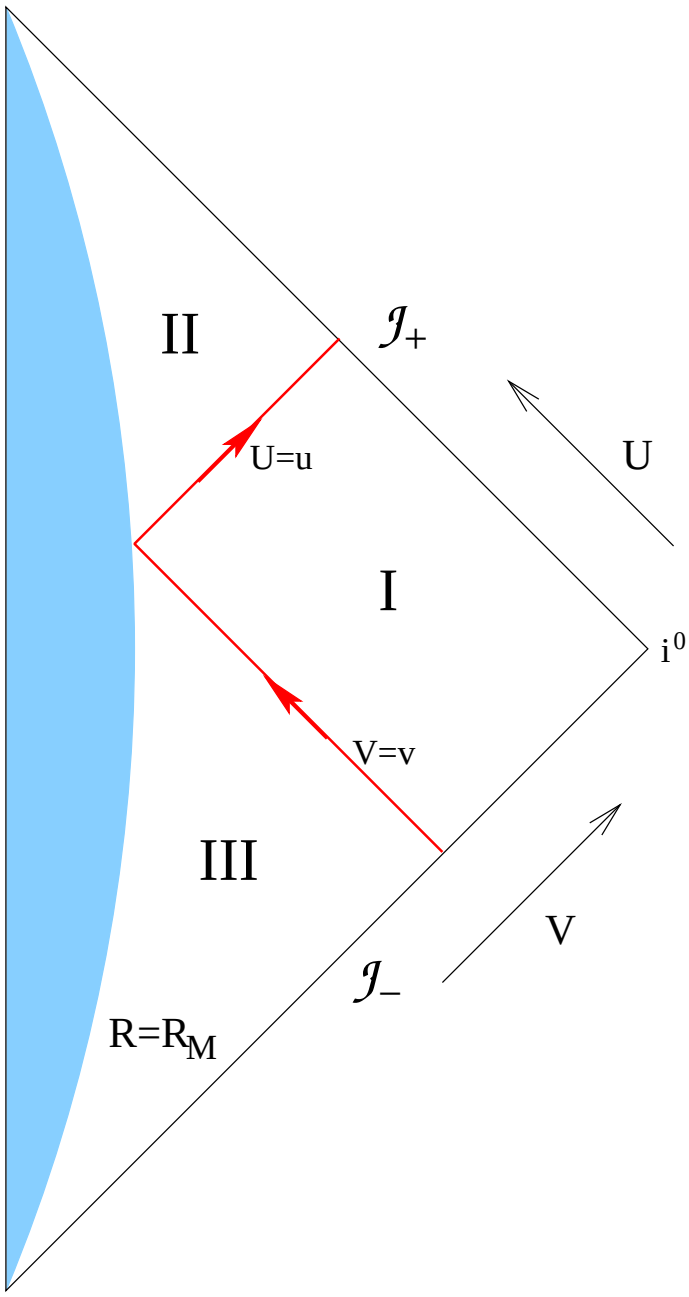

Figure 5.1: *Penrose-like diagram of the spacetime* $\bar{\mathcal{M}}$. *The shaded region lies inside the mirror with the radius* $R = R_M$. *The ingoing shell trajectory defined by* $V = v$ *starts at past light-like infinity* $\mathcal{I}_-$, *becomes an outgoing shell trajectory* $U = u$ *at the mirror and ends up at future light-like infinity* $\mathcal{I}_+$. *The region outside the shell is denoted by I, that inside the outgoing (ingoing) shell by II (III). The arrows show the directions in which the double-null coordinates* U *and* V *increase.*

κ is the Kruskal function which is defined in the Appendix E. The radial function $\bar{R}$ is computed by subtracting $\bar{U}$ from $\bar{V}$ in eq. (5.1) to get rid of the T and solving the resulting equation for R, where the definition of the Kruskal function is needed. The other metric function $\bar{A}$ is found by comparing the line elements (5.4) and that for the Schwarzschild coordinates T and R,

$$ds^2 = \frac{R-2E}{R}dT^2 - \frac{R}{R-2E}dR^2 - R^2 d\Omega^2, \tag{5.7}$$

by using the inverse of the coordinate transformation given by (5.1).

In the regions II and III inside the shell we choose the Double-Null Minkowski (DNM) coordinates $\tilde{U}$, $\tilde{V}$ that are defined by

$$\tilde{U} = T_M - R, \quad \tilde{V} = T_M + R, \tag{5.8}$$

where T_M, R are the Minkowski coordinates. The metric with respect to the DNM coordinates is given by

$$ds^2 = \tilde{A}d\tilde{U}d\tilde{V} - \tilde{R}^2 d\Omega^2, \tag{5.9}$$

where

$$\tilde{A} = 1, \quad \tilde{R} = \frac{\tilde{V}-\tilde{U}}{2}. \tag{5.10}$$

$\tilde{R}$ and $\tilde{A}$ are the limits of $\bar{R}$ and $\bar{A}$ as $E \to 0$, so the DNM metric can be considered as the DNEF metric for which $E = 0$.

Let us now turn to the shell trajectory. It starts in $\bar{\mathrm{I}} \subset \bar{\mathcal{M}}_E$ as an ingoing spherically symmetric null hypersurface defined by the equation

$$\bar{V} = \bar{v}, \quad \bar{v} \in \mathbb{R}. \tag{5.11}$$

The trajectory of the same shell in $\bar{\mathrm{III}} \subset \bar{\mathcal{M}}_0$ is also an ingoing null hypersurface and has the equation

$$\tilde{V} = \tilde{v}, \quad \tilde{v} \in \mathbb{R}. \tag{5.12}$$

Similarly, the outgoing part of the trajectory is characterized by

$$\bar{U} = \bar{u} \tag{5.13}$$

in $\bar{\mathrm{I}}$ and by

$$\tilde{U} = \tilde{u} \tag{5.14}$$

in region $\bar{\mathrm{II}} \subset \bar{\mathcal{M}}_0$. The shell trajectories given by eqs. (5.11) and (5.13) meet at the mirror, hence the parameters $\bar{v}$ and $\bar{u}$ must be related by some equation. This equation can be easily found by solving

$$\bar{R}(\bar{U} = \bar{u}, \bar{V} = \bar{v}) = 2E\kappa \left(e^{\frac{\bar{v}-\bar{u}}{4E}}\right) = R_M \tag{5.15}$$

for $\bar{u}$, yielding

$$\bar{u} = \bar{v} - 2R_M^*. \tag{5.16}$$

The parameter $\bar{u}$ is thus determined by the solution parameters $\bar{v}$, E and the radius of the mirror, R_M. There is, however, no such relation between the parameters $\tilde{u}$ and $\tilde{v}$ inside the shell. We thus find that the independent parameters are $E \in (0, \infty)$, $\bar{v} \in \mathbb{R}$, $\tilde{v} \in \mathbb{R}$ and $\tilde{u} \in \mathbb{R}$.

We now cut the pieces I from $\mathcal{M}_E$ and II, III from $\mathcal{M}_0$ and paste them together along the shell's trajectory to the solution spacetime $\mathcal{M}$, as displayed in fig. (5.1). The pasting is always possible and uniquely defined by the requirement that points with the same value of R coincide. Hence, the shell spacetime is given by the union $\mathcal{M} = \mathrm{I} \cup \mathrm{II} \cup \mathrm{III}$ of the three pieces I, II and III. Its maximal analytic extension is that of the union, such that the scris are well-defined.

Which of the parameters $\tilde{u}$, $\tilde{v}$, $\bar{v}$ and E we obtained are truly physical parameters and which of them are only gauge? Of course, the Schwarzschild mass E is a physical, even geometrical quantity that determines the geometry of Schwarzschild spacetime. The two parameters $\tilde{u}$ and $\tilde{v}$, however, are only gauge, because all pieces $\mathrm{II}(\tilde{v})$ ($\mathrm{III}(\tilde{u})$) cut from $\bar{\mathcal{M}}_0$ are isometric for each $\tilde{v}$ ($\tilde{u}$). Also all pieces $\mathrm{I}(\bar{v})$ cut from $\bar{\mathcal{M}}_E$ with fixed mass E are isometric. So it seems that E is the only non-gauge parameter. But the position of the shell with respect to the reference frame of the observers at infinity is also a physical parameter. Indeed, $\bar{v}$ is the value of the advanced time when the shell is sent in, measurable by the observers near i^0 of $\mathcal{M}$ and is therefore the second observable. We thus have two coordinates describing the physical phase space:

$$E \in (0, \infty), \quad \bar{v} \in (-\infty, \infty). \tag{5.17}$$

Notice that in order to avoid solutions whose horizon is outside the mirror, the domain of E is further restricted to the interval $[0, \frac{R_M}{2})$. The two remaining parameters $\tilde{u}$, $\tilde{v}$ are gauge.

5.3 Regular DNEF gauge

The authors of [HK02a] call the DNEF and DNM coordinates used above *singular gauge*, because the metric with respect to them is not continuous at the shell. Coordinates with respect to which the metric is C^0 at the shell and smooth outside the shell are called *regular*. Together with some boundary conditions at the mirror and at infinity a regular gauge is the necessary requirement for defining a covariant gauge fixing [HK01]. Such a covariant gauge fixing is needed for the construction of a Kuchař decomposition (cf. [Háj00a], [HK00] and the references cited therein). Since we will need the Kuchař decomposition in the course of the reduction of the canonical action to the true degrees of freedom, we construct such a regular gauge for our system.

The regular coordinates we are looking for are denoted by U and V. U shall be a retarded null coordinate, whereas V is required to be a advanced null coordinate. In the regular coordinate the equation of the shell reads

$$V = v, \quad U = u \tag{5.18}$$

in the ingoing and outgoing parts of the trajectory, respectively. The relation between u and v will be determined below.

Outside the shell (in region I) we simply choose the DNEF coordinates from above:

$$U = \bar{U}, \quad V = \bar{V}. \tag{5.19}$$

Thus the relation between the shell parameters reads

$$u = \bar{u}, \quad v = \bar{v}, \tag{5.20}$$

hence, according to eq. (5.16), it holds that

$$u = v - 2R^*_M. \tag{5.21}$$

In the coordinates U and V the region I is defined by the inequalities

$$U < u, \quad V > v. \tag{5.22}$$

The metric coefficients in region I are given by eq. (5.6) for R and by eq. (5.5) for A.

We now construct an extension of the coordinates U, V into the inside for which the metric is continuous at the shell. The coordinates must be therefore at least C^1 at the shell. We consider the extension of U, V from I to II across the outgoing shell. The

extension from I to III is completely analogous. To this end we apply the method defined by the lemma 2 in [HK02a]. For the retarded null coordinate U in II we put

$$U = \tilde{U} - \tilde{u} + \bar{u}, \tag{5.23}$$

which implies that U is smooth at the outgoing shell. The inverse transformation reads

$$\tilde{U} = U + \tilde{u} - \bar{u}. \tag{5.24}$$

The extension of the advanced null coordinate is more intricate. According to the aforementioned lemma the coordinates must satisfy that the function R is continuous at the shell,

$$R_I(\bar{U} = \bar{u}, \bar{V}) = R_{II}(\tilde{U} = \tilde{u}, \tilde{V}), \tag{5.25}$$

where R_I, R_{II} are the radial functions in the respective regions. R_I is given by

$$R_I = 2E\kappa \left(e^{\frac{\bar{V}-\bar{U}}{4E}} \right), \tag{5.26}$$

whereas R_{II} reads

$$R_{II} = \frac{\tilde{V} - \tilde{U}}{2}. \tag{5.27}$$

These two equations and eq. (5.25) imply that

$$2E\kappa \left(e^{\frac{V-u}{4E}} \right) = \frac{\tilde{V} - \tilde{u}}{2} \tag{5.28}$$

at the outgoing shell. Solving this equation for V yields

$$V = \tilde{V} + u - \tilde{u} + 4E \ln \left| \frac{\tilde{V} - \tilde{u}}{4E} - 1 \right|. \tag{5.29}$$

using that $\tilde{V} = \tilde{U} + 2R$ and the inverse transformation (5.24), one finds that

$$V = U + 2R + 4E \ln \left| \frac{U - u + 2R}{4E} - 1 \right|. \tag{5.30}$$

The radial metric function R is hence easily found to be

$$R(U, V) = \frac{u - U}{2} + 2E\kappa \left(e^{\frac{V-u}{4E}} \right). \tag{5.31}$$

The other metric function $A(U,V)$ can be obtained by comparing the double-null line element

$$ds^2 = A(U,V)dUdV - R^2(U,V)d\Omega^2 \tag{5.32}$$

and that defined by the retarded EF coordinates,

$$ds^2 = d\tilde{U}^2 + 2d\tilde{U}dR - R^2d\Omega^2, \tag{5.33}$$

where $d\tilde{U}$ and dR are given by

$$d\tilde{U} = dU, \quad dR = -\frac{1}{2}dU + \frac{1}{2}\left(1 - \frac{1}{\kappa\left(e^{\frac{V-u}{4E}}\right)}\right)dV, \tag{5.34}$$

yielding

$$A(U,V) = 1 - \frac{1}{\kappa\left(e^{\frac{V-u}{4E}}\right)}. \tag{5.35}$$

It is straightforward to show that the metric is continuous at the shell $U = u$ in the new coordinates U, V.

The domains of U and V in II are constrained by the shell ($U > u, V > v$) and by the mirror ($R > R_M$). Solving the inequality $R > R_M$ for V, we obtain

$$V > V^*(U) \doteq U + 2R_M + 4E\ln\left|\frac{U-u+2R_M}{4E} - 1\right|. \tag{5.36}$$

Hence, the domains of the new coordinates in II read

$$U > u, \quad V > V^*(U). \tag{5.37}$$

The extension of the coordinates U and V to the region III inside the ingoing shell is obtained in a similar way. We find that the coordinates read

$$V = \tilde{V} - \tilde{v} + \bar{v} \tag{5.38}$$

and

$$U = \tilde{U} + v - \tilde{v} - 4E\ln\left|\frac{\tilde{v}-\tilde{U}}{4E} - 1\right| = V - 2R - 4E\ln\left|\frac{v-V+2R}{4E} - 1\right|. \tag{5.39}$$

The equation of the shell is hence

$$V = v. \tag{5.40}$$

The metric functions in III read

$$R = \frac{V - v}{2} + 2E\kappa\left(e^{\frac{v-U}{4E}}\right) \tag{5.41}$$

and

$$A = 1 - \frac{1}{\kappa\left(e^{\frac{v-U}{4E}}\right)}. \tag{5.42}$$

The functions A and R are obviously continuous at the shell $V = v$.
The domains of U and V in the region III are found to be

$$U < U^*(V), \quad V < v, \tag{5.43}$$

where

$$U^*(V) \doteq V - 2R_M - 4E \ln\left|\frac{v - V + 2R_M}{4E} - 1\right|. \tag{5.44}$$

We summarize the results in the three regions. The transformation from the singular to the regular gauge reads:

$$U = \begin{cases} \bar{U} & : \quad U < u, V > v, \\ \tilde{U} & : \quad U > u, V > V^*, \\ \tilde{U} + v - \tilde{v} - 4E \ln\left|\frac{\tilde{v}-\tilde{U}}{4E} - 1\right| & : \quad U < U^*, V < v, \end{cases} \tag{5.45}$$

$$V = \begin{cases} \bar{V} & : \quad U < u, V > v, \\ \tilde{V} + u - \tilde{u} + 4E \ln\left|\frac{\tilde{V}-\tilde{u}}{4E} - 1\right| & : \quad U > u, V > V^*, \\ \tilde{V} & : \quad U < U^*, V < v. \end{cases} \tag{5.46}$$

The metric functions with respect to the new coordinates are given by:

$$A(U,V) = \begin{cases} 1 - \frac{1}{\kappa\left(e^{\frac{V-U}{4E}}\right)} & : \quad U < u, V > v, \\ 1 - \frac{1}{\kappa\left(e^{\frac{V-u}{4E}}\right)} & : \quad U > u, V > V^*, \\ 1 - \frac{1}{\kappa\left(e^{\frac{v-U}{4E}}\right)} & : \quad U < U^*, V < v, \end{cases} \tag{5.47}$$

$$R(U,V) = \begin{cases} 2E\kappa\left(e^{\frac{V-U}{4E}}\right) & : \quad U < u, V > v, \\ \frac{u-U}{2} + 2E\kappa\left(e^{\frac{V-u}{4E}}\right) & : \quad U > u, V > V^*, \\ \frac{V-v}{2} + 2E\kappa\left(e^{\frac{v-U}{4E}}\right) & : \quad U < U^*, V < v. \end{cases} \tag{5.48}$$

Obviously, the functions A, R are continuous at the ingoing ($V = v$) as well as at the outgoing ($U = u$) shell. The relation between the shell parameters v and u is given by eq. (5.21), $u = v - 2R^*_M$. We will call the gauge corresponding to the above defined coordinates U and V *regular DNEF gauge* in the following. This gauge will be used in many calculations.

Chapter 6

The central-regular gauge

6.1 Motivation

In this chapter we modify the regular DNEF gauge found in the last chapter. The resulting gauge is then not the simplest possible for this model, but it has several very useful properties. First of all, the *metric* with respect to the chosen coordinates is *continuous* even *at the shell*. The second useful feature of the gauge affects the Hamiltonian formulation of the theory that is treated in the next chapter. The volume part in the ADM action is given by the integral over some quantity, the integral being taken over the radial coordinate ρ from $\rho = \rho_M$ to $\rho = \infty$, where ρ_M is the solution of the equation $R(\rho, \tau) = R_M$ for ρ. In a general gauge (as, e.g. the regular DNEF gauge,) ρ_M is not constant but depends on the time coordinate τ, having the unattractive effect that the variation of ρ_M does not vanish. But since ρ_M is contained in the action as the lower boundary of an integral over some complicated quantity, the variational equation becomes very involved. We, therefore, demand that the coordinates near the mirror are such that $\rho_M = const$. The coordinates that are constructed in this chapter satisfy this requirement, they are *regular at the centre* in this sense. The third property is that the components of the Cartesian transform of the metric are all only small perturbations in a given parameter s of the components of the Minkowski metric, such that they *satisfy the condition for DeWitt's semiclassical theory*, cf. chapter 9. The explicit proof is written down in the appendix C.

The price of these three restrictions on the gauge choice is that the coordinates cannot be chosen to be double null (DN) everywhere in spacetime, because one has to interpolate between the DN coordinates at the shell and at the mirror, which is not feasible with coordinates that remain DN in the interpolation region. In fact, it is impossible to construct DN coordinates that have the first two properties from above at the same time.

This chapter is organized as follows: in section 2 we remind the reader of our notation concerning the various regions of spacetime and we introduce the singular DNEF coordinates from chapter 5 as the reference coordinates on the spacetime manifold. The splitting of spacetime into smaller parts is continued in section 3, where we define the wedges,

where the interpolation of the coordinates takes place. The next section is devoted to the construction of the new coordinates, the so-called central-regular (CR) coordinates. In section 5 we settle the notation for three types of coordinate systems and the associated metrics. In section 6 we explicitly write down the components of the CR metric in each region separately and show that they are continuous at the shell.

6.2 Regions in spacetime, reference coordinates

The spacetime $(\mathcal{M}, g)$ is split into three disjoint parts by the trajectory of the shell. The region outside the shell is denoted by I, the region between the outgoing shell and the mirror by II and that between the ingoing shell and the mirror by III, as displayed in figure 5.1 of chapter 5. The reference coordinates on our spacetime manifold $\mathcal{M}$ for the outside I are the double null Eddington Finkelstein (DNEF) coordinates $\bar{U}$ and $\bar{V}$ (cf. section 2 of chapter 5), being defined by the Schwarzschild coordinates T and R by

$$\bar{U} \doteq T - R^* = T - R - 2E \ln \left| \frac{R}{2E} - 1 \right| \tag{6.1}$$

and

$$\bar{V} \doteq T + R^* = T + R + 2E \ln \left| \frac{R}{2E} - 1 \right|. \tag{6.2}$$

The equation of the ingoing shell is given by

$$\bar{V} = \bar{v}, \tag{6.3}$$

while that of the outgoing shell reads

$$\bar{U} = \bar{u}, \tag{6.4}$$

where $\bar{u}$ and $\bar{v}$ are constant numbers. The shell's trajectory defines the boundaries of the region I and restricts thus the allowed values of the coordinates to the domains

$$-\infty < \bar{U} \leq \bar{u}, \quad \bar{v} \leq \bar{V} < \infty. \tag{6.5}$$

In the two inside regions II and III of $\mathcal{M}$ the reference coordinates are the double null (retarded and advanced) Minkowski (DNM) coordinates $\tilde{U}$ and $\tilde{V}$ defined by their relation to the polar Minkowski coordinates T_M and R:

$$\tilde{U} \doteq T_M - R, \tag{6.6}$$

$$\tilde{V} \doteq T_M + R. \tag{6.7}$$

The trajectory of the shell is given by

$$\tilde{V} = \tilde{v} \tag{6.8}$$

in region III and by

$$\tilde{U} = \tilde{u} \tag{6.9}$$

in region II, respectively. Again, $\tilde{u}$ and $\tilde{v}$ are constants and again, the shell's trajectory confines the possible values of the reference coordinates in the regions II and III. There is another boundary given by the mirror. The radius of the (spherical) mirror is

$$R = R_M. \tag{6.10}$$

It follows from eqs. (6.6) and (6.7), that there is a dependence between $\tilde{U}$ and $\tilde{V}$ at the mirror:

$$\tilde{V} = \tilde{U} + 2R_M. \tag{6.11}$$

The shell hits the mirror in the point with the coordinates

$$\tilde{V} = \tilde{v}, \quad \tilde{U} = \tilde{u}, \tag{6.12}$$

therefore, it follows from eq. (6.11) that

$$\tilde{v} = \tilde{u} + 2R_M. \tag{6.13}$$

The reference coordinates in region II are thus restricted to the domains

$$\tilde{u} \leq \tilde{U} < \infty, \quad \tilde{U} + 2R_M \leq \tilde{V} < \infty. \tag{6.14}$$

In region III the domains are given by

$$-\infty < \tilde{U} \leq \tilde{u}, \quad \tilde{U} + 2R_M \leq \tilde{V} \leq \tilde{v}. \tag{6.15}$$

There is also a relation between $\bar{u}$ and $\bar{v}$:

$$\bar{v} = \bar{u} + 2R_M^*, \quad R_M^* \doteq R_M + 2E \ln \left| \frac{R_M}{2E} - 1 \right|. \tag{6.16}$$

This follows from $R = R_M$ at the mirror and from the definition of the DNEF coordinates, eqs. (6.1) and (6.2).

Of course, the DNEF coordinates in I and the DNM coordinates in II and III do not join continuously at their respective boundaries. The consequence thereof is that $\tilde{v} \neq \bar{v}$ and $\tilde{u} \neq \bar{u}$ in general, and that the metric with respect to the coordinates has jumps at the shell. We observe that the reference coordinates are very closely related to the singular DNEF coordinates defined in chapter 5. The difference is that there the coordinates $\tilde{U}$, $\tilde{V}$ are not reference coordinates on the shell spacetime manifold $\mathcal{M}$ but in the pieces II and III cut from the Minkowski spacetime manifold $\mathcal{M}_0$, such that there is no relation between the coordinates in II and III. In the present case, however, the reference coordinates are defined in the union II $\cup$ III $\subset \mathcal{M}$.

6.3 The three wedges of II and III

The two inside regions II and III are cut into three sections each by two straight lines each emanating form the point $(\tilde{U}, \tilde{V}) = (\tilde{u}, \tilde{v})$ to infinity. These lines are denoted by $k_\pm, l_\pm$, where $k_\pm$ lies nearer to the mirror than $l_\pm$ and where a plus (minus) sign denotes, that the line is in region II (III). The regions between the shell's trajectory and the line $l_\pm$ are denoted by $Z_\pm$, those between the two lines by $K_\pm$ and those between the lines $k_\pm$ and the mirror by $M_\pm$ (See figure (6.1)).

The equations of the lines are given by

$$l_+ : \ \tilde{U}_l \doteq \beta(\tilde{V} - \tilde{v}) + \tilde{u}, \tag{6.17}$$

$$k_+ : \ \tilde{U}_k \doteq \alpha(\tilde{V} - \tilde{v}) + \tilde{u}, \quad 0 < \beta < \alpha < 1 \tag{6.18}$$

in region II and by

$$l_- : \ \tilde{U}_{l_-} \doteq \frac{1}{\alpha}(\tilde{V} - \tilde{v}) + \tilde{u}, \tag{6.19}$$

$$k_- : \ \tilde{U}_{k_-} \doteq \frac{1}{\beta}(\tilde{V} - \tilde{v}) + \tilde{u} \tag{6.20}$$

in region III.

The reason for splitting up the interior parts of the spacetime is that we want the coordinates to be double null Minkowski at the mirror and DN and C^1 at the shell. But this is impossible if the coordinates are DN everywhere (we do not prove this here). Therefore, we have to choose a section of the interior parts, where the coordinates interpolate between those near the shell and those near the mirror but are not DN anymore. This

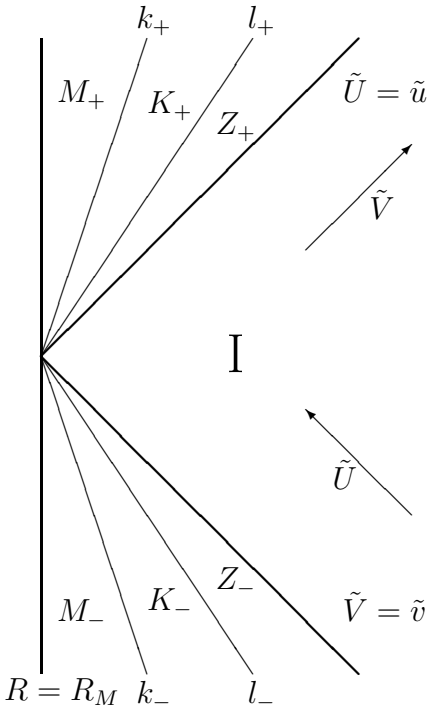

Figure 6.1: *The spacetime pieces II and III inside the shell are split into three parts each by the wedges $K_\pm$. The three wedges with a plus sign constitute region II, those with a minus sign are the parts of region III. The arrows show the directions of the increasing reference coordinates $\tilde{U}$ and $\tilde{V}$. The mirror is represented by the line $R = R_M$, whereas the ingoing (outgoing) shell's trajectory is given by the line $\tilde{V} = \tilde{v}$ ($\tilde{U} = \tilde{u}$).*

section should touch as little of the mirror and the shell as possible and should have a simple description. The choice above seems to satisfy these requirements sufficiently well. More reasons for the specific choices of the coordinates in the various regions are written down in the corresponding subsections.

6.4 Central-regular (CR) coordinates

We give the explicit construction of the *CR coordinates* U and V that are smooth everywhere except at the shell where they are C^1. The construction is written down for each part separately. The coordinates are double null (DN) everywhere except in the wedges $K_\pm$.

The equation of the shell in the new coordinates is given by

$$V = v \tag{6.21}$$

in the ingoing and

$$U = u \tag{6.22}$$

in the outgoing case, where u and v are constant numbers.

6.4.1 Region I

Outside the shell we simply choose for U and V the DNEF coordinates $\bar{U}$ and $\bar{V}$ given by eqs. (6.1) and (6.2), so

$$U = \bar{U}, \quad V = \bar{V} \tag{6.23}$$

in region I, where the domains of $\bar{U}$ and $\bar{V}$ are given by eq. (6.5). It follows that

$$u = \bar{u}, \quad v = \bar{v}, \tag{6.24}$$

and, therefore, using eq. (6.16),

$$v = u + 2R^*_M \tag{6.25}$$

in region I.

There is another reason to choose the DNEF coordinates outside the shell besides their simplicity: The proper Schwarzschild time difference between the two events at which the shell meets the observer seems to be the most sensible to define the sojourn time. The polar transform (cf. section 5.2 of this chapter) of the DNEF coordinates yields that $\tau = \frac{V+U}{2}$ is the Schwarzschild time T. So the choice of DNEF coordinates outside the shell is in this sense a preferred one.

6.4.2 Region II, Z_+

The coordinates in this wedge are required to be DN and (at least) C^1 at the shell. Outside the shell we have already found suitable coordinates. These are now to be extended across the shell, similar to the construction of the regular DNEF coordinates in section 3 of chapter 5. The retarded null coordinate U does not pose a problem, we simply put

$$U = \tilde{U} - \tilde{u} + \bar{u}; \tag{6.26}$$

because of eq. (6.23), U is then of course smooth at the shell.

The advanced null coordinate V reads:

$$V = \tilde{V} - \tilde{u} + \bar{u} + 4E \ln \left| \frac{\tilde{V} - \tilde{u}}{4E} - 1 \right| = \tilde{V} - \tilde{u} + \bar{u} + L, \tag{6.27}$$

where the useful abbreviation

$$L = L(\tilde{V}) \doteq 4E \ln \left| \frac{\tilde{V} - \tilde{u}}{4E} - 1 \right| \tag{6.28}$$

has been introduced. Equations (6.26) and (6.27) give thus the DN coordinates in Z_+ expressed in the reference coordinates that take values from the domains

$$\tilde{U} \in (\tilde{u}, \tilde{U}_{l_+}), \quad \tilde{V} \in (\tilde{v}, \infty). \tag{6.29}$$

The relation between the shell parameters $v, \tilde{v}$ in Z_+ reads

$$v = \tilde{v} - \tilde{u} + \bar{u} + L|_{\tilde{V}=\tilde{v}} = \tilde{v} - \tilde{u} + \bar{u} + c, \tag{6.30}$$

where the constant c is given by

$$c \doteq L|_{\tilde{V}=\tilde{v}} = 4E \ln \left| \frac{R_M}{2E} - 1 \right|. \tag{6.31}$$

We will show later that the metric with respect to these coordinates is continuous, implying that the coordinates are at least C^1 at the shell as requested.

6.4.3 Region II, M_+

In the innermost wedge we take DNM coordinates, where $\tilde{U}$ is shifted by $-\tilde{u} + \bar{u}$ and $\tilde{V}$ by the constant $c - \tilde{u} + \bar{u}$. The coordinates in M_+ are hence

$$U = \tilde{U} - \tilde{u} + \bar{u}, \quad V = \tilde{V} - \tilde{u} + \bar{u} + c, \tag{6.32}$$

where the reference coordinates take values from the intervals

$$\tilde{U} \in (\tilde{U}_{k_+}, \tilde{V} - 2R_M), \quad \tilde{V} \in (\tilde{v}, \infty). \tag{6.33}$$

On account of the additional number c the coordinates in M_+ and Z_+ are equal at the shell $\tilde{V} = \tilde{v}$, i.e. at the mirror, so there are no jumps of the coordinates in the point at

the mirror where the shell is reflected. The equation between the shell parameters reads, again:

$$v = \tilde{v} - \tilde{u} + \bar{u} + c. \tag{6.34}$$

The metric with respect to the coordinates (6.32) is of course the DNM metric, where R is shifted by some constant. Its polar transform to the coordinates τ, X is the polar Minkowski metric, where the coordinates τ and X differ from the Minkowski coordinates T_M, R by additive constants. The equation of the mirror $R = R_M$ does not depend on the Minkowski time T_M because the coordinate lines are orthogonal in M_+. This is not the case if one took the coordinates defined by eqs. (6.26, 6.27) from Z_+ in M_+, because then the equation $R = R_M$ would depend on the corresponding time coordinate. The Hamiltonian action we intend to use and reduce in the next chapter depends on the radius of the mirror, $\rho_M(\tau)$, that appears as the lower boundary of the ρ-integral in the volume part of the Liouville form. If ρ_M depends on τ, then the variation of ρ_M (ρ_M is a dependent dynamical variable due to $R(\rho_M) = R_M$ and must be, therefore, varied) does not vanish, which leads to the unwelcome consequence that the equations of motion become very complicated. We therefore want to restrict the gauge freedom at the mirror such that $\delta\rho_M = 0$, i.e. that ρ_M is constant and thus has zero variation. The Minkowski coordinates (and those related to them by additive constants) have this nice property. This issue will be discussed in more detail when we turn our attention to the canonical analysis in the next chapter.

6.4.4 Region II, K_+

This wedge is the most intricate part of the gauge because it involves a smooth interpolation of the coordinates U and V between their values in the two adjacent wedges Z_+ and M_+. The cost of the interpolation is that the coordinates in K_+ are no more DN.

The retarded coordinate U again poses no problem; we make the natural choice

$$U = \tilde{U} - \tilde{u} + \bar{u}; \tag{6.35}$$

U is thus smooth at the boundaries k_+ and l_+ of the wedge K_+. The coordinate V depends on both reference coordinates through the interpolating function $\omega_+ = (L - c)q_+$ that is defined in the appendix D by eq. (D.11). V is given by

$$V = \tilde{V} - \tilde{u} + \bar{u} + L + \omega_+, \tag{6.36}$$

the domains of the reference coordinates being

$$\tilde{U} \in (\tilde{U}_{l_+}, \tilde{U}_{k_+}), \quad \tilde{V} \in (\tilde{v}, \infty). \tag{6.37}$$

At the outer boundary l_+ of K_+, $q_+ = 0$, so $V = \tilde{V} - \tilde{u} + \bar{u} + L$ smoothly goes over to the V of Z_+. Similarly, at the inner boundary k_+, $q_+ = -1$, such that $V = \tilde{V} - \tilde{u} + \bar{u} + c$. The properties of the interpolating function are discussed in detail in the appendix D.

The equation between the shell parameters is

$$v = \tilde{v} - \tilde{u} + \bar{u} + L|_{\tilde{V}=\tilde{v}} + \omega_+|_{\tilde{V}=\tilde{v}} = \tilde{v} - \tilde{u} + \bar{u} + c. \tag{6.38}$$

From the equations (6.24, 6.30, 6.34, 6.38) follows that the relations between the shell parameters in the whole region II are given by

$$u = \bar{u}, \quad v = \tilde{v} - \tilde{u} + \bar{u} + c = \bar{v}. \tag{6.39}$$

Furthermore, since $\tilde{v} - \tilde{u} = 2R_M$,

$$v = u + 2R_M + c = u + 2R_M^* \tag{6.40}$$

holds in the region II, as required.

6.4.5 Region III, Z_-

The construction of the coordinates in the three wedges of III is completely analogous, so we write down only the results. The DN coordinates in Z_- read

$$U = \tilde{U} - \tilde{v} + \bar{v} - L', \quad V = \tilde{V} - \tilde{v} + \bar{v}, \tag{6.41}$$

where

$$L' = L'(\tilde{U}) \doteq 4E \left| \frac{\tilde{v} - \tilde{U}}{4E} - 1 \right|. \tag{6.42}$$

The domains of the reference coordinates are given by

$$\tilde{U} \in (-\infty, \tilde{u}), \quad \tilde{V} \in (\tilde{V}_{l_-}, \tilde{v}), \tag{6.43}$$

where

$$\tilde{V}_{l_-} \doteq \alpha(\tilde{U} - \tilde{u}) + \tilde{v} \tag{6.44}$$

is the solution of eq. (6.19) for $\tilde{V}$.

The equation between the shell parameters reads

$$u = \tilde{u} - \tilde{v} + \bar{v} - L'|_{\tilde{U}=\tilde{u}} = \tilde{u} - \tilde{v} + \bar{v} - c', \tag{6.45}$$

where c' is defined by

$$c' \doteq L'|_{\tilde{U}=\tilde{u}} = 4E \ln \left| \frac{R_M}{2E} - 1 \right| = c. \tag{6.46}$$

6.4.6 Region III, M_-

The DN coordinates read

$$U = \tilde{U} - \tilde{v} + \bar{v} - c', \quad V = \tilde{V} - \tilde{v} + \bar{v}. \tag{6.47}$$

Their domains are

$$\tilde{U} \in (-\infty, \tilde{u}), \quad \tilde{V} \in (\tilde{U} + 2R_M, \tilde{V}_{k_-}), \tag{6.48}$$

where

$$\tilde{V}_{k_-} \doteq \beta(\tilde{U} - \tilde{u}) + \tilde{v} \tag{6.49}$$

is the solution of eq. (6.20) for $\tilde{V}$.

The equation between the shell parameters is given by

$$u = \tilde{u} - \tilde{v} + \bar{v} - c'. \tag{6.50}$$

6.4.7 Region III, K_-

In this wedge the interpolation is performed as in K_+. The coordinate V is given by

$$V = \tilde{V} - \tilde{v} + \bar{v}, \tag{6.51}$$

whereas the other coordinate U interpolates between its values in M_- and Z_-:

$$U = \tilde{U} - \tilde{v} + \bar{v} - c' + \omega_-. \tag{6.52}$$

$\omega_- = (L' - c')q_-$ is defined in the appendix D by eq. (D.13). As $\tilde{V} = \tilde{V}_{k_-}$, the function q_- obtains the value 0, so V is smooth at the line k_-. Similarly, at the line l_-, where

$\tilde{V} = \tilde{V}_{l_-}$, it holds that $q_- = -1$. Hence, V is also smooth there. The domains of the reference coordinates in K_- are

$$\tilde{U} \in (-\infty, \tilde{u}), \quad \tilde{V} \in (\tilde{V}_{k_-}, \tilde{V}_{l_-}). \tag{6.53}$$

The equations between the shell parameters are again given by

$$u = \tilde{u} - \tilde{v} + \bar{v} - L'|_{\tilde{U}=\tilde{u}} + \omega|_{\tilde{U}=\tilde{u}} = \tilde{u} - \tilde{v} + \bar{v} - c'. \tag{6.54}$$

It follows from eqs. (6.24, 6.45, 6.50, 6.54) that

$$u = \tilde{u} - \tilde{v} + \bar{v} - c' = \bar{u}, \quad v = \bar{v} \tag{6.55}$$

in the whole of III. Furthermore,

$$u = v - 2R_M - c = v - 2R^*_M \tag{6.56}$$

is also satisfied in the region III, as expected.

6.5 Metric components: general UV-type, polar and Cartesian form

In this section we write down the general form of the line element in the various regions. The next section is then devoted to the explicit expressions for the metric components. We state the line elements in the UV-type, polar and Cartesian coordinates.

6.5.1 UV-type coordinates

We call the coordinates (U, V, θ, ϕ) and the corresponding line elements *UV-type*. In the last section we have constructed the coordinates U and V in all parts of the spacetime. These coordinates are DN everywhere except in the two interpolation wedges $K_\pm$. We therefore, treat the DN and the non-DN cases separately in the following. If the coordinates are DN, then the corresponding line element has the general form

$$ds^2 = AdUdV - R^2 d\Omega^2, \quad d\Omega^2 \doteq d\theta^2 + \sin^2\theta\, d\phi^2, \tag{6.57}$$

where (U, V, θ, ϕ) is the DN coordinate system and where A and R are functions of U and V alone.

If U and V are not double null, then the line element reads

$$ds^2 = F dU^2 + H dV^2 + 2G dU dV - R^2 d\Omega^2, \tag{6.58}$$

where, as above, the metric components F, G, H and R depend on U and V only.

6.5.2 Polar coordinates

Consider the transformation

$$\tau = \frac{V+U}{2}, \quad X = \frac{V-U}{2}, \quad \theta = \theta, \quad \phi = \phi \tag{6.59}$$

to the *polar coordinates* (τ, X, θ, ϕ). Its inverse

$$U = \tau - X, \quad V = \tau + X, \quad \theta = \theta, \quad \phi = \phi \tag{6.60}$$

changes the UV-type line elements to

$$ds^2 = A\,(d\tau^2 - dX^2) - R^2 d\Omega^2 \tag{6.61}$$

in the DN case and to

$$ds^2 = Q\,d\tau^2 + J\,d\tau\,dX + W\,dX^2 - R^2 d\Omega^2 \tag{6.62}$$

in the non-DN case. Here, the abbreviations

$$Q \doteq F + 2G + H, \quad J \doteq 2(H - F), \quad W \doteq F - 2G + H \tag{6.63}$$

have been introduced. The metric components A, Q, J, W and R depend on τ and X only.

6.5.3 Cartesian coordinates

Consider the *Cartesian coordinates* $(\tau, \vec{X})$, where

$$\vec{X} = X\vec{n}, \quad \vec{n} \doteq (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta). \tag{6.64}$$

Here, $\vec{n}$ is a vector parametrizing the unit sphere with unit length, $\vec{n}\cdot\vec{n} = 1$. We define the following two vectors,

$$\vec{m} \doteq \frac{\partial\vec{n}}{\partial\theta} = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta), \tag{6.65}$$

$$\vec{l} \doteq \frac{1}{\sin\theta}\frac{\partial \vec{n}}{\partial \phi} = (-\sin\phi, \cos\phi, 0) \tag{6.66}$$

that constitute an orthonormal system together with $\vec{n}$. From

$$d\vec{n} = \vec{m}\, d\theta + \sin\theta\, \vec{l}\, d\phi, \tag{6.67}$$

follows hence that

$$d\vec{n} \cdot d\vec{n} = d\theta^2 + \sin^2\theta d\phi^2 = d\Omega^2. \tag{6.68}$$

From eq. (6.64) we obtain

$$d\vec{X} = X d\vec{n} + \vec{n} dX, \tag{6.69}$$

hence

$$d\vec{X} \cdot d\vec{X} = X^2 d\Omega^2 + dX^2. \tag{6.70}$$

From the inverse of eq. (6.64),

$$X = \sqrt{\vec{X}\cdot\vec{X}}, \quad \theta = \arctan\frac{\sqrt{X_1^2 + X_2^2}}{X_3}, \quad \phi = \arctan\frac{X_2}{X_1}, \quad \vec{X} \doteq (X_1, X_2, X_3), \tag{6.71}$$

it follows that

$$dX = \frac{1}{X}\vec{X}\cdot d\vec{X}. \tag{6.72}$$

Now we can express $d\Omega^2$ in terms of the spatial Cartesian coordinates $\vec{X}$:

$$d\Omega^2 = \frac{1}{X^2}\left(d\vec{X}^2 - \frac{1}{X^2}(\vec{X}\cdot d\vec{X})^2\right). \tag{6.73}$$

The line element for the DN case reads thus

$$ds^2 = A d\tau^2 - \frac{R^2}{X^2} d\vec{X}^2 + \left(\frac{R^2}{X^2} - A\right)\left(\frac{\vec{X}\cdot d\vec{X}}{X}\right)^2. \tag{6.74}$$

For example, the Minkowski line element is recovered for $A = 1, R = X$. A and R are functions of τ and X; they depend on the spatial coordinates through the absolute value

X only. For the sake of simplicity however, we write down the metric components as functions of U and V in the following even in the Cartesian case. The expressions with respect to the Cartesian coordinates are then found by replacing U and V by τ and X using the transformation (6.60).

The non-DN case is given by

$$ds^2 = Qd\tau^2 + \frac{J}{X}d\tau\vec{X}\cdot d\vec{X} - \frac{R^2}{X^2}d\vec{X}^2 + \left(\frac{R^2}{X^2} + W\right)\left(\frac{\vec{X}\cdot d\vec{X}}{X}\right)^2 . \tag{6.75}$$

The Minkowski line element is again recovered if the metric functions are $Q = 1, J = 0, W = -1, R = X$, as can be easily checked.

If one transforms the CR metric into the Cartesian form, one obtains the Cartesian central regular (CCR) metric, that satisfies the conditions for DeWitt's method. This will be shown in the appendix C.

6.6 Metric components: explicit UV-type form

We write down the explicit form of the metric components in UV-type form (cf. the line elements (6.57) and (6.58) defined in the last section) in each of the seven parts of the spacetime. We show that this metric is smooth at the boundaries of the various regions, except at the shell, where it is still continuous. The metric is called *central-regular (CR) metric*.

6.6.1 Region I

The two non-vanishing metric functions A and R with respect to the coordinates U and V in the outside region are given by

$$A = 1 - \frac{2E}{R} \tag{6.76}$$

and

$$R = 2E\kappa\left(e^{\frac{V-U}{4E}}\right), \tag{6.77}$$

where κ is the Kruskal function defined by equation (E.1) in the appendix E. A and R have been already explicitly constructed in section 2 of chapter 5.

6.6.2 Region II, Z_+

The metric component R in the line element (6.57) is found by replacing $\tilde{V}$ by $\tilde{U}+2R$ in eq. (6.27) and solving the resulting quantity for R:

$$R = \frac{u-U}{2} + 2E\kappa\left(e^{\frac{V-u}{4E}}\right). \tag{6.78}$$

A is determined by comparing the flat line element with respect to the retarded Minkowski coordinates (cf. chapter 5), leading to

$$A = 1 - \frac{1}{\kappa\left(e^{\frac{V-u}{4E}}\right)}. \tag{6.79}$$

At the outgoing shell $U=u$, the metric components A and R of Z_+ and that of region I are equal, as can be easily checked. Hence, the metric is continuous there, as requested.

6.6.3 Region II, M_+

In the wedge M_+ the metric functions are given by the components of the double-null Minkowski metric, where the radial function R differs form the Minkowski radial function by the constant $\frac{c}{2}$,

$$R = \frac{\tilde{V}-\tilde{U}}{2} = \frac{V-U-c}{2}, \quad A = 1. \tag{6.80}$$

6.6.4 Region II, K_+

In order to find the metric components in this wedge we need the UV-line element (6.58). By comparison of this line element with the components of the flat DN reference metric, the functions F, G and H are found to be

$$F = -\frac{V_{,\tilde{U}}}{V_{,\tilde{V}}}, \quad G = \frac{1}{2V_{,\tilde{V}}}, \quad H = 0, \quad R = \frac{\tilde{V}-\tilde{U}}{2} = \frac{\tilde{V}(V,U)-U-\tilde{u}+\bar{u}}{2}. \tag{6.81}$$

Here, the derivatives of V with respect to the reference coordinates are given by

$$V_{,\tilde{U}} = (L-c)q_{+,\tilde{U}} = -(L-c)\frac{\psi(x=\tilde{U}-z;a,z)}{K}, \tag{6.82}$$

$$\begin{aligned} V_{,\tilde{V}} &= 1 + (1+q_+)L_{,\tilde{V}} + (L-c)q_{+,\tilde{V}} \\ &= 1 + (1+q_+)L_{,\tilde{V}} + \frac{(L-c)\psi}{2K}\left[(\alpha-\beta)\frac{\tilde{U}-z}{a} + \alpha + \beta\right], \end{aligned} \tag{6.83}$$

where eqs. (D.7) and (D.33) from the appendix D have been used.

We now demonstrate that the metric components go smoothly over to those in the adjacent regions M_+ and Z_+. The boundary to the region M_+ is given by the line k_+. There we obtain from the equations (6.32, 6.36, 6.81, 6.82, 6.83 and D.6) that

$$F = 0, \quad G = \frac{1}{2}, \quad R = \frac{V-U-c}{2}, \tag{6.84}$$

but these functions just correspond to the metric components ($A = 2G = 1$) in region M_+. Furthermore, the interpolation function is smooth at the boundaries and its derivatives all vanish there, so the metric components smoothly go over to that in M_+.

The boundary to Z_+ is given by the line l_+. Using the same set of equations as above, we find

$$F = 0, \quad G = \frac{1}{2}\left(1 - \frac{4E}{\tilde{V}-\tilde{u}}\right), \quad R = \frac{\tilde{V}-U-\tilde{u}+\bar{u}}{2}. \tag{6.85}$$

Solving eq. (6.27) for $\tilde{V}$ yields

$$\tilde{V} = \tilde{u} + 4E\kappa\left(e^{\frac{V-\bar{u}}{4E}}\right). \tag{6.86}$$

But the reference coordinate $\tilde{V}$ in K_+ and Z_+ is the same, hence $A = 2G$ and R of eq. (6.85) coincide with A and R (eqs. (6.79, 6.78)) from Z_+ at l_+:

$$R = \frac{\tilde{V}-U-\tilde{u}+\bar{u}}{2} = \frac{u-U}{2} + 2E\kappa\left(e^{\frac{V-u}{4E}}\right), \tag{6.87}$$

$$A = 1 - \frac{4E}{\tilde{V}-\tilde{u}} = 1 - \frac{1}{\kappa\left(e^{\frac{V-u}{4E}}\right)}, \tag{6.88}$$

where it has been used that $u = \bar{u}$. The same arguments as for the boundary k_+ hold of course also for l_+, hence, the metric functions are smooth at l_+.

6.6.5 Region III, Z_-

Similarly to the treatment in region Z_+, we obtain the metric functions

$$R = \frac{V - v}{2} + 2E\kappa \left(e^{\frac{v-U}{4E}} \right) \tag{6.89}$$

and

$$A = 1 - \frac{1}{\kappa \left(e^{\frac{v-U}{4E}} \right)}. \tag{6.90}$$

Of course they join continuously with the components of region I at the shell.

6.6.6 Region III, M_-

We obtain the same line element as in region M_+, with $c = c'$.

6.6.7 Region III, K_-

The metric components in the line element (6.58) are given by:

$$F = 0, \quad G = \frac{1}{2U_{,\tilde{U}}}, \quad H = -\frac{U_{,\tilde{V}}}{U_{,\tilde{U}}}. \tag{6.91}$$

In these equations, the coordinate U is given by (6.52). Its derivatives read (using eqs. (D.13, D.35 and D.36) from the appendix D):

$$U_{,\tilde{V}} = -(L' - c')\frac{\psi(\tilde{V} - y)}{K} \tag{6.92}$$

and

$$U_{,\tilde{U}} = 1 + L'_{,\tilde{U}} q_- + \frac{(L' - c')\psi(\tilde{V} - y)}{2K} \left[(\alpha - \beta)\frac{\tilde{V} - y}{a} + \alpha + \beta \right]. \tag{6.93}$$

The same smoothness arguments as in the analysis of region K_+ lead to the conclusion that the metric in K_- goes smoothly over into the metrics of the adjacent regions.

6.7 Conclusions

We have shown that the CR metric is smooth, except at the shell, where it is only continuous. We have also demonstrated that the metric has the desired behaviour at the mirror.

The following table shows the main features of the three gauges we have been considering so far:

Gauge	**Double-null**	C^0 **at shell**	**Regular at mirror**
Singular DNEF	yes	no	yes
Regular DNEF	yes	yes	no
Central-regular	no*	yes	yes

Table 6.1: *The main properties of the three gauges that have been used so far. *) but DN at the mirror, at the shell and at* ∞.

We observe that it seems to be impossible to satisfy all of the three conditions at once. Each of the three gauges has its advantage and will be therefore used in the corresponding situation.

Chapter 7

The canonical action and its reduction

7.1 Introduction

In this chapter we reduce the canonical action for the system consisting of the shell, its surrounding gravitational field and the mirror to an action containing as variables only the true degrees of freedom. These true degrees of freedom are Dirac observables that will play the role of the observables in the corresponding quantum theory. We adopt the reduction method developed by Hájíček, Kiefer and Kouletsis (HKK) in refs. [HK01], [HK02b], [KH02] and [Háj03] to our model including a spherical mirror as the inner boundary. Their method can be straightforwardly applied to our system if a suitable restriction on the allowed gauges near the mirror is imposed and if some further assumptions are made.

The next section is devoted to the original LWF [LWF98] action and the resulting equations of motion. It is followed by a summary of the reduction method due to HKK (section 3). In the final section we add the mirror to the LWF system and explain the differences that appear on that account. We show, after making some further assumptions, that the method by HKK also works in our case, the reduced action containing the expected degrees of freedom: the total energy of the shell and its canonical conjugate, the asymptotic advanced (or retarded) time.

7.2 The LWF action

Louko, Whiting and Friedman [LWF98] have found the ADM action for spherically symmetric spacetimes containing a spherically symmetric thin shell made of null dust. In this section we write down the most important formulae that we will need from their paper.

The spacetime geometry is given by the ADM metric

$$ds^2 = N^2 dt^2 - \Lambda^2 (d\rho + N^\rho dt)^2 - R^2 d\Omega^2, \tag{7.1}$$

where N, N^ρ, Λ, R are continuous functions of t and ρ. N, Λ and R are also required to be positive. The shell history is denoted by $\rho = r(t)$. p is its conjugate momentum. Quantities Q evaluated at the shell are denoted by $Q(r)$. Derivatives with respect to ρ are abbreviated by a prime, Q', those with respect to t by an overdot, $\dot{Q}$.

Including the null shell, the Hamiltonian bulk action reads

$$S_\Sigma = \int dt \left[p\dot{r} + \int_{\mathbb{R}} d\rho \left(P_\Lambda \dot{\Lambda} + P_R \dot{R} - N\mathcal{H} - N^\rho \mathcal{H}_\rho \right) \right], \tag{7.2}$$

where the constraint functions are given by

$$\mathcal{H} = \frac{\Lambda P_\Lambda^2}{2R^2} - \frac{P_\Lambda P_R}{R} + \frac{RR''}{\Lambda} - \frac{RR'\Lambda'}{\Lambda^2} + \frac{R'^2}{2\Lambda} - \frac{\Lambda}{2} + \frac{\eta p}{\Lambda}\delta(\rho - r), \tag{7.3}$$

which is the so-called *superhamiltonian*, and

$$\mathcal{H}_\rho = P_R R' - P'_\Lambda \Lambda - p\delta(\rho - r) \tag{7.4}$$

(the *supermomentum*) and where N, N^ρ are Lagrange multipliers and $\eta \doteq \mathrm{sign}(p)$. The canonically conjugate momenta of the metric functions R and Λ are given by

$$P_\Lambda = -\frac{R}{N}(\dot{R} - N^\rho R') \tag{7.5}$$

and

$$P_R = -\frac{\Lambda}{N}(\dot{R} - N^\rho R') - \frac{R}{N}\left[\dot{\Lambda} - (N^\rho \Lambda)'\right]. \tag{7.6}$$

The constraint equations are

$$\mathcal{H} = 0, \quad \mathcal{H}_\rho = 0. \tag{7.7}$$

The functions N, N^ρ, R, Λ are to be smooth functions of ρ everywhere, except at the shell, where they are only continuous and may have finite jumps in their first derivatives. Also the conjugate momenta P_R, P_Λ are smooth except at the shell, where they have finite discontinuities. The most singular contributions come from the explicit matter delta-terms in the constraints and the implicit delta-functions appearing in R'' and P'_Λ. However, it is argued in [LWF98], that these discontinuities pose no problem and that the action is well-defined. The LWF action is a generalization of Kuchař' canonical action for

pure spherical gravity obtained in [Kuc94]. By similar methods the canonical spacetime dynamics including massive (time-like) instead of null shells has been found [FLWH97].

Variation of the action with respect to the canonical variables and the Lagrange multipliers yields the dynamical,

$$\dot{\Lambda} = N\left(\frac{\Lambda P_\Lambda}{R^2} - \frac{P_R}{R}\right) + (N^\rho \Lambda)', \tag{7.8}$$

$$\dot{R} = -\frac{N P_\Lambda}{R} + N^\rho R', \tag{7.9}$$

$$\dot{P}_\Lambda = \frac{N}{2}\left[-\frac{P_\Lambda^2}{R^2} - \left(\frac{R'}{\Lambda}\right)^2 + 1 + \frac{2\eta p}{\Lambda^2}\delta(\rho - r)\right] - \frac{N' R R'}{\Lambda^2} + N^\rho P_\Lambda', \tag{7.10}$$

$$\dot{P}_R = N\left[\frac{\Lambda P_\Lambda^2}{R^3} - \frac{P_\Lambda P_R}{R^2} - \left(\frac{R'}{\Lambda}\right)'\right] - \left(\frac{N' R}{\Lambda}\right)' + (N^\rho P_R)', \tag{7.11}$$

$$\dot{r} = \frac{\eta N(r)}{\Lambda(r)} - N^\rho(r), \tag{7.12}$$

$$\dot{p} = p\left(N^\rho - \frac{\eta N}{\Lambda}\right)'(r), \tag{7.13}$$

and the constraint equations (7.7).

The possible occurrence of surface terms has not been taken into account yet, but we will do it after having imposed the fall-off conditions on the metric variables at the infinities $\rho \to \pm\infty$, that have been given by LWF and Kuchař [Kuc94]:

$$\Lambda(t,\rho) \approx 1 + \frac{M_\pm}{\rho} + O(|\rho|^{-1-\epsilon}), \tag{7.14}$$

$$R(t,\rho) \approx |\rho| + O(|\rho|^{-\epsilon}), \tag{7.15}$$

$$P_\Lambda(t,\rho) \approx O(|\rho|^{-\epsilon}), \tag{7.16}$$

$$P_R(t,\rho) \approx O(|\rho|^{-1-\epsilon}), \tag{7.17}$$

$$N(t,\rho) \approx N_{\pm\infty} + O(|\rho|^{-\epsilon}), \tag{7.18}$$

$$N^\rho(t,\rho) \approx O(|\rho|^{-\epsilon}), \tag{7.19}$$

where $M_\pm$ and $N_{\pm\infty}$ are functions of t, and where $\epsilon \in (0,1]$. With these fall-off conditions the two asymptotic regions ($\rho \to \pm\infty$) are asymptotically flat. $N_{\pm\infty}$ are the rates at which the asymptotic Minkowski times $T_{\pm\infty}$ evolve with respect to the coordinate time t. There are only two surface terms appearing in the variation of the action if one imposes the above fall-off conditions. The variation of the term $\frac{NRR'\Lambda'}{\Lambda^2}$ in the superhamiltonian constraint $-N\mathcal{H}$ with respect to Λ leads to the non-vanishing terms (cf. [Kuc94])

$$N_{\pm\infty} \lim_{\rho\to\pm\infty} \left(\frac{RR'}{\Lambda^2} \delta\Lambda \right) = N_{\pm\infty} \delta M_{\pm}. \tag{7.20}$$

The last equality follows from the fall-off conditions. The surface terms are therefore proportional to the variation of the mass function M defined in equation (7.89) at the the infinities, $M_{\pm}$, as has been pointed out by Kuchař in [Kuc94]. When the equations of motion hold, the masses $M_{\pm}$ are equal to the Schwarzschild energies $E_{\pm\infty}$. The surface terms (7.20) can be canceled by adding the so-called ADM boundary term (see also [RT74]) to the bulk action:

$$S_{\partial\Sigma} = -\int dt (N_{\infty} E_{\infty} + N_{-\infty} E_{-\infty}), \tag{7.21}$$

the resulting action being

$$\begin{aligned} S &= \int dt \left[p\dot{r} - N_{-\infty} E_{-\infty} - N_{+\infty} E_{+\infty} \right] \\ &\quad + \int dt \left[\int_{-\infty}^{\infty} d\rho \left(P_{\Lambda} \dot{\Lambda} + P_R \dot{R} - N\mathcal{H} - N^{\rho} \mathcal{H}_{\rho} \right) \right]. \end{aligned} \tag{7.22}$$

The left infinity (as ρ goes to $-\infty$) will become unimportant as soon as we introduce the mirror.

LWF have also incorporated fall-off conditions at $\rho = 0$ for the systems with $\mathbb{R}^3$ spatial topology:

$$\begin{aligned} \Lambda(t,\rho) &\approx \Lambda_0 + O(\rho^2), && (7.23)\\ R(t,\rho) &\approx R_1 \rho + O(\rho^3), && (7.24)\\ P_{\Lambda}(t,\rho) &\approx P_{\Lambda_2} \rho^2 + O(\rho^4), && (7.25)\\ P_R(t,\rho) &\approx P_{R_1} \rho + O(\rho^3), && (7.26)\\ N(t,\rho) &\approx N_0 + O(\rho^2), && (7.27)\\ N^{\rho}(t,\rho) &\approx N_1^{\rho} \rho + O(\rho^3), && (7.28) \end{aligned}$$

where $\Lambda_0 > 0$, $R_1 > 0$, P_{Λ_2}, P_{R_1}, $N_0 > 0$ and N_1^{ρ} are functions of t. These fall-offs imply that the mass left of the shell vanishes when the equations of motion hold. The classical solutions thus describe a shell with flat interior. There is only need for one ADM boundary term:

$$S_{\partial\Sigma} = -\int dt N_{\infty} E_{\infty}. \tag{7.29}$$

The LWF action for $\mathbb{R}^3$ spatial topology hence reads

$$S = \int dt \left[p\dot{r} - N_\infty E_\infty + \int_0^\infty d\rho \left(P_\Lambda \dot{\Lambda} + P_R \dot{R} - N\mathcal{H} - N^\rho \mathcal{H}_\rho \right) \right]. \tag{7.30}$$

7.3 Canonical reduction of the LWF action using the methods by HKK

In this section we give a short review of the reduction methods introduced by Hájíček and Kiefer in [HK01] and refined by Hájíček and Kouletsis in refs. [HK02b] and [KH02]. The aim of these methods of reduction is to find an action containing only the true degrees of freedom as variables without losing too much information about the spacetime geometry. There exists a particular form of the gravitational action that is effectively reduced, but still contains some information about the geometry of the spacetime. This is the so-called *Kuchař decomposition* (see [HK00] and the references therein). Kuchař variables are split into pure gauge ones (the so-called embeddings), dependent ones that are canonically conjugate to the embeddings, and physical degrees of freedom (Dirac observables).

Our aim is thus to bring the action (7.30) into this form. The method to achieve this involves the choice of a particular gauge and the solving of the constraints. This procedure has been successfully accomplished for a single spherical thin null shell in [HK01]. The calculations were involved, included some miraculous simplifications and finally yielded a very simple result: The true degrees of freedom were the total energy of the shell and the asymptotic advanced (retarded) time in the ingoing (outgoing) case. In the papers [HK02b], denoted by HKo2 in the following and [KH02] (HKo3), the method has been generalized to any double-null (DN) gauge and to any number of shells, and some understanding has been obtained about the general structure of the reduction process. We write down the most important results for us from the two above cited papers in the next subsections.

7.3.1 Transformation of the Liouville form at the constraint surface

In HKo2 the integrand in the LWF action (7.30) is reduced to a form consisting only of boundary terms. This involves a transformation of the Liouville form Θ at the constraint surface Γ that is defined by the solution of $\mathcal{H} = 0$ and $\mathcal{H}_\rho = 0$ and uses the nice property of the pull-back Θ_Γ of Θ to Γ that it depends only on the Dirac observables. Although Θ_Γ is thus a gauge invariant quantity, the original variables depend on the gauge degrees of freedom. The transformation must be therefore still calculated explicitly. It is found by using a Kuchař decomposition in an arbitrary DN gauge, represented by the coordinates U and V. The choice of Dirac observables is left open; they are denoted by o^k, $k = 1, 2$.

The final variables are thus the Dirac observables o^k, the embeddings $U(\rho), V(\rho)$ and their momenta $P_U(\rho)$, $P_V(\rho)$, which vanish at the constraint surface.

A double-null metric has the general form

$$ds^2 = A(U,V;o)\,dU dV - R^2(U,V;o)\,d\Omega^2, \tag{7.31}$$

the metric components being dependent on the the Dirac observables o^k. Derivatives of the functions A and R with respect to the coordinates U and V are abbreviated by $A_{,U}$, etc.

The Liouville form at the constraint surface for a single shell is given by

$$\Theta_\Gamma = pdr - N_\infty E_\infty + \int_0^\infty d\rho\,(P_\Lambda d\Lambda + P_R dR). \tag{7.32}$$

It can be split into a boundary part (the first two terms on the right hand side) and two volume parts, which are associated with the inside and the outside of the shell, respectively,

$$\Theta_a^b = \int_a^b d\rho\,(P_\Lambda d\Lambda + P_R dR), \tag{7.33}$$

where $a = 0, b = r$ for the inside and $a = r, b = \infty$ for the outside part. The volume parts are to be transformed to a sum of boundary terms that do not depend on the gauge degrees of freedom and their momenta anymore. This must be feasible, because the Liouville form at the constraint surface is gauge invariant!

We now summarize how the transformation has been achieved by HKK. For any DN gauge, the equations (47-49) of [HK02b] (or eqs. (32-34) of [HK01]) represent the condition that the transformation is performed at the constraint surface. The following ansatz is useful for the transformation of the volume parts:

$$\Theta_a^b|_\Gamma = \int_a^b d\rho\,[(f dU + g dV + h_i do^i)' + d\varphi], \tag{7.34}$$

where

$$f = \frac{RR_{,U}}{2}\ln\left(-\frac{U'}{V'}\right) + F(U,V,o^i), \tag{7.35}$$

$$g = \frac{RR_{,V}}{2}\ln\left(-\frac{U'}{V'}\right) + G(U,V,o^i), \tag{7.36}$$

$$h_i = \frac{RR_{,i}}{2}\ln\left(-\frac{U'}{V'}\right) + H_i(U,V,o^i), \tag{7.37}$$

$$\varphi = RR_{,U}U' - RR_{,V}V' - \frac{R}{2}(R_{,U}U' + R_{,V}V')\ln\left(-\frac{U'}{V'}\right) - FU' - GV' + \phi(U, V, o^i), \quad (7.38)$$

and where the functions F, G and H_i must satisfy the following differential equations:

$$F_{,V} - G_{,U} = \frac{R}{2A}(2AR_{,UV} - A_{,U}R_{,V} - A_{,V}R_{,U}), \quad (7.39)$$

$$F_{,i} - H_{i,U} = \frac{R}{2A}(2AR_{,iU} - A_{,i}R_{,U} - A_{,U}R_{,i}), \quad (7.40)$$

$$G_{,i} - H_{i,V} = -\frac{R}{2A}(2AR_{,iV} - A_{,i}R_{,V} - A_{,V}R_{,i}), \quad (7.41)$$

$$\phi = 0, \quad (7.42)$$

that are valid for any DN gauge. The equations (47-49) of [HK02b] provide the integrability conditions for this system of differential equations. The volume parts are hence transformed to boundary terms:

$$\Theta^b_a|_\Gamma = (fdU + gdV + h_i do^i - \varphi d\rho)|^{\rho=b}_{\rho=a}. \quad (7.43)$$

The entire Θ_Γ can thus be brought to a sum of boundary terms with contributions from the regular centre ($\rho = 0$), from infinity and from the shell, at least in the cases for which the equations (7.39) - (7.41) can be solved.

7.3.2 Properties of the functions F, G and H_i

In HKo2 the authors find that any two sets of solutions (F^0, G^0, H^0_i) and (F, G, H_i) of the eqs. (7.39) - (7.41) are related by

$$F = F^0 + W_{,U}, \quad G = G^0 + W_{,V}, \quad H_i = H^0_i + W_{,i} + C_i, \quad (7.44)$$

where W is an arbitrary function of U, V and o^i, and C_i depends on the Dirac observables alone but is otherwise arbitrary. They also argue that each such solution (7.44) leads to the same Liouville form $\Theta^b_a|_\Gamma$. However, different solutions lead not to the same *boundary* Liouville form (7.43) but only to equivalent ones, that differ by an exact form $d[w(b(t), t) - w(a(t), t)]$, where $w(\rho, t) \doteq W(U(\rho, t), V(\rho, t), o^i(t))$.

Gauge transformations also affect the functions F, G and H_i, although the Liouville form itself does not change. The results found in HKo2 are

$$F = \tilde{F}\tilde{X}_{,U} - \frac{R^2}{4}\frac{\tilde{X}_{,UU}}{\tilde{X}_{,U}}, \tag{7.45}$$

$$G = \tilde{G}\tilde{Y}_{,V} - \frac{R^2}{4}\frac{\tilde{Y}_{,VV}}{\tilde{Y}_{,V}}, \tag{7.46}$$

$$H_i = \tilde{H}_i + \tilde{F}\tilde{X}_{,i} + \tilde{G}\tilde{Y}_{,i} - \frac{R^2}{4}\left(\frac{\tilde{X}_{,Ui}}{\tilde{X}_{,U}} - \frac{\tilde{Y}_{,Vi}}{\tilde{Y}_{,V}}\right), \tag{7.47}$$

where the transformation between two sets of DN coordinates,

$$\tilde{U} = \tilde{X}(U, o), \quad \tilde{V} = \tilde{Y}(V, o), \tag{7.48}$$

also depends on the Dirac observables o^i. The eqs. (7.45) - (7.47) lead to the transformation of the functions $\tilde{F}, \tilde{G}$ and $\tilde{H}_i$ that solve the eqs. (7.39) - (7.41) for the gauge $\tilde{U}, \tilde{V}$ to the functions F, G and H_i that solve them for the gauge U, V. So if one knows the functions F, G, H_i in one gauge, one can obtain it in any other (DN) gauge, for which the coordinate transformation can be given explicitly. In fact, one can always find a gauge $\tilde{U}$, $\tilde{V}$ such that the right hand sides of the eqs. (7.39) - (7.41) become trivial.

For example in the flat interior one can choose for $\tilde{U}$, $\tilde{V}$ the advanced and retarded Minkowski null (DNM) coordinates which yields the trivial solution

$$\tilde{F} = \tilde{G} = \tilde{H}_i = 0, \quad \forall i. \tag{7.49}$$

In the curved spacetime outside of the shell we can take the double-null Eddington-Finkelstein (DNEF) coordinates defined by eqs. (5.1) of chapter 5:

$$U = T - R - 2E \ln\left|\frac{R}{2E} - 1\right|, \tag{7.50}$$

$$V = T + R + 2E \ln\left|\frac{R}{2E} - 1\right|, \tag{7.51}$$

where T, R are the Schwarzschild coordinates. The metric components in this gauge read

$$R(U, V) = 2E\kappa\left(e^{\frac{V-U}{4E}}\right), \quad A(U, V) = 1 - \frac{2E}{R}. \tag{7.52}$$

If one chooses these coordinates as the $\tilde{U}$ and $\tilde{V}$ from above, one obtains the simple solution

$$\tilde{F} = \tilde{G} = 0, \quad \tilde{H}_i = \frac{\tilde{V} - \tilde{U}}{2} E_{,i}. \tag{7.53}$$

The drawback of the (singular) DNEF gauge is that the above solution diverges at the horizon. But this singularity can be removed by subtracting a suitable W-term, as demonstrated in HKo2. But for shells not on a horizon the eqs. (7.53) are sufficiently suitable.

7.3.3 Explicit calculation of the Liouville form Θ_Γ

The solutions (7.49) and (7.53) are used to compute the boundary Liouville form (7.43) explicitly. This has been accomplished by Kouletsis and Hájíček in HKo3 ([KH02]) for any number of shells. We adopt their results to the case of a single shell with flat interior spacetime.

The Liouville form is given by

$$\begin{aligned} \Theta_\Gamma &= p dr - N_\infty E_\infty + \Theta_0^r|_\Gamma + \Theta_r^\infty|_\Gamma \\ &= \Theta_0 + \Theta_r + \Theta_\infty, \end{aligned} \tag{7.54}$$

where

$$\Theta_0 \doteq -(f dU + g dV + h_i do^i)|_{\rho=0} \tag{7.55}$$

and

$$\Theta_\infty \doteq \lim_{\rho\to\infty} (f dU + g dV + h_i do^i) - N_\infty E_\infty \tag{7.56}$$

are the contributions from the centre and from infinity, respectively. These quantities are most easily calculated in the singular DNEF gauge. Here we subsume the DN Minkowski (DNM) gauge into the DNEF gauges — it is the special case for which $E = 0$. In this gauge the coordinates are singular at the horizon. One can justify that one may nevertheless use such a singular gauge as follows: The Liouville form at the constraint surface Θ_Γ is point-wise gauge invariant. At those points for which the DNEF gauge is regular, the integrand of Θ_Γ in any volume part has the same value as for a regular C^1 gauge. At the points for which the embedding intersects a horizon, the DNEF gauge is singular. The value of the integrand in such a point is the limit from the left or right, since the integrand is continuous in a regular gauge. Therefore, the difference of the boundary terms that one obtains by integrating over any volume part is gauge invariant and can thus be computed also in the singular DNEF gauge.

More care is to be taken with the expression of the shell variables r and p in terms of the embeddings and the Dirac observables, because at the shell the DNEF coordinates (outside) and the DNM coordinates (inside) the shell are not continuous. The results of HKo3 are as follows: If (U, V) are C^1 DN coordinates, $(\tilde{U}, \tilde{V})$ DNM coordinates left from the (outgoing) shell and $(\bar{U}, \bar{V})$ DNEF coordinates right from the shell, then the expressions for the shell variables are given by

$$p_{\text{out}} = -R(r)\Delta_r(R_{,U}U'), \quad \tilde{U}(r) = \tilde{u}, \quad \bar{U}(r) = \bar{u} \tag{7.57}$$

for the outgoing and, similarly, by

$$p_{\text{in}} = R(r)\Delta_r(R_{,V}V'), \quad \tilde{V}(r) = \tilde{v}, \quad \bar{V}(r) = \bar{v} \tag{7.58}$$

for the ingoing case. Here the abbreviation

$$\Delta_r(X) \doteq \lim_{\rho \searrow r} X - \lim_{\rho \nearrow r} X \tag{7.59}$$

has been used. For the new variables that describe the shell it generally holds that $\tilde{u} \neq \bar{u}$, $\tilde{v} \neq \bar{v}$, because the coordinates are not continuous at the shell.

Now we are ready to compute the three contributions Θ_0, Θ_r and Θ_∞ to the Liouville form along the hypersurfaces Σ defined by the embedding $(U(\rho), V(\rho))$.

The contribution from the shell

We start with the contribution from the shell. According to eqs. (14)-(18) of HKo3, Θ_r for a shell not on a horizon can be brought to the form (eq. (19) in HKo3)

$$\begin{aligned} \Theta_r &= -\Delta_r \left[\frac{1}{4} \ln \left(-\frac{U'}{V'} \right) d(R^2) + F du + G dv + H_i do^i \right] \\ &\quad + \Delta_r \left[R R_{,U} U' dr - R R_{,V} V' dr \right] + p dr. \end{aligned} \tag{7.60}$$

Further transformations depend on the direction of the motion of the shell. We shall assume that it is outgoing; the ingoing case is treated in an analogous way. The jumps in eq. (7.60) must be calculated, where it is assumed that the embedding is C^1. The result is given by eq. (34) of HKo3:

$$\Theta_r = -\frac{1}{4} d\left(R^2(r)\right) \Delta_r(\ln|R - 2E|) - \Delta_r(F du + G dv + H_i do^i) \tag{7.61}$$

and holds at each intersection of an outgoing shell with the hypersurface Σ that is not a point of a Schwarzschild horizon. Now one can plug in the expressions for the functions

F, G and H_i on both sides of the shell. Left from the shell the spacetime is flat, hence the functions are given by eqs. (7.49), and right from the shell the functions for the singular DNEF gauge are given by the eqs. (7.53). Further simplifications, where one utilizes some properties of the Kruskal function, lead to the final results

$$\Theta_r = u dE \tag{7.62}$$

for the outgoing and

$$\Theta_r = v dE \tag{7.63}$$

for the ingoing shells. The eqs. (7.62) and (7.63) are valid as long as the shell is not on a horizon. The case for shells on a horizon has been carefully analysed in HKo3.

The contributions from the centre and from infinity

The contribution from the centre is easily computed using the solution (7.49) for the functions F, G, H_i inside the shell. At $\rho = 0$ the geometric radius R vanishes, $R(\rho = 0) = 0$, and the boundary condition $U'(0) = -V'(0)$ must be imposed in order that the embedded hypersurfaces avoid conical singularities at the centre. All these conditions yield the result that the contribution from the centre vanishes:

$$\Theta_0 = 0. \tag{7.64}$$

For the computation of the contribution from infinity the authors of HKo3 restrict the foliation at infinity such that it is parallel to the hypersurfaces given by the slices of constant Schwarzschild time coordinate. This means that the Schwarzschild coordinates as functions of the embedding have the following fall-off at infinity:

$$R(\rho) \to \rho + \mathcal{O}(\rho^{-1}), \quad T(\rho) \to T_\infty + \mathcal{O}(\rho^{-1}). \tag{7.65}$$

The authors further assume that the foliation parameter t is equal to the asymptotic Minkowski time T_∞ at infinity. Hence, $N_\infty = 1$. Finally, they replace the ADM mass E_∞ by the total energy E (the Schwarzschild mass) of the system. The functions F, G and H_i are given by the eqs. (7.53). Putting all these results together and using the asymptotic expansions of R, U, V and their derivatives, they obtain for the contribution from infinity, eq. (7.56),

$$\Theta_\infty = \lim_{\rho\to\infty} \frac{R}{2} \ln\left(-\frac{U'}{V'}\right) \left(R_{,U} dU + R_{,V} dV + R_{,i} do^i\right) = 0. \tag{7.66}$$

Hence, also the total contribution to the Liouville form from infinity vanishes.

The reduced action

We have seen that only the contribution from the shell survives, leading to the total Liouville form at the constraint surface:

$$\Theta_\Gamma = v dE, \quad \Theta_\Gamma = u dE. \tag{7.67}$$

The first equation holds for ingoing, the second for outgoing shells.

The reduced action for a single ingoing shell reads thus

$$S_{\text{in}} = \int dt(v\dot{E}), \tag{7.68}$$

whereas that for a single outgoing shell is given by

$$S_{\text{out}} = \int dt(u\dot{E}). \tag{7.69}$$

A natural choice for the conjugate pair of Dirac observables is obviously (E, v) or (E, u) in the respective cases. These pairs span the physical phase space of the model, the domains of the variables being

$$E \in [0, \infty), \quad v \in \mathbb{R}, \tag{7.70}$$

$$E \in [0, \infty), \quad u \in \mathbb{R}, \tag{7.71}$$

respectively.

7.4 The action for the system including the mirror and its reduction

7.4.1 Gauge conditions

We include the spherical mirror centered at $R = 0$ and with the fixed radius $R = R_M$ and accordingly change the LWF action (7.30). The inclusion of the mirror restricts the possible values of the coordinates ρ and t. From the equation $R(t, \rho) = R_M$ one can infer the boundary values of ρ and t; the equation corresponds to a curve in the (t, ρ)-plane, denoted by $\rho_M(t)$. The allowed values of the coordinates are thus

$$t \in \mathbb{R}, \quad \rho \in [\rho_M(t), \infty). \tag{7.72}$$

Consequently, we have to replace the lower boundary of the ρ-integration in the LWF action by $\rho = \rho_M(t)$. But a time-dependent boundary in the integration imposes a severe problem: $\rho_M(t)$ becomes a dependent dynamical variable in the variation of the action. From $R(\rho_M) = R_M = const.$ follows that

$$\delta R(\rho_M) = \delta R_M = 0. \tag{7.73}$$

But the equality

$$\delta R(\rho_M) = (\delta R)_M + (R')_M \delta\rho_M \tag{7.74}$$

implies that

$$\delta\rho_M = -\left(\frac{\delta R}{R'}\right)_M \tag{7.75}$$

does not vanish in general. If the action is varied with respect to R, a boundary term of the form

$$-\frac{\delta S}{\delta\rho_M}\left(\frac{\delta R}{R'}\right)_M \tag{7.76}$$

is contributed, that is exceedingly intricate. We, therefore, rather prefer ρ_M to be constant, such that $\delta\rho_M = 0$.

This property can be obtained by imposing suitable boundary conditions on the coordinates at the mirror. Good boundary conditions seem to be the following: we require the coordinates to be the *polar Minkowski coordinates at the mirror* (or coordinates that differ from them only by additive constants):

$$t|_M = T_M, \quad \rho|_M = R_M. \tag{7.77}$$

Then, since R_M is constant, $\delta\rho_M = 0$, as desired. Of, course, the coordinates must be still C^1 across the shell and smooth everywhere else. Such coordinates do exist, as we have shown earlier, in chapter 6, by explicitly constructing the central-regular gauge, i.e. its polar transform, but the cost is that the coordinates for the corresponding metric in UV-form cannot be double-null everywhere anymore. As we will observe later, the restriction of the choice of coordinates at the mirror has beautiful consequences in addition to making the variation of ρ_M vanish.

The problem of C^1 DN-gauges (as e.g. the regular DNEF gauge) is that in the case where the mirror is present, the curve defined by the equation $R(t, \rho) = R_M$ necessarily has a cusp at the point where the shell is reflected by the mirror. The cusp can be removed only if the coordinates are allowed to be non-DN in some regions of the spacetime.

A second problem affects the applicability of the DeWitt method that is presented in chapter 9. The solution $\rho_M(t,\rho)$ of the equation $R(\rho,t) = R_M$ explicitly depends on the solution parameters E and v. If E is changed, then the shape of the cusp changes accordingly, and if v varies then the position of the cusp is moved. Even in the limit $E \to 0$ the curve depends on the choice of v, so there are different flat metrics for different v. But DeWitt's method requires a fixed background manifold common to all solutions. Therefore, in order to be able to apply DeWitt's methods, we need a restriction on the possible gauges at the mirror, like that given by eq. (7.77).

The modified LWF action thus reads:

$$S = \int dt \left[p\dot{r} - N_\infty E_\infty + \int_{\rho_M}^{\infty} d\rho \left(P_\Lambda \dot{\Lambda} + P_R \dot{R} - N\mathcal{H} - N^\rho \mathcal{H}_\rho \right) \right], \tag{7.78}$$

where the superhamiltonian is modified in comparison to that of LWF:

$$\mathcal{H} = \frac{\Lambda P_\Lambda^2}{2R^2} - \frac{P_\Lambda P_R}{R} + \frac{RR''}{\Lambda} - \frac{RR'\Lambda'}{\Lambda^2} + \frac{R'^2}{2\Lambda} - \frac{\Lambda}{2} + \frac{|p|}{\Lambda}\delta(\rho - r), \tag{7.79}$$

so that there is no extra parameter η describing the direction of the shell, and the action is valid for both the in- and the outgoing movement of the shell. The other terms are identical to those in the unmodified LWF action.

7.4.2 Surface terms

Varying the action (7.78) with respect to the canonical variables leads to new surface terms from the mirror:

$$B_M \doteq \left[\frac{NR}{\Lambda}\delta R' - \frac{NRR'}{\Lambda^2}\delta\Lambda - \frac{N'R}{\Lambda}\delta R + N^\rho P_R \delta R - N^\rho \Lambda \delta P_\Lambda \right]_{\rho=\rho_M}. \tag{7.80}$$

Notice that in contrast to the case in the previous section we do not impose any fall-off conditions on the canonical variables at $\rho = \rho_M$. But the surface terms B_M can be removed with the help of the gauge condition from the preceding section.

From the boundary conditions on the coordinates at the mirror follows that in any allowed coordinates the metric near the mirror is given by the (polar) Minkowski line element (omitting the angular part)

$$ds_2^2 = dT_M^2 - dR^2. \tag{7.81}$$

Here, T_M is the Minkowski time coordinate. Introducing the foliation $T_M(t,\rho)$, $R(t,\rho)$ and expressing the differentials in the line element in terms of the coordinates t, ρ yields

$$ds_2^2 = (\dot{T}_M^2 - \dot{R}^2)dt^2 - 2(\dot{R}R' - \dot{T}_M T'_M)dtd\rho - (R'^2 - T'^2_M)d\rho^2. \tag{7.82}$$

Comparison of the coefficients of this line element with those of the corresponding ADM line element,

$$ds_2^2 = (N^2 - \Lambda^2 (N^\rho)^2) dt^2 - 2N^\rho \Lambda^2 dt d\rho - \Lambda^2 d\rho^2, \tag{7.83}$$

leads to the following set of equations:

$$\begin{aligned} \dot{T}_M^2 - \dot{R}^2 &= N^2 - \Lambda^2 (N^\rho)^2, & (7.84)\\ \dot{T}_M T_M' - \dot{R} R' &= -N^\rho \Lambda^2, & (7.85)\\ R'^2 - T_M'^2 &= \Lambda^2. & (7.86) \end{aligned}$$

But we required that the coordinates be Minkowski at the mirror, so $t = T_M$ and $\rho = R$, which implies that the ADM metric coefficients at the mirror read

$$N = \dot{T}_M = 1, \quad N^\rho = 0, \quad \Lambda = R' = 1. \tag{7.87}$$

Inserting these equations into eq. (7.80) yields that the boundary term from the mirror vanishes:

$$B_M = 0. \tag{7.88}$$

Restricting the gauge at the mirror is hence doubly helpful!

From the eqs. (7.87) and from the definition of P_Λ, eq. (7.5), it also follows that the so-called mass function [LWF98],

$$M = \frac{R}{2} \left(1 - \left(\frac{R'}{\Lambda} \right)^2 + \left(\frac{P_\Lambda}{R} \right)^2 \right). \tag{7.89}$$

vanishes at the mirror,

$$M_M = 0. \tag{7.90}$$

7.4.3 Boundary conditions

Even with vanishing surface terms B_M, the variation of the action (7.78) does not lead to the correct equations of motion yet. An additional boundary condition has to be imposed in order that the shell is really reflected at the mirror and does not pass through it unhindered. To incorporate the reflection, the condition

$$|p|_M = constant \tag{7.91}$$

is needed, meaning that the absolute value of the momentum of the shell does not change, when the shell is reflected by the mirror.

The boundary condition (7.91) is indeed responsible for gluing the otherwise independent in- and outgoing trajectories together. We can show this as follows: Let (U, V) be the regular DNEF coordinates. They have been defined in chapter 5 by eqs. (5.45, 5.46). The formulae for the shell momentum p expressed in the embedding variables given by eqs. (8) and (9) of HKo3 read

$$p_{\text{out}} = -R(r)\Delta_r\left(R_{,U}\right)U'(r), \quad p_{\text{in}} = R(r)\Delta_r\left(R_{,V}\right)V'(r). \tag{7.92}$$

The derivatives of R in this gauge are given by

$$R_{,U} = -\frac{A}{2}, \quad \rho > r, \quad R_{,U} = -\frac{1}{2}, \quad \rho < r \tag{7.93}$$

for the outgoing and by

$$R_{,V} = \frac{A}{2}, \quad \rho > r, \quad R_{,V} = \frac{1}{2}, \quad \rho < r \tag{7.94}$$

for the ingoing shells. Inserting these into the eqs. (7.92) and evaluating them at the mirror yields

$$(p_{\text{out}})_M = -\frac{R_M}{2}((A(r))_M - 1)(U'(r))_M, \quad (p_{\text{in}})_M = \frac{R_M}{2}((A(r))_M - 1)(V'(r))_M. \tag{7.95}$$

Hence, we obtain that, if (7.91) holds, then also

$$V'(r)|_M = U'(r)|_M \tag{7.96}$$

must be satisfied. But since, according to eqs. (12) and (13) in HKo3, $U(r) = u$ and $V(r) = v$ at the mirror, the equation (7.96) is equivalent to

$$v = u + c, \tag{7.97}$$

where c is a not yet specified constant. Hence, the additional boundary condition glues the trajectories together. However, the value of the constant c, which should be equal to R^*_M, remains unspecified by the condition (7.91).

Since $M_M = 0$, as we have seen above, there is no shell passing through the mirror in either direction because otherwise the total mass at the mirror would change. Thus, the additional boundary condition and the vanishing of the mass function at the mirror are reconcilable. In addition, the requirement that the shell is reflected by the mirror glues

the in- and outgoing shell trajectories together in a unique way. We know from chapter 5 that the relation between the shell parameters v and u is given by $u = v - 2R_M^*$. Hence, the value of the constant c is determined:

$$c = 2R_M^*. \tag{7.98}$$

7.4.4 Reduction

We reduce the action (7.78) using the same methods as in the case without the mirror. The shell's trajectory results from glueing together an in- and an outgoing one at the mirror, in contrast to the system without the mirror, where the shell is either in- or outgoing. Thus, in our case, it depends on where the embedding hypersurface Σ lies, which part of the shell's trajectory it intersects. Σ can even go through the point where the shell hits the mirror. The diagram fig. (7.1) shows the three possible cases for the embedding hypersurface.

The reduction method of HKK can be performed on each of these three types of hypersurface separately. The calculation is completely analogous to that of HKK except of two things: first, the integrals in the volume parts of the Liouville form have the lower boundary ρ_M instead of 0. This is a minor difference and poses no difficulties, as we will show later.

The second difference however is more imminent, and it requires an additional assumption in order that the occurring problem can be solved. The gauge condition we imposed at the mirror implied that the coordinates can not be double-null (DN) everywhere. One can restrict the region, where they are not DN to an area that is away from both the mirror and the shell's trajectory. We have constructed such a gauge, the CR coordinates, explicitly in chapter 6. The figure (7.1) shows the 'wedges' in which the coordinates are not DN in this gauge.

The problem is that the ansatz and the differential equations for the functions F, G and H_i found by HKK hold only in the case of DN gauges. However, the resulting Liouville form at the constraint surface *is* gauge invariant and must hence be the same for any gauge. We therefore make the following **assumption**:

> *The reduction method using the ansatz (7.34) by HKK works in any gauge for which the coordinates are DN at the mirror, at the shell and at infinity. The resulting Liouville form at the constraint surface is given by the sum of the boundary terms that were present in the action initially and of those that result from the integration of the volume parts, i.e. eq. (7.43), where the possible boundaries of the integration are either $(\rho_{min}, r(t))$, where ρ_{min} is the minimal value of the radial coordinate ρ and $r(t)$ the position of the shell, or $(r(t), \infty)$.*

Since the interpolation wedges do not intersect neither the mirror nor the shell, the integration boundaries do not lie in a region where the coordinates are not DN, so our interpolation gauge satisfies the conditions of the assumption. Taking the assumption for granted, we can compute the Liouville form at the constraint surface in any such gauge we want, be the gauge DN or not outside the 'delicate' regions. We make the same simple choice as in the previous section: DNEF coordinates outside and DNM inside the shell.

After the preceding discussion, along any embedding hypersurface Σ, the only difference to the system without the mirror is the different lower integration boundary in the volume part corresponding to the spacetime interior of the shell. Hence we can use the results for the contributions from infinity and from the shell from the previous section:

$$\Theta_\infty = 0 \tag{7.99}$$

and

$$\Theta_r = v dE \tag{7.100}$$

for a hypersurface intersecting the ingoing shell, or

$$\Theta_r = u dE \tag{7.101}$$

for a Σ intersecting an outgoing one. Therefore, we only need to compute the contribution from the mirror:

$$\Theta_M = -\left(f dU + g dV + h_i do^i - \varphi d\rho_M\right)_{\rho_M}. \tag{7.102}$$

The last term in the bracket on the right hand side vanishes because ρ_M is constant. In the DNM gauge the functions F, G and H_i all vanish, so only the terms proportional to the logarithm $\ln\left(-\frac{U'}{V'}\right)$ survive. We get rid of the logarithm by imposing that the embedding hypersurfaces are parallel to the surfaces of constant Minkowski time at the mirror, which is equivalent to the condition

$$T'_M(\rho_M) = \frac{V'(\rho_M) + U'(\rho_M)}{2} = 0, \tag{7.103}$$

such that the hypersurfaces avoid conical singularities. In our case this requirement is automatically fulfilled because of the gauge conditions we have imposed at the mirror earlier. The contribution from the mirror hence vanishes:

$$\Theta_M = 0. \tag{7.104}$$

We thus obtain the same total Liouville form at the constraint surface as in the system without the mirror. For a hypersurface intersecting an ingoing shell the reduced action is hence given by

$$S = \int dt(v\dot{E}), \tag{7.105}$$

while that for a Σ intersecting the trajectory of an outgoing shell reads

$$S = \int dt(u\dot{E}). \tag{7.106}$$

If Σ goes through the point where the shell bounces from the mirror one has to take limits. Depending on which limit (left or right) is taken, one obtains one of the two actions above.

We have shown above (eq. (7.97)) that the two asymptotic times u and v only differ by a constant:

$$u = v - 2R^*_M. \tag{7.107}$$

Therefore, the two actions (7.105) and (7.106) are equivalent, and thus one of them alone governs the entire trajectory of the shell. The condition (7.107) (or (7.97)) represents the boundary condition (7.91) in the reduced theory. We will work with the action (7.105) in the following, such that the phase space variables of our system are E and v. They have the following domains:

$$E \in [0, \infty), \quad v \in \mathbb{R}. \tag{7.108}$$

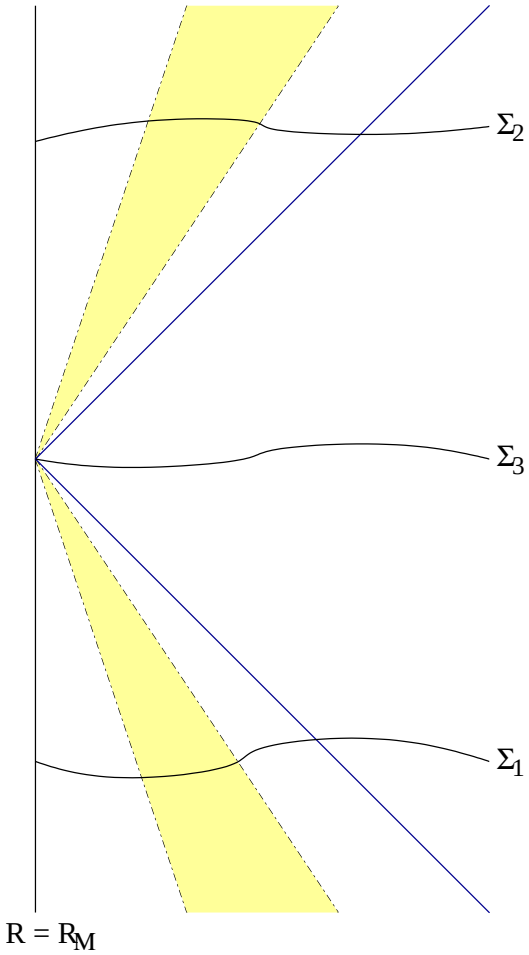

Figure 7.1: *Schematic diagram of the trajectory of the shell bouncing at the mirror at the radius $R = R_M$. Three embedding hypersurfaces Σ are drawn. Σ_1 intersects the ingoing shell, Σ_2 the outgoing one. Σ_3 goes through the point where the shell hits the mirror. The shaded region shows the two 'wedges' where in the central-regular gauge the coordinates are not double-null.*

Chapter 8

Sojourn time

8.1 Introduction

This chapter is devoted to the main findings of this thesis. The sojourn time is explicitly computed and the rather surprising results are discussed. The chapter is organized as follows: The gauge invariant definition of the scattering time from chapter 4 is recalled in section 2. A well-defined quantum theory is constructed by representing the Dirac observables E and $D \doteq Ev$ by a corresponding pair of self-adjoint operators in section 3. Suitable wave packets, the so-called quasi-polynomial packets, are then introduced in section 4. In the quantum theory the scattering time is represented by the self-adjoint operator $\hat{t}_s$. Formulae for the mean value of $\hat{t}_s$ and the spread of the proper times $\hat{s}_\pm$ valid for any kind of wave packet are written down in section 5. In section 6 they are explicitly evaluated analytically for the quasi-parabolic wave packets. In section 7 the results of our numerical computations are written down. They are visualized by diagrams displaying the plots of the sojourn time and the spread of the proper time s_+ as functions of the mean energy or the width of the wave packets. The last section is devoted to a short summary and discussion of the most important results.

8.2 The scattering time

The Schwarzschild energy E and its canonical conjugate, the asymptotic advanced time v are Dirac observables of our simple scattering system consisting of a single self-gravitating spherical thin null shell reflected by a fixed spherical mirror. The classical proper scattering time t_s has been found earlier in chapter 4. It is a function of the observable E and the constants R_M and R_O:

$$t_s = 2\sqrt{1 - \frac{2E}{R_O}} \left(R_O - R_M + 2E \ln \left| \frac{R_O - 2E}{R_M - 2E} \right| \right). \tag{8.1}$$

In section 4 of chapter 4 we have pointed out that the scattering time Δs, that has been defined using explicit coordinates, is not gauge invariant. But we have found a definition (in section 5 of chapter 4) that is both gauge invariant and yields the same quantity, $t_s = \Delta s$. Hence, also t_s is a Dirac observable. We have defined the two gauge invariant quantities $s_\pm$, that are the proper times measured by an ideal clock carried along by a static observer at the instants at which the he sees the shell pass by inwards (s_-) and outwards (s_+). They read:

$$s_- = v - R_O, \quad s_+ = v - R_O + t_s. \tag{8.2}$$

The proper time s_- has been identified with the Minkowski time in the past of the shell. The difference of the two proper times is the scattering time t_s:

$$s_+ - s_- = t_s. \tag{8.3}$$

The two observables $s_\pm$ depend on both basic Dirac observables E and v, whereas t_s depends only on the energy E.

In the next section we turn the proper times $s_\pm$ into operators by substituting the Dirac observables E and v by their representative operators $\hat{E}$ and $\hat{v}$.

8.3 Quantum mechanics

In this section we construct a quantum mechanical description of our system. The Dirac observables E and v are to be represented by a pair of self-adjoint operators. But there is a severe problem: the domain of E is the positive half-axis, whereas that of its conjugate v is $\mathbb{R}$. This leads to the well-known conclusion that the momentum operator $\hat{v} = -i\frac{\partial}{\partial p}$ cannot be made self-adjoint on the half-axis. This follows from von Neumann's theorem on the deficiency indices of this $\hat{v}$, cf. e.g. [RS75, BFV01]. See also the appendix F, where we give a short introduction on this topic.

We therefore need another pair of Dirac observables, that can be made into self-adjoint operators. This can be achieved using the following trick. The pair of observables E, v is replaced by the pair E, $D \doteq Ev$ that has the Poisson bracket

$$\{E, D\} = E. \tag{8.4}$$

D is called a *dilatation* and is of course also a Dirac observable. The corresponding Lie algebra generates a group of symplectic transformations on the phase space that preserves the boundary $E = 0$. The group is the so-called *affine group* $\mathcal{A}$. It has three irreducible unitary representations. In the first one the spectrum of the operator $\hat{E}$ is the positive real axis, in the second one $\hat{E}$ is the zero operator and in the third, the spectrum is $(-\infty, 0]$. We must thus choose the first representation.

The Hilbert space $\mathcal{K}$ of state vectors consists of the complex functions $\phi(p)$ of $p \in [0, \infty)$ that have finite norm with respect to the scalar product

$$(\phi(p), \psi(p)) = \int_0^\infty \frac{dp}{p}\, \bar{\phi}(p)\psi(p), \tag{8.5}$$

that is left invariant by the affine group. Here, p is the eigenvalue of the energy operator $\hat{E}$ belonging to an eigenstate $\phi(p)$ in the $\hat{E}$-representation,

$$\hat{E}\phi(p') = p\phi(p') = p\delta(p - p'). \tag{8.6}$$

The action of the dilatation operator $\hat{D}$ on a Hilbert state reads

$$\hat{D}\phi(p) = -ip\partial_p\phi(p). \tag{8.7}$$

The quantum representative of the Poisson bracket (8.4) is given by the commutator

$$[\hat{E}, \hat{D}] = i\hat{E}. \tag{8.8}$$

The observable $v = \frac{D}{E}$, is represented by the operator

$$\hat{v} = \widehat{E^{-\frac{1}{2}}}\hat{D}\widehat{E^{-\frac{1}{2}}}, \tag{8.9}$$

where the factor ordering has been chosen such that the operator is symmetric with respect to the scalar product, which can be easily checked. This operator possesses, however, *no self-adjoint extension*, again because of von Neumann's theorem. This is demonstrated in section 4 of appendix F. But, due to the lack of time in the completion of this thesis, we continue to work with this $\hat{v}$. We justify this decision as follows: Even if the observable v is not represented by a self-adjoint operator, the quantum theory constructed above is well-defined. The advanced time v appears only in the proper times $s_\pm$ but not in the scattering time t_s, so the effect on the sojourn time of using a non self-adjoint operator should not be significant. Of course, the operators $s_\pm$ that depend explicitly on $\hat{v}$ are now not self-adjoint, such that their mean value and spread are to be interpreted with some care. But we do not expect a significant qualitative difference in the results if one uses a really self-adjoint $\hat{v}$. We could define a self-adjoint operator corresponding to the observable v by taking the square of (8.9), denoted by $\hat{Q}$, which has a self-adjoint extension, according to section 4 of the appendix mentioned above. Then we could define the operator $\hat{v}$ by taking the square root of $\hat{Q}$ and by applying the spectral theorem. Of course, one must be careful in the definition of the square root, because the domain of v is the whole real axis. In any case this modification would require a lot of time and would also, presumably, make some of the calculations of the next sections much more intricate. So we postpone it to a later time.

The action of our operator $\hat{v}$ on a Hilbert state is given by

$$\hat{v}\phi(p) = \frac{i}{2p}\phi(p) - i\partial_p\phi(p). \tag{8.10}$$

$\hat{v}$ thus acts as the differential operator

$$\hat{v} = \frac{i}{2p} - i\partial_p \tag{8.11}$$

in the energy representation.

8.4 Wave packets

In this chapter we work with wave packets in the energy space defined by

$$\psi(p) = 3\frac{\sqrt{70}}{(b-a)^{\frac{9}{2}}}\sqrt{p}\left(p^4 - 2(a+b)p^3 + (a^2+4ab+b^2)p^2 - 2ab(a+b)p + a^2b^2\right)\ \chi_{[a,b]}, \tag{8.12}$$

where $\chi_{[a,b]}$ is the characteristic function of the interval $[a,b]$ and where

$$0 \leq a < b \ll \frac{R_O}{2}. \tag{8.13}$$

These packets are localized in a finite region in energy space. We will call them *quasi-polynomial wave packets* in the following, because they differ from polynomial packets only by the factor $\sqrt{p}$. This factor has been deliberately introduced in order to obtain similar results to that one would obtain using polynomial wave packets and the 'standard' scalar product without the p in the denominator of the measure, instead. The figure (8.1) shows a sample quasi-polynomial wave packet.

Our main motivation to take these wave packets is that they are the lowest order polynomial (up to the factor $\sqrt{p}$) wave packets that lie in the domain of the differential operators ∂_p and ∂_p^2 and have compact support in energy space. Polynomials of lower order can not satisfy the conditions $\psi(a) = \psi(b) = \psi'(a) = \psi'(b) = 0$, where $\psi' \doteq \frac{\partial^2\psi}{\partial p^2}$. E.g. for a rectangular packet that is the simplest possible wave packet — it is made of step functions — already the first derivative has delta-functions at $p = a$ and $p = b$ coming from the derivative of the characteristic function $\chi_{[a,b]}$ that are not compensated by ψ, because the rectangular packet is not continuous there. The consequence is that ψ' is not in the Hilbert space. A polynomial of fourth order, however, can be chosen such that the conditions at a and b are satisfied, and, hence, the derivative of ψ is continuous there. This is necessary

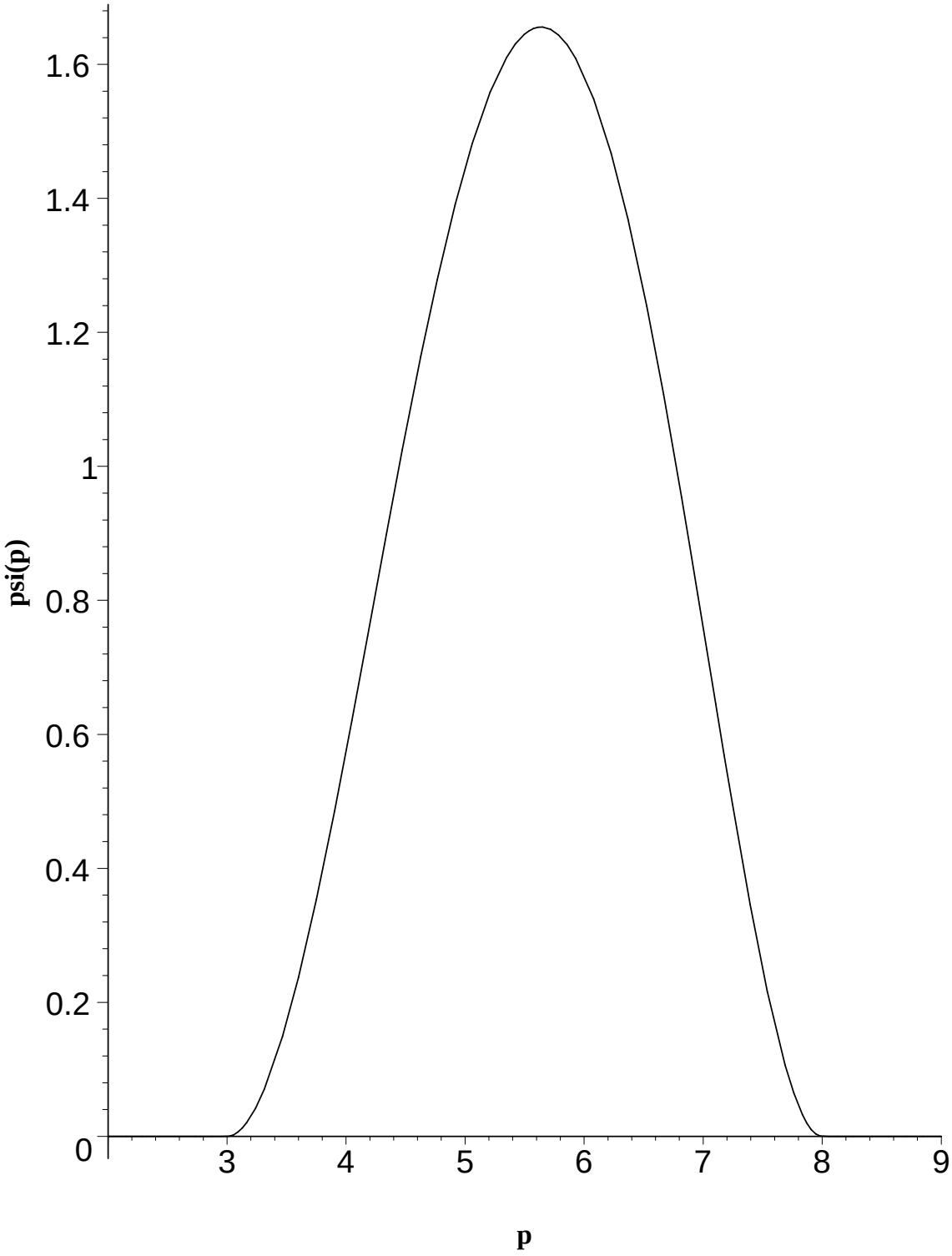

Figure 8.1: *The graph of the function $\psi(p)$ describing a quasi-polynomial wave packet is plotted as a function of the energy p. The parameters a and b that define the lower and upper boundaries of the interval on which the amplitude of the packet is different from zero have the values $a = 3$ and $b = 8$.*

because some of the operators in our theory are second-order differential operators, i.e. $\widehat{v^2}$ and $\widehat{D^2}$.

It is also very useful that one can restrict the contributing energies to a finite interval such that one can investigate what happens if a specific energy is in that interval. This is especially important in the case of the mean square deviation of the operator corresponding to s_+ that diverges if the wave packet contains the state with the energy equal to $\frac{R_M}{2}$, as we will show in the next section.

The mean energy of a quasi-polynomial packet reads

$$\langle E\rangle = \int_0^\infty \frac{dp}{p}|\psi|^2 p = \frac{b+a}{2}, \tag{8.14}$$

and the expectation value of the operator $\widehat{E}^2$ is given by

$$\langle E^2\rangle = \frac{3b^2+3a^2+5ab}{11}. \tag{8.15}$$

The mean square deviation is thus

$$(\Delta E)^2 = \frac{(b-a)^2}{44}. \tag{8.16}$$

The action of the dilatation operator $\hat{D}$ and its square $\widehat{D^2}$ on the wave packet is given by

$$\hat{D}\psi = -ip\partial_p\psi, \tag{8.17}$$

$$\widehat{D^2}\psi = -p\partial_p\psi - p^2\partial_p^2\psi. \tag{8.18}$$

Its expectation values read

$$\langle D\rangle = 0 \tag{8.19}$$

and

$$\langle D^2\rangle = \frac{3}{4}\frac{(5a^2+6ab+5b^2)}{(b-a)^2} = (\Delta D)^2. \tag{8.20}$$

We will not make use of the operator $\hat{D}$ from now on. It has been only introduced in order to construct a well-defined quantum theory.

The operator corresponding to the advanced time has been defined in eq. (8.9). Its action on the wave packet is given by

$$\hat{v}\psi = \frac{i}{2p}\psi - i\partial_p\psi, \tag{8.21}$$

whereas that of its square reads

$$\widehat{v^2}\psi = -\frac{3}{4p^2}\psi + \frac{1}{p}\partial_p\psi - \partial_p^2\psi. \tag{8.22}$$

We first compute the expected value of $\hat{v}$. The resulting integral vanishes:

$$\langle v\rangle = 0. \tag{8.23}$$

The expected value of the squared operator is hence equal to the mean square deviation of $\hat{v}$ and reads

$$\langle v^2\rangle = (\Delta v)^2 = \frac{12}{(b-a)^2}. \tag{8.24}$$

The spreads of $\hat{E}$ and $\hat{v}$ are related by

$$\Delta E\Delta v = \sqrt{\frac{3}{11}} \quad \Rightarrow \quad \Delta E \propto \frac{1}{\Delta v}. \tag{8.25}$$

8.5 General formulae

We represent the proper times before (s_-) and after (s_+) the scattering by quantum operators $\hat{s}_\pm$ and write down the important results relating to the expected value and the spread of these operators, valid for any kind of wave packet. We use these formulae in the next section, where we calculate the expected values explicitly for quasi-polynomial wave packets.

The two proper times $s_\pm$ defined in eqs. (8.2) of section 3 are represented by the operators (in the $\hat{E}$-representation)

$$\hat{s}_- = -R_O + \hat{v} = -R_O + \frac{i}{2p} - i\partial_p \tag{8.26}$$

and

$$\hat{s}_+ = -R_O + \hat{v} + \hat{t}_s = -R_O + \frac{i}{2p} - i\partial_p + \hat{t}_s, \tag{8.27}$$

where

$$\hat{t}_s = 2\sqrt{1-\frac{2p}{R_O}}\left(R_O - R_M + 2p\ln\left|\frac{R_O-2p}{R_M-2p}\right|\right) \tag{8.28}$$

is the *scattering time operator* in the $\hat{E}$-representation. Since $\hat{t}_s$ depends only on one of the basic operators, i.e. $\hat{E}$, there are no factor ordering ambiguities, and there are no difficulties with the non-self-adjointness of our $\hat{v}$.

The expectation value of $\hat{s}_-$ in any wave packet is given by

$$\langle s_-\rangle = -R_O + \langle v\rangle, \tag{8.29}$$

that of its square by

$$\langle s_-^2\rangle = R_O^2 + \langle v^2\rangle - 2R_O\langle v\rangle, \tag{8.30}$$

its mean square deviation reads thus

$$(\Delta s_-)^2 = (\Delta v)^2. \tag{8.31}$$

The mean value of the other operator is

$$\langle s_+\rangle = -R_O + \langle v\rangle + \langle t_s\rangle, \tag{8.32}$$

where $\langle t_s\rangle$ is the expectation value of the scattering time operator which is equal to the *sojourn time*. The mean value of $\widehat{s^2}_+$ is slightly more involved, because the factor ordering becomes important. The result is, after choosing the obvious factor ordering,

$$\langle s_+^2\rangle = R_O^2 - 2R_O\langle v\rangle + \langle v^2\rangle + \langle t_s^2\rangle - 2R_O\langle t_s\rangle + \langle t_s v\rangle + \langle v t_s\rangle. \tag{8.33}$$

The mean square deviation of $\hat{s}_+$ reads therefore

$$(\Delta s_+)^2 = (\Delta v)^2 + (\Delta t_s)^2 + \langle t_s v\rangle + \langle v t_s\rangle - 2\langle v\rangle\langle t_s\rangle. \tag{8.34}$$

The difference of the expected proper times $\langle s_+\rangle$ and $\langle s_-\rangle$ measured by the observer at the radius R_O turns out to be equal to the *sojourn time*:

$$\langle s_+\rangle - \langle s_-\rangle = \langle t_s\rangle. \tag{8.35}$$

Our interest is focussed to this quantity and the *spreads of the proper time before* (Δs_-) *and after* (Δs_+) *the reflection*:

$$\Delta s_- = \Delta v, \tag{8.36}$$

$$\Delta s_+ = \sqrt{(\Delta v)^2 + (\Delta t_s)^2 + \langle t_s v\rangle + \langle v t_s\rangle - 2\langle v\rangle\langle t_s\rangle}. \tag{8.37}$$

Especially interesting is the case where the energies in the wave packets exceed the cut-off at $E = \frac{R_M}{2}$ that has been introduced in the classical theory. The classical scattering time diverges at the cut-off and is no longer defined for energies that are even higher. We will observe that this is not the case for the sojourn time.

But before we turn our attention to the explicit calculation of these quantities, we have a closer look at the term $\langle v t_s\rangle$ appearing in eq. (8.37), because it is the only expression that is responsible for divergences. In order to compute this term, the following property is useful: One can show that for our (non-self-adjoint) operator $\hat{v}$ it holds that:

$$[\hat{E}, \hat{v}] = i. \tag{8.38}$$

The action of any regular function F of the operator $\hat{E}$ alone on a eigenstate ϕ of $\hat{E}$ is, according to spectral analysis, given by,

$$F(\hat{E})\phi = F(E)\phi. \tag{8.39}$$

Hence, one obtains, after a simple calculation, the useful relation

$$\langle vF\rangle = \langle Fv\rangle - i\langle F'\rangle, \quad F' \doteq \frac{\partial F}{\partial E}. \tag{8.40}$$

From eq. (8.40) it follows that

$$\langle v t_s\rangle = \langle t_s v\rangle - i\langle t_s'\rangle. \tag{8.41}$$

The first term on the right hand side is regular everywhere. This is, however, not true for the second term and stems from the fact that the derivative of the scattering time with respect to the energy contains singularities. The derivative reads

$$\begin{aligned} t_s' \;=\; & -\frac{2}{R_O\sqrt{1-\frac{2E}{R_O}}}\left(R_O - R_M + 2E\ln\left|\frac{R_O-2E}{R_M-2E}\right|\right) \\ & +4\sqrt{1-\frac{2E}{R_O}}\ln\left|\frac{R_O-2E}{R_M-2E}\right| \\ & 8E\sqrt{1-\frac{2E}{R_O}}\left(\frac{|1,R_O-2E|}{|R_O-2E|} - \frac{|1,R_M-2E|}{|R_M-2E|}\right), \end{aligned} \tag{8.42}$$

where the abbreviation

$$|1,x| \doteq \frac{d|x|}{dx} \tag{8.43}$$

has been introduced. $|1, x|$ is equal to the sign of x if $x \neq 0$ and is undefined for $x = 0$. The first two terms on the right hand side of eq. (8.42) are as regular as t_s and thus pose no problem. The expression in the bracket of the third term, however, is highly singular if $E = \frac{R_M}{2}$ or $E = \frac{R_O}{2}$. Indeed, the expression $\frac{|1,x-q|}{|x-q|}$ has an (essential) singularity at the point $x = q$:

$$\lim_{x \searrow q} \frac{|1, x-q|}{|x-q|} = +\infty, \quad \lim_{x \nearrow q} \frac{|1, x-q|}{|x-q|} = -\infty. \tag{8.44}$$

The integral

$$\int_{q-z}^{q+z} dx\, g(x)\, \frac{|1, x-q|}{|x-q|}, \quad z > 0 \tag{8.45}$$

for an arbitrary function $g(x)$ thus diverges if $g(q) \neq 0$. But the wave packets we use lead to integrals of this form with non-vanishing functions g at the point $E = \frac{R_M}{2}$. The consequence is that the expectation value of t_s' diverges as soon as the wave packet contains states with $E = \frac{R_M}{2}$. Hence, also the spread of $\hat{s}_+$ diverges for such a wave packet.

Of course, as pointed out earlier, all results involving the operator $\hat{v}$ have to be interpreted with some care. But we expect that the results that one would obtain using a self-adjoint $\hat{v}$ would not be qualitatively different from ours.

8.6 Formulae for quasi-polynomial wave packets

We explain how the quantities $\langle t_s \rangle$, Δs_- and Δs_+ are computed explicitly for the quasi-polynomial wave packets given by eq. (8.12). We present a quite detailed description of the calculation of the sojourn time $\langle t_s \rangle$. The results for $\langle t_s^2 \rangle$ and $\langle t_s v \rangle$ are found using similar methods. The quantity $\langle v t_s \rangle$ diverges in some cases as has been pointed out in the previous section. In all other cases it cancels the contribution from the term $\langle t_s v \rangle$. We have made this observation by performing numerical computations.

The expected value of $\hat{t}_s$ consists, up to constant factors, of integrals of the types

$$I_1 \doteq \int dp\, \Pi(p) \sqrt{1 - \alpha p}, \tag{8.46}$$

and

$$I_{L,i} \doteq \int dp\, p \Pi(p) \sqrt{1 - \alpha p} \ln |2p - \beta_i| \tag{8.47}$$

where

$$\alpha = \frac{2}{R_O}, \quad \beta_1 = R_O, \quad \beta_2 = R_M, \tag{8.48}$$

and where $\Pi(p)$ is a polynomial in p the coefficients of which are determined by the explicit form of the wave packet ψ, eq. (8.12),

$$\Pi(p) \doteq \tilde{c}_8 p^8 + ... + \tilde{c}_0. \tag{8.49}$$

$\langle t_s \rangle$ is hence given by

$$\langle t_s \rangle = 2(R_O - R_M)\, I_1|_a^b + 4\left(I_{L,1}|_a^b - I_{L,2}|_a^b \right). \tag{8.50}$$

We first compute the part I_1 that does not contain logarithms. In order to transform the integrand of I_1 into a pure polynomial, we make the substitution

$$y^2 = 1 - \alpha p, \quad dp = -\frac{2y}{\alpha} dy. \tag{8.51}$$

The integration after the substitution yields

$$I_1 = -\frac{2}{\alpha} \int dy\, y^2\, \Pi\left(\frac{1-y^2}{\alpha} \right) = -R_O \tilde{Q}(y), \tag{8.52}$$

where

$$\tilde{Q}(y) \doteq \frac{c_{16}}{19} y^{19} + ... + \frac{c_0}{3} y^3. \tag{8.53}$$

Here the coefficients $c_0, ..., c_{16}$ are defined the substitution (8.51) and by the definition of Π, eq. (8.49)

$$\Pi\left(\frac{1-y^2}{\alpha} \right) = c_{16} y^{16} + ... + c_0. \tag{8.54}$$

The logarithmic terms $I_{L,i}$ require a little bit more work. After the substitution (8.51) we obtain that

$$I_{L,i} = -\frac{2}{\alpha} \int dy\, y^2 (1-y^2) \Pi\left(\frac{1-y^2}{\alpha} \right) \ln |\Sigma_i(y)|, \tag{8.55}$$

where

$$\Sigma_i = -\frac{2}{\alpha} y^2 + \frac{2}{\alpha} - \beta_i. \tag{8.56}$$

By integration by parts the $I_{L,i}$ can be brought to the form

$$I_{L,i} = -\frac{2}{\alpha}\left[Q(y)\ln|\Sigma_i(y)| - \int dy\, Q(y)\frac{\Sigma_i'(y)}{\Sigma_i(y)}\right] \doteq -\frac{2}{\alpha}\left[K_i - J_i\right], \tag{8.57}$$

where

$$\begin{aligned} Q(y) &= \frac{1}{\alpha}\int dy\,(1-y^2)y^2\Pi\left(\frac{1-y^2}{\alpha}\right) \\ &= \frac{1}{\alpha}\left[\tilde{Q}(y) - W(y)\right], \\ W(y) &\doteq \frac{c_{16}}{21}y^{21} + ... + \frac{c_0}{5}y^5. \end{aligned} \tag{8.58}$$

K_i reads thus

$$\begin{aligned} K_i &= \frac{1}{\alpha}\left[\tilde{Q}(y) - W(y)\right]\ln|\Sigma_i(y)| \\ &= \frac{1}{\alpha}\left[b_{21}y^{21} + ... + b_3y^3\right]\ln|\Sigma_i(y)|, \\ b_m &\doteq \frac{c_{m-3} - c_{m-5}}{m}, \quad c_k = 0 \ : \ k < 0. \end{aligned} \tag{8.59}$$

The integrand of J_i is now a rational function, because the quotient $\frac{\Sigma_i'(y)}{\Sigma_i(y)}$ is given by

$$\frac{\Sigma_i'(y)}{\Sigma_i(y)} = \frac{2y}{y^2 - \sigma_i}, \quad \sigma_i \doteq 1 - \frac{\alpha\beta_i}{2}. \tag{8.60}$$

The J_i thus read

$$J_i = \frac{2}{\alpha}\int dy\,\frac{b_{21}y^{22} + ... + b_3y^4}{y^2 - \sigma_i}. \tag{8.61}$$

The integrals that appear in eq. (8.61) are of the form

$$L_{n,i} \doteq \int dy\,\frac{y^n}{y^2 - \sigma_i} = -\frac{y^{(n+1)}}{2\sigma_i}\,\Phi\left(\frac{y^2}{\sigma_i}, 1, \frac{n+1}{2}\right), \tag{8.62}$$

where $\Phi(z,h,\nu)$ is Lerch's Φ function (cf. A. Erdélyi, Higher transcendental functions, Vol. 1). The first few L_n interesting for us are given by

$$
\begin{aligned}
L_4 &= \frac{y^3}{3} + \sigma y - \sigma^{\frac{3}{2}} \mathrm{ArTh}\left(\frac{y}{\sqrt{\sigma}}\right), \\
L_5 &= \frac{y^4}{4} + \frac{\sigma}{2} y^2 + \frac{\sigma^2}{2} \ln(y^2 - \sigma), \\
L_6 &= \frac{y^5}{5} + \frac{\sigma}{3} y^3 + \sigma^2 y - \sigma^{\frac{5}{2}} \mathrm{ArTh}\left(\frac{y}{\sqrt{\sigma}}\right), \\
L_7 &= \frac{y^6}{6} + \frac{\sigma}{4} y^4 + \frac{\sigma^2}{2} y^2 + \frac{\sigma^3}{2} \ln(y^2 - \sigma).
\end{aligned} \tag{8.63}
$$

The J_i hence read

$$
J_i = \frac{2}{\alpha} \sum_{k=3}^{21} b_k L_{(k+1),i}. \tag{8.64}
$$

Having obtained these results we are able to combine the terms in $I_{L,i}$ to

$$
I_{L,i} = -\frac{R_O^2}{2} \sum_{m=3}^{21} b_m \left[y^m \ln |\Sigma_i(y)| - 2L_{(m+1),i} \right]. \tag{8.65}
$$

The sojourn time eq. (8.50) is now easily put together.

Using similar methods one can calculate the expectation values $\langle t_s v \rangle$ and $\langle t_s^2 \rangle$. The square logarithmic terms in the latter quantity can be removed by repeated integration by parts.

The expected values of $\hat{t}_s$, $\hat{t_s^2}$ and $\widehat{t_s v}$ are *finite* for all wave packets that are a superposition of states with energies from the interval $[0, \frac{R_O}{2})$. This is in strong contrast to the classical case where the scattering time diverges as the energy approaches the value for which the Schwarzschild radius of the shell is equal to the radius of the mirror, $\frac{R_M}{2}$. We call this energy the *Schwarzschild energy corresponding to the mirror with radius* R_M and denote it in the following by

$$
E_{\mathrm{Schw}} \doteq \frac{R_M}{2}. \tag{8.66}
$$

In the next section we give some numerical examples to illustrate these findings, but one can also easily convince oneself without numerically calculating $\langle t_s \rangle$ that it remains finite in the case of wave packets for which $a < E_{\mathrm{Schw}}$, $b > E_{\mathrm{Schw}}$ by looking at the various terms contributing to the mean sojourn time. None of them can diverge even in this case.

The quantity $\langle v t_s \rangle$, however, diverges for wave packets that contain the Schwarzschild energy E_{Schw}, as it has already been pointed out at the end of the previous section. Consequently, it is not compensated by the term $\langle t_s v \rangle$ that remains finite, such that the proper time of such packets has an infinite spread after the scattering. In the case of wave packets that consist exclusively of energies below (or above) E_{Schw}, the spread of the proper time after the scattering is finite.

8.7 Numerics, tables and diagrams

8.7.1 The Planck regime

In this section we illustrate the results from the preceding section by presenting some numerical computations. In contrast to the other parts of this thesis, we use units for which $c = \hbar = 1$, $G = \ell_{\mathrm{Pl}}^2$, where

$$\ell_{\mathrm{Pl}} = \sqrt{\frac{\hbar G}{c^3}}, \tag{8.67}$$

is the *Planck length*, which in SI units amounts of $\ell_{\mathrm{Pl}} = 1.62 \cdot 10^{-35} m$. Quantum effects in gravitational systems become important only at the Planck scale which is very small and far beyond experimental reach at present. All quantities we are interested in are therefore given in units at the Planck scale, that are, besides the Planck length, the *Planck mass*,

$$m_{\mathrm{Pl}} = \sqrt{\frac{\hbar c}{G}} = \frac{1}{\ell_{\mathrm{Pl}}} = 2.18 \cdot 10^{-8} kg, \tag{8.68}$$

and the *Planck time*

$$t_{\mathrm{Pl}} = \sqrt{\frac{\hbar G}{c^5}} = \ell_{\mathrm{Pl}} = 5.39 \cdot 10^{-44} s. \tag{8.69}$$

We have reintroduced Newton's constant G, thus we have to adjust the notation of the scattering time, because of $E = GM = \ell_{\mathrm{Pl}}^2 M$, where M is the *Schwarzschild mass* of the shell. The scattering time reads:

$$t_s = 2\sqrt{1 - \frac{2M}{R_O}\ell_{\mathrm{Pl}}^2}\left(R_O - R_M + 2M\ell_{\mathrm{Pl}}^2 \ln\left|\frac{R_O - 2M\ell_{\mathrm{Pl}}^2}{R_M - 2M\ell_{\mathrm{Pl}}^2}\right|\right). \tag{8.70}$$

The distances R_M or R_O are given in units of the Planck length,

$$R_O = \alpha_O \ell_{\mathrm{Pl}}, \quad R_M = \alpha_M \ell_{\mathrm{Pl}}, \tag{8.71}$$

where $\alpha_O \geq \alpha_M > 0$ are dimension-less numbers. The Schwarzschild mass M can be expressed in units of the Planck mass,

$$M = q m_{\mathrm{Pl}} = \frac{q}{\ell_{\mathrm{Pl}}}, \tag{8.72}$$

where q is dimension-less, whereas times as the scattering t_s or the asymptotic advanced time v are expressed in units of the Planck time $t_{\mathrm{Pl}} = \ell_{\mathrm{Pl}}$:

$$t_s = 2\sqrt{1 - \frac{2q}{\alpha_O}} \left(\alpha_O - \alpha_M + 2q \ln \left| \frac{\alpha_O - 2q}{\alpha_M - 2q} \right| \right) \ell_{\mathrm{Pl}}, \tag{8.73}$$

$$v = \omega t_{\mathrm{Pl}} = \omega \ell_{\mathrm{Pl}}. \tag{8.74}$$

Obviously, t_s is then of the order of the Planck time.

We investigate the dependence of the three interesting quantities $\langle t_s \rangle$, Δs_- and Δs_+ on the mean energy $\langle E \rangle = \ell_{\mathrm{Pl}}^2 \langle M \rangle = \langle q \rangle m_{\mathrm{Pl}}$ and the width of the packets. The computations have been done using MAPLE. For all calculations we chose the following values for the two radii:

$$R_M = 20\,\ell_{\mathrm{Pl}}, \quad R_O = 2000\,\ell_{\mathrm{Pl}}, \tag{8.75}$$

which correspond to $\alpha_M = 20$, $\alpha_O = 2000$. The Schwarzschild energy $E_{\mathrm{Schw}} = \ell_{\mathrm{Pl}}^2 M_{\mathrm{Schw}}$ is hence given by $M_{\mathrm{Schw}} = q_{\mathrm{Schw}} m_{\mathrm{Pl}}$, where $q_{\mathrm{Schw}} = \frac{\alpha_M}{2} = 10$, i.e. the Schwarzschild radius of a shell with this energy coincides with the radius of the mirror. The interesting shells are thus those that have Schwarzschild radii of the order of magnitude of the radius of the mirror. We, therefore, choose the parameters a and b appearing in the definition of the quasi-polynomial wave packet (8.12) such that they are not too different from the Schwarzschild energy. Thus q_a and q_b, where

$$q_a \doteq \frac{a}{\ell_{\mathrm{Pl}}} = \frac{m_a}{m_{\mathrm{Pl}}}, \quad m_a \doteq \frac{a}{\ell_{\mathrm{Pl}}^2} \tag{8.76}$$

and

$$q_b \doteq \frac{b}{\ell_{\mathrm{Pl}}} = \frac{m_b}{m_{\mathrm{Pl}}}, \quad m_b \doteq \frac{b}{\ell_{\mathrm{Pl}}^2}, \tag{8.77}$$

may not be too different from $q_{\mathrm{Schw}} = 10$.

The radius of the observer's position has been chosen such that it is much larger than the values of a and b we intend to take. This is because the computations are not valid anymore if the energies are not small compared to the Schwarzschild energy of the observer, $\frac{R_O}{2}$, for the observer is supposed to reside in a region where the gravitational effect of the shell on the spacetime is only weak. One effect that occurs if the energies in the wave packet are too high compared to R_O is that the sojourn time becomes negative.

The results concerning the spreads of the operators $\hat{s}_\pm$ have to be interpreted with some care, according to our observations in the previous sections. But we expect that the qualitative results do not change significantly if the operator $\hat{v}$ is replaced by one that is really self-adjoint. The results on the sojourn time are then of course not affected.

8.7.2 Numerics

We first investigate the behaviour of $\langle t_s \rangle$, Δs_- and Δs_+ for wave packets with the same width in energy space but varying mean energy. We take packets for which the two parameters a and b in eq. (8.12) satisfy $b - a = 2\ell_{\mathrm{Pl}}$, i.e. $q_b - q_a = 2$. Hence, according to eq. (8.16), the spread of the energy $\Delta E = \Delta M \ell_{\mathrm{Pl}}^2$, with $\Delta M = m_{\mathrm{Pl}} \Delta q$, reads $\Delta q = 0.30$. The expected value of $\hat{v}$ is zero for all quasi-polynomial packets. The root mean square deviation is given by $\Delta v = \ell_{\mathrm{Pl}} \Delta\omega$, $\Delta\omega = 1.73$. It is equal to the spread Δs_- of the proper time before the scattering. The following table (8.1) displays the results for $\langle t_s \rangle$ and Δs_+.

q_a	q_b	$\langle q \rangle = \langle M \rangle / m_{\mathrm{Pl}}$	$\langle t_s \rangle$ $[t_{\mathrm{Pl}}]$	Δs_+ $[t_{\mathrm{Pl}}]$
0	2	1	3976.89	5.49
2	4	3	4013.53	6.13
4	6	5	4055.80	7.17
6	8	7	4108.27	9.35
8	10	9	4191.65	20.05
9	11	10	4285.81	diverges
10	12	11	4241.84	7.52
11	13	12	4232.47	1.87
12	14	13	4233.75	2.04
13	15	14	4238.50	2.45
14	16	15	4244.83	2.69
16	18	17	4259.58	2.91
50	52	51	4482.07	2.24
82	84	83	4596.65	1.76

Table 8.1: *Numerical results for the sojourn time $\langle t_s \rangle$ and the spread of $\hat{s}_+$ are written down in units of the Planck time for packets with constant energy width and varying mean energy. The Schwarzschild energy is at $\langle q \rangle = 10$. There the sojourn time has a local maximum, and the spread of the proper time of the outgoing shell diverges.*

The sojourn time grows until it reaches a *local finite maximum at the Schwarzschild energy* corresponding to $q_{\mathrm{Schw}} = \frac{\alpha_M}{2} = 10$. Beyond this energy it then decreases, only to begin to grow slowly again. The diagram fig. (8.2) illustrates this behaviour.

Also the spread of $\hat{s}_+$ increases near the Schwarzschild energy. For the packets that contain the Schwarzschild energy it diverges as expected. Above the Schwarzschild energy it first decreases very fast, then reaches a minimum, slowly increases again to reach a much less distinctive maximum, after which it slightly decreases again. The dependence of Δs_+ on the mean energy is displayed in the diagram (8.3).

The next table (8.2) shows the dependence of $\langle t_s \rangle$ on the width of the packets while the

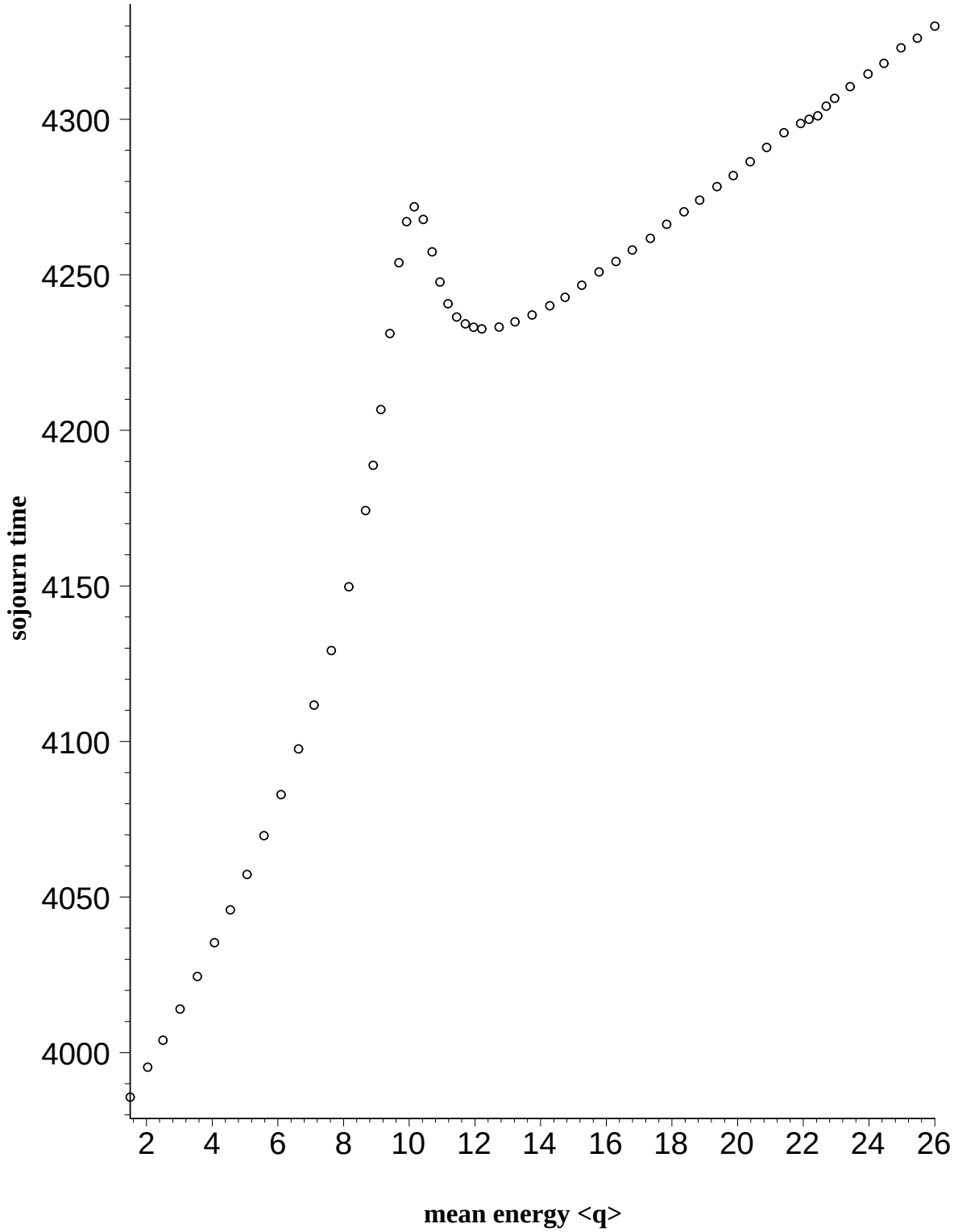

Figure 8.2: *The sojourn time $\langle t_s \rangle$ is plotted as a function of the mean energy $\langle q \rangle$, where the energy width of the quasi-polynomial packets is constant: Δq =0.30. The Schwarzschild energy is given by $q_{Schw} = 10$. The peak in $\langle t_s \rangle$ at $\langle q \rangle = 10$ is well visible. The sojourn time is finite for all $\langle q \rangle$, in contrast to the classical scattering time that diverges at and is not defined beyond the Schwarzschild energy. The dimensionless variable q is related to the energy E by $q = \frac{M}{m_{Pl}} = \frac{E}{\ell_{Pl}}$, where m_{Pl} and ℓ_{Pl} are the Planck mass and length, respectively. The sojourn time is given in units of the Planck time t_{Pl}.*

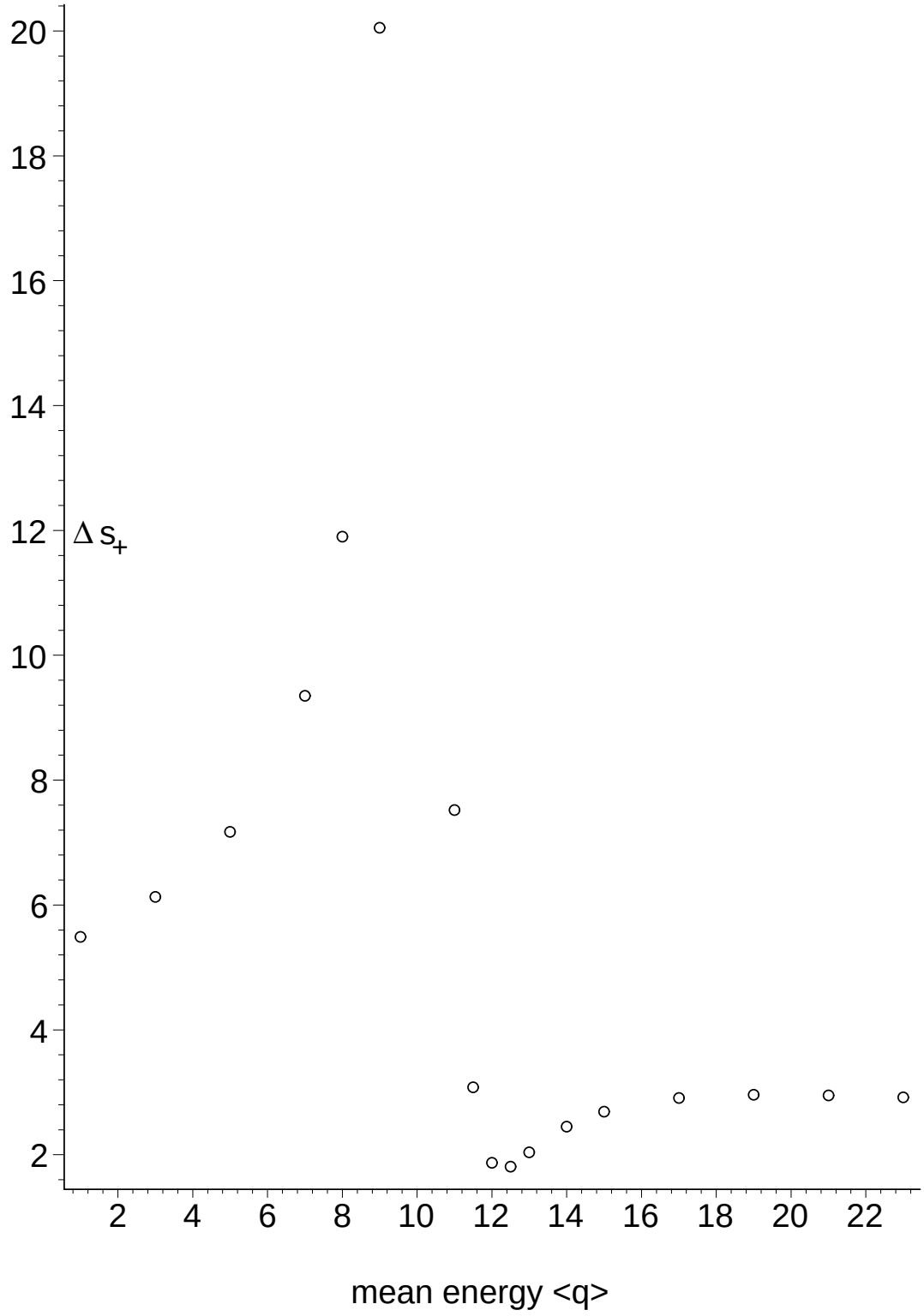

Figure 8.3: *The spread of the proper time measured by the observer after the scattering Δs_+ is plotted as a function of the mean energy $\langle q \rangle$, where the energy width of the quasi-polynomial packets remains constant at $\Delta q = 0.30$. If the wave packets contain the Schwarzschild energy $q_{Schw} = 10$, which is true for $\langle q \rangle \in (9, 11)$, then the spread diverges.*

mean energy is constant and equal to the Schwarzschild energy, $\langle q \rangle = q_{\text{Schw}} = 10$. The width of the packets is varied by choosing different values for $q_b - q_a$. The spread Δs_+ is infinite for each of these packets. Δs_- is of course equal to Δv, as always.

q_a	q_b	Δq	$\langle t_s \rangle$ $[t_{\text{Pl}}]$	Δs_- $[t_{\text{Pl}}]$
0	20	3.02	4194.03	0.17
4	16	1.81	4214.44	0.29
8	12	0.60	4258.22	0.87
9	11	0.30	4285.81	1.73
9.9	10.1	0.03	4377.45	17.32

Table 8.2: *Numerical results for the sojourn time $\langle t_s \rangle$ are written down in units of the Planck time for packets with constant mean energy $\langle q \rangle = 10$ and varying energy width.*

As the packets become more and more narrow in energy space, the sojourn time increases. This is because the contribution from the singular energy q_{Schw} becomes more and more important. The diagram fig. (8.4) illustrates the dependence of $\langle t_s \rangle$ on the width of the packets. For a packet with mean energy $\langle q \rangle = 10$ and zero energy width ($\Delta q = 0$), the sojourn time diverges, corresponding to the classical scattering time for a shell with mass $M = 10 m_{\text{Pl}}$.

The following table (8.3) displays the results for packets with constant mean energy $\langle M \rangle = 5 m_{\text{Pl}}$ that contain only states with energies below the Schwarzschild energy. In this case the spread of the proper time after the scattering is finite.

q_a	q_b	Δq	$\langle t_s \rangle$ $[t_{\text{Pl}}]$	Δs_- $[t_{\text{Pl}}]$	Δs_+ $[t_{\text{Pl}}]$
0	10	1.51	4058.69	0.35	36.47
4	6	0.30	4055.80	1.73	7.17
4.9	5.1	0.03	4055.69	17.32	17.34

Table 8.3: *Numerical results for the sojourn time and proper time spreads for packets with mean energy $\langle q \rangle = 5$ and varying energy width are written down.*

The sojourn time slightly decreases as the packets become narrower, because the contribution from the energies near the Schwarzschild energy becomes less important. The spread Δs_- of the proper time before the scattering is proportional to the inverse of the spread of the packet in energy space, as expected. Δs_+, on the other hand, displays a less simple behaviour that is shown in fig. (8.5).

The last table (8.4) shows the results for packets of variable width and constant mean energy $\langle M \rangle = 15 m_{\text{Pl}}$ that consist entirely of states with energies that lie above the Schwarzschild energy.

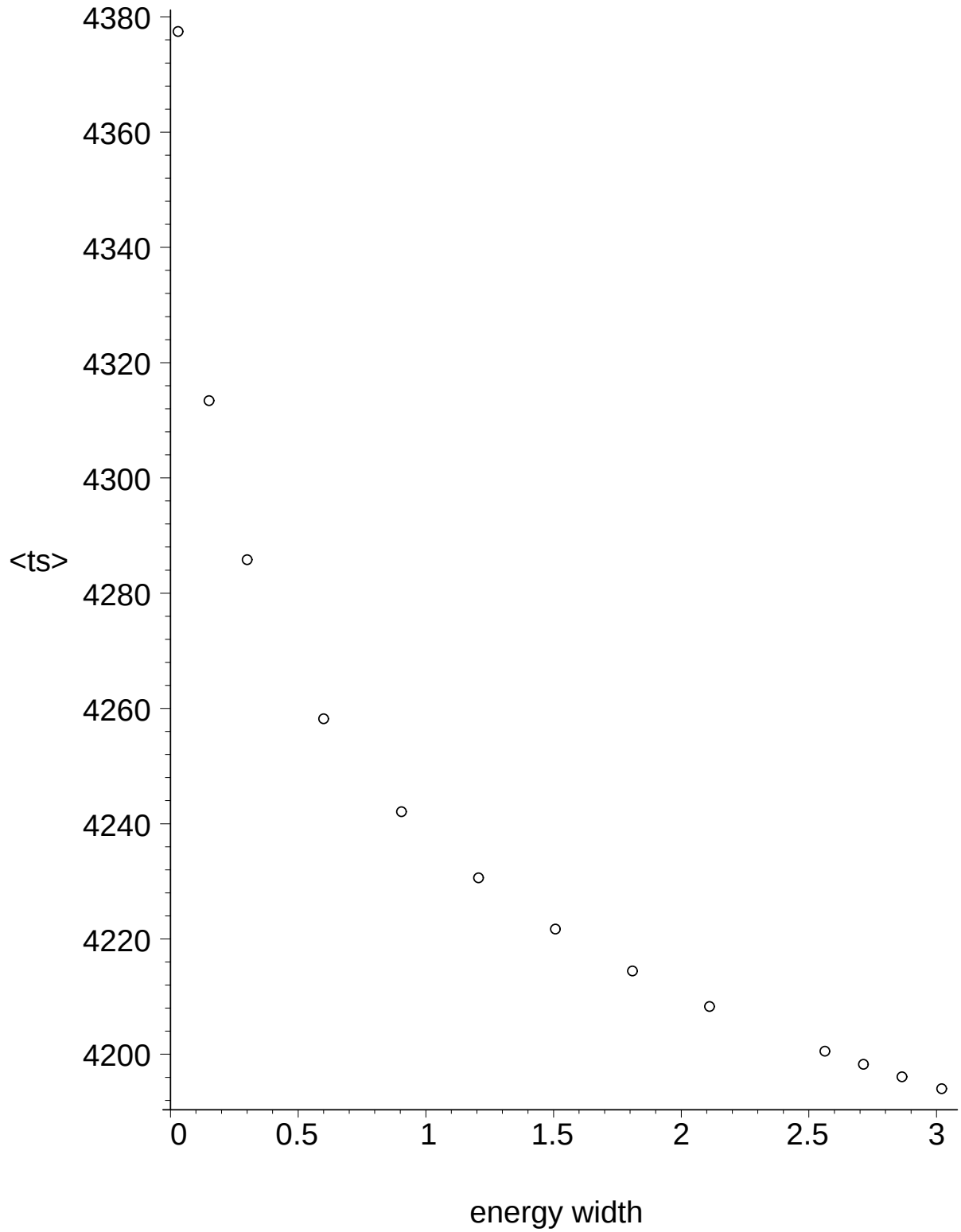

Figure 8.4: *The sojourn time $\langle t_s \rangle$ is plotted as a function of the energy width Δq of the wave packets, while the expected energy remains constant at $\langle q \rangle = 10$.*

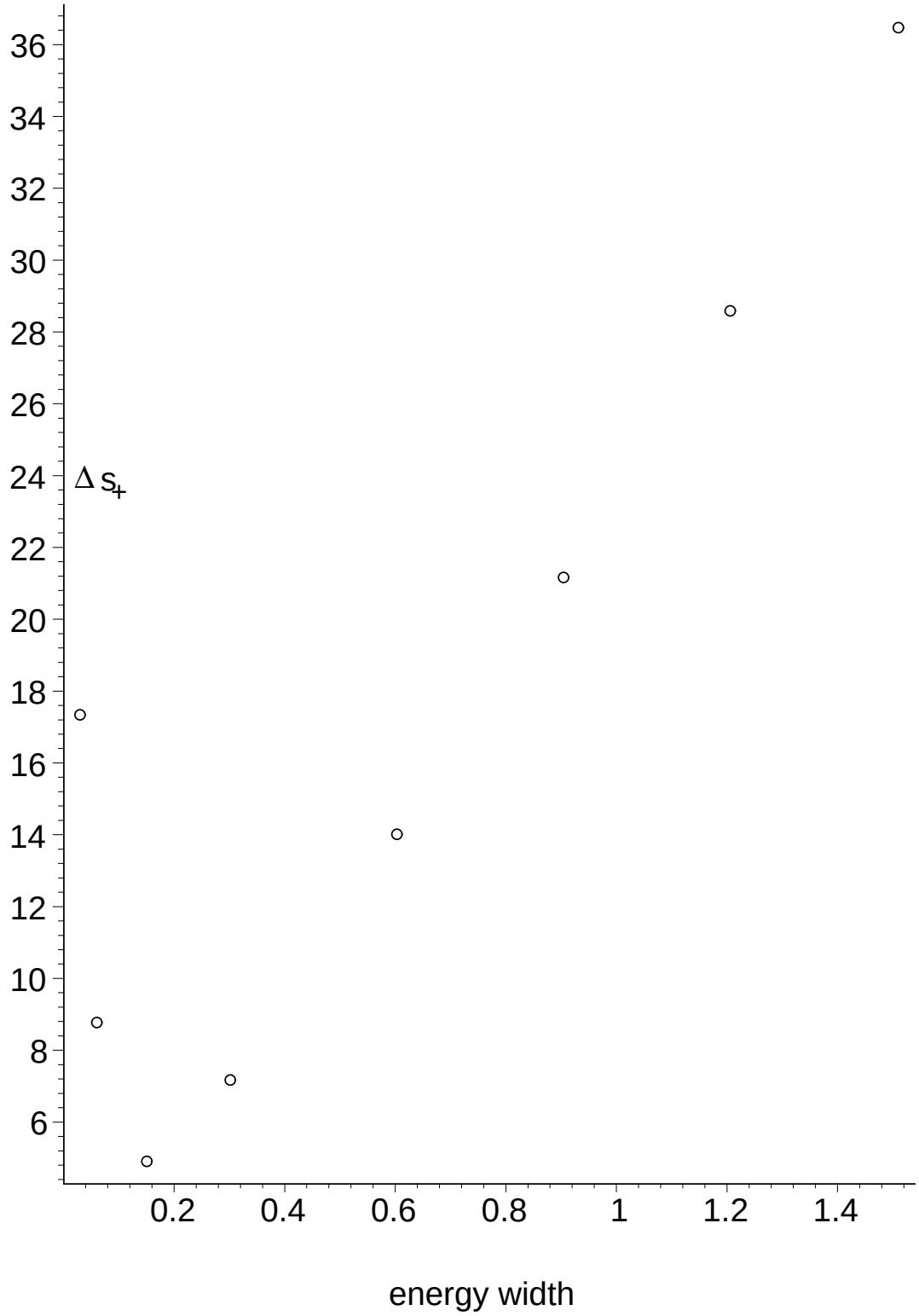

Figure 8.5: *The spread of the proper time measured by the observer after the scattering Δs_+ is plotted as a function of the energy width Δq of the wave packets, while the expected energy remains constant at $\langle q \rangle = 5$.*

q_a	q_b	Δq	$\langle t_s \rangle$ $[t_{\mathrm{Pl}}]$	Δs_- $[t_{\mathrm{Pl}}]$	Δs_+ $[t_{\mathrm{Pl}}]$
10	20	1.51	4245.91	0.35	9.58
14	16	0.30	4244.83	1.73	2.69
14.9	15.1	0.03	4244.79	17.32	17.33

Table 8.4: *The table shows numerical results for the sojourn time and the spreads of the proper time before and after the scattering for packets with mean energy* $\langle q \rangle = 15$ *and varying energy width.*

The sojourn time decreases very slightly as the packets become more and more narrow. The spread Δs_+ shows the same behaviour as in the case of the low-energy packets.

8.8 Summary and discussion

The most important observations that we have made in this chapter are the following: The *sojourn time* $\langle t_s \rangle$ is *always finite*, even if the wave packet consists of energies for which the corresponding classical scattering time, eq. (8.1), diverges. The sojourn time also remains finite beyond the Schwarzschild energy and has a local maximum at E_{Schw}. The spread of the proper time of the incoming packets Δs_- is equal to the spread of $\hat{v}$, and is therefore proportional to the inverse of ΔE. It is thus always finite. The spread of the outgoing packets, Δs_+, however displays a non-trivial dependence on the form of the wave packets. It diverges if the packets contain the Schwarzschild energy $E_{\mathrm{Schw}} = \frac{R_M}{2}$, otherwise it is finite. Thus, even if parts of a wave packet fall under their Schwarzschild horizon, the sojourn time remains finite, but the spread of the proper time measured by the observer after the scattering diverges. If, however, the entire packet crosses the horizon, also the spread becomes finite.

There is an interesting analogy to the theory of quantum scattering of a particle on a potential. The (finite) peak of the sojourn time at the Schwarzschild energy looks like a *resonance*. The peak is more sharply defined if the packet is narrower in energy space and becomes less visible as the packets widen. This behaviour is illustrated in the diagram fig. (8.6). We can thus say that the sojourn time of a null shell scattered by a mirror has a *resonance at the Schwarzschild energy* for which the classical scattering time diverges. E_{Schw} has been introduced as an energy cut-off in the classical system. In the corresponding quantum theory it becomes a resonance.

In the introduction of this thesis we have stated the hypothesis that the scattering times could be long enough that the black hole has enough time to evaporate almost completely by emitting Hawking radiation, such that only a small remnant of the initial system remains. The computations we have done in the previous subsection, however, seem to imply that the sojourn time is much too short to allow for that scenario. We want to elucidate this statement by a (although highly improbable) example. Consider a null

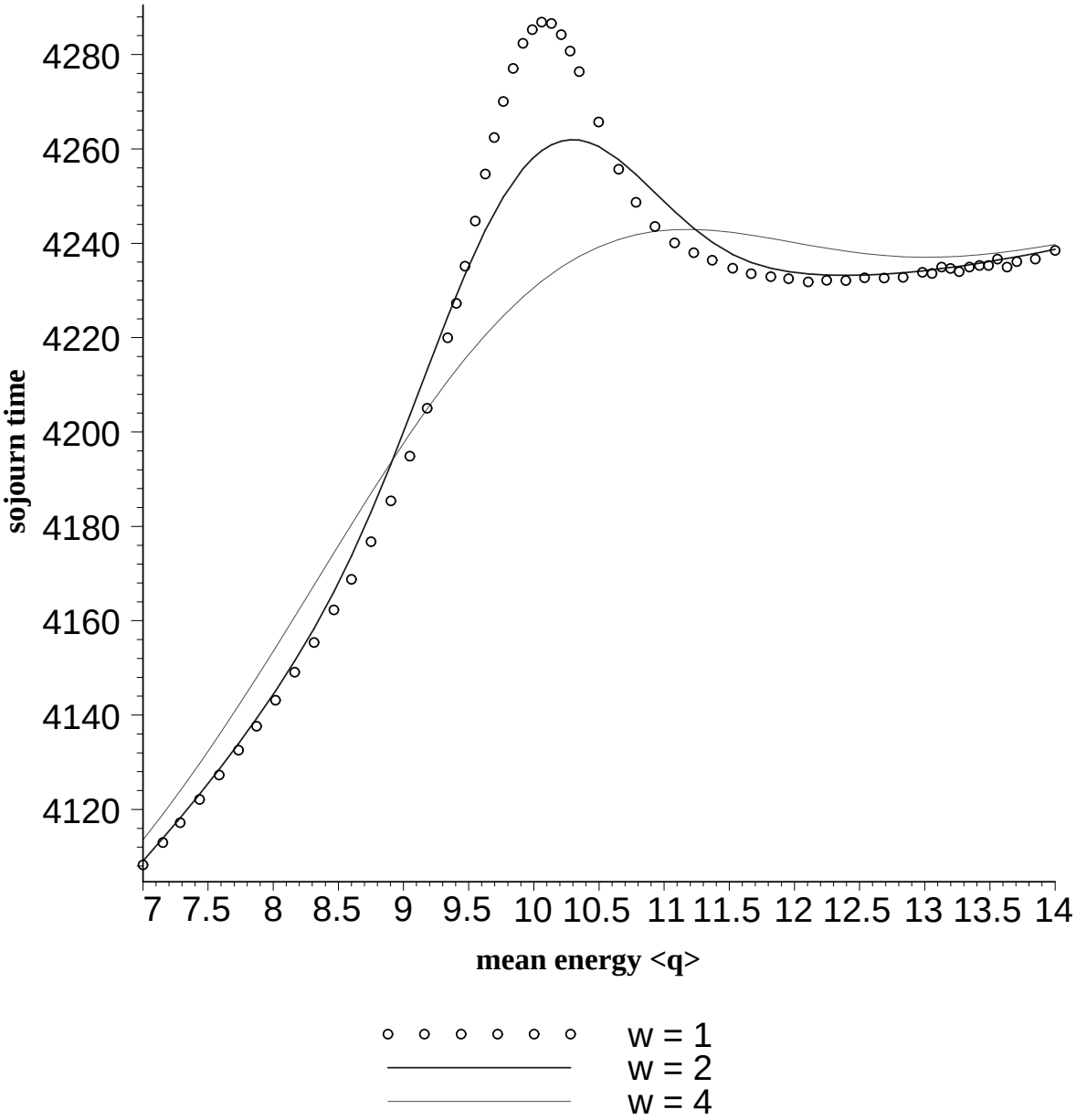

Figure 8.6: *The sojourn time $\langle t_s \rangle$ is plotted as a function of the expected energy $\langle q \rangle$ for three wave packets of different energy widths Δq. The distinctiveness of the resonance at the Schwarzschild energy at $\langle q \rangle = 10$ dependent on the width of the packets is well visible. The curves have been plotted from $\langle q \rangle = 7$ to $\langle q \rangle = 14$. The curve with the sharpest peak comes from the packets with the parameter $w = 1$, where w and Δq are related by $w = \sqrt{11}\Delta q$. The two other curves are for broader wave packets with $w = 2$ and $w = 4$, respectively. The three packets have the energy widths $\Delta q = 0.30$, $\Delta q = 0.60$ and $\Delta q = 1.21$.*

shell with one solar mass that is scattered by a mirror in a neighbouring galaxy one Mpc (about 3.3×10^6 light years) away. The sojourn time measured by the observer residing in our galaxy is then about 6.6×10^6 years. But this is roughly the time light needs to cover the way from the observer to the mirror and back. Thus, there is no time at all for black hole radiation, and the hypothesis above is not satisfied in our model. There seem to be two possible reasons for how that comes about. The first one is that the corresponding theory for a star looks significantly different from that for a null shell, such that our construction does not apply for astrophysical situations. The second reason may be that our calculations have been too optimistic. We have defined the sojourn time $\langle t_s \rangle$ simply by taking the expectation value of an operator that has been obtained by the replacement of the Dirac observables in the scattering time t_s by the corresponding operators. We then have computed the thus defined sojourn time in energy regions (beyond the Schwarzschild energy) for which its classical counterpart is not even defined and have found, that it is still finite there, which is our justification for using it. But it is not clear if such an extrapolation is the right method. Maybe a different way to define the scattering times will prove to be more accurate, e.g. the perturbative approach that is the subject of the next chapter.

A further problem in our definition of the sojourn time has not been mentioned before. Since the operator $\hat{t}_s$ does not depend on the advanced time $\hat{v}$, it does not feel the spatial extension of the wave packets. For example, if an eigenstate of energy (the energy being different from the Schwarzschild energy) is taken instead of a packet, then the sojourn time is finite according to our formula. But the shell state is time independent and the times of intersection of the shell with the observer are not well defined. We have tried to solve this issue by introducing the two proper times $s_\pm$ measured by the observer at the instants he encounters the shell. But still, the sojourn time itself did not depend on the width of the packet in position space. This is a second hint that our method is not perfect yet. Our construction is thus to be looked at as a first attempt to compute the duration of the collapse-reemergence scenario.

Chapter 9

The perturbative approach

9.1 Introduction

This chapter is devoted to an alternative approach to quantize the shell-mirror model. In contrast to the treatment in the preceding chapters this approach is perturbative. It uses the fact that if the gravitational fields are very weak, then they can be considered as small disturbances on a fixed, classical background such that one can make use of semi-classical methods. For systems where an infinite-dimensional gauge group is present, as it is case for our model, the methods worked out by DeWitt [DeW63] seem to be the most promising ones.

Before we turn our attention to the perturbative approach we briefly repeat our motivations to choose our system. We have simplified the collapse model of the single self-gravitating null shell to a simple scattering system that avoids the singularity by introducing a perfectly reflecting spherical mirror at the fixed radius R_M. In order that the shell does not fall under its Schwarzschild horizon the energy cut-off $E < \frac{R_M}{2}$ is imposed. The shell is thus always reflected by the mirror, and by this in- and outgoing shell trajectories are glued together in a unique way. In the last chapter, we then have relaxed the condition that the shell energy be smaller than the Schwarzschild energy in the quantum theory and obtained results, that could not be expected from the classical model.

A second simplification that is necessary for a semi-classical treatment is provided by a more severe restriction of the shell's energy E to very small values, i.e. $E \ll \frac{R_M}{2}$, such that the metric of the spacetime can be interpreted as a small disturbance on a flat metric in suitable coordinates. We have found such coordinates, so we are in the fortunate situation to make use of DeWitt's [DeW63] method of quantizing, in a covariant way, a system that has an infinite-dimensional transformation group.

In the last chapter we have explicitly computed the sojourn time and obtained quite surprising results. The sojourn time has been defined by replacing the classical scattering time t_s by the operator $\hat{t}_s$ and evaluating it in specific wave packets. At the end of that

chapter we have discussed that, although interesting, the results are not yet very satisfying and that it is not clear whether our construction can be really taken seriously. DeWitt's method thus could help us to find out if our approach is correct, or to show us what went wrong, at least in the low-energy regime. One of the problems mentioned in the last chapter is that the sojourn time does not depend on the width of the wave packets in position space, which makes our theory not very plausible if the packets are not well localized in space. We therfore hope that the perturbative approach presented in this section will once lead us to a quantity that can be interpreted as the scattering time of a quantum system and also depends on the shape of the wave packets in position space.

Our findings in this chapter are also interesting simply because of the fact that our model is an example where DeWitt's method can be explicitly applied. We have, however, not done explicit calculations of the sojourn time in the perturbative approach yet.

The following section is devoted to a very short summary of DeWitt's methods. It is followed by a section where we give the necessary conditions on a pair of metrics to be usable for DeWitt's approximation and state a theorem that there exists a coordinate system such that the conditions are satisfied. Then, in section 4, it is specified what is needed from these methods for the shell-mirror model. That section is followed by a discussion of the classical dynamics of the fields in the chosen coordinates. The next section (6) is devoted to the two definitions of the scattering time already encountered in chapter 4. Then, infinitesimal gauge transformations are introduced (section 7) and it is shown how these affect the fields and the scattering time (section 8).

9.2 A brief summary of DeWitt's method

DeWitt's method [DeW63] provides a manifestly covariant quantum theory for systems that can be considered a small disturbance on a classical background and have an infinite-dimensional transformation group. In this very short summary of the method we adopt the terminology and the condensed notation from the aforementioned article and from a classic paper by the same author, [DeW67]. Since the use of the method is not restricted to the gravitational field alone, the following notation is very general, holding for a large class of physical systems.

The most fundamental assumption is that every isolated dynamical system is described by an *action functional*

$$S = \int dx L[\phi^i, \phi^i_{,\mu}], \quad dx \doteq \prod_\mu dx^\mu. \tag{9.1}$$

L is the Lagrangian and depends on a set of real dynamical (field) variables $\phi^i(x^\mu)$ and

their first derivatives. We adopt DeWitt's notation for functional derivatives:

$$\frac{\delta S}{\delta \phi^i} \doteq S_{,i}. \tag{9.2}$$

The dynamical equations of the system are determined by the *stationary action principle* which states that the admissible values for the fields are the solutions of the equation

$$S_{,i} = 0. \tag{9.3}$$

Each distinct solution corresponds to a classical dynamical history. Let ϕ^i be such a solution and $\phi^i + \delta\phi^i$ a small (infinitesimal) perturbation thereof which also obeys the dynamical equations. Then

$$0 = S_{,i}[\phi + \delta\phi] = S_{,i}[\phi] + S_{,ij}[\phi]\delta\phi^j \;\Rightarrow\; S_{,ij}\delta\phi^j = 0, \tag{9.4}$$

which is the *equation of small disturbances*, since $\delta\phi^i$ can be interpreted as a small disturbance in the system that *propagates* on the *background field* ϕ^i. The second functional derivative of the action, $S_{,ij}$ plays a fundamental role, insofar as it is the differential operator governing the propagation of infinitesimal disturbances on an arbitrary background field.

The invariance group of the gravitational field is infinite-dimensional and non-Abelian. The change produced in the field by an infinitesimal group transformation is expressed in the form

$$\delta\phi^i = \int R^i_{\alpha'}\delta\xi^{\alpha'}dx' \doteq R^i_\alpha\delta\xi^\alpha, \quad f^{\alpha'} \doteq f^\alpha(x'), \tag{9.5}$$

where the $R^i_{\alpha'}$ are linear combinations of the delta function and its derivatives and where the $\delta\xi^{\alpha'}$ are arbitrary infinitesimal functions of x^μ, known as *group parameters*. The domain inside which the group parameters are non-zero is finite but otherwise arbitrary.

In this section lower case Latin letters are field indices, whereas Greek letters from the beginning of the alphabet are group indices. Letters form the middle $(\mu, \nu, ...)$ of the Greek alphabet are coordinate indices. In the case of general relativity this distinction is a bit confused because coordinate, group and field $(i = (\mu\nu))$ indices are all the same.

A functional A of the fields is regarded as a *physical observable* if it is a group invariant, the condition for this property to hold being

$$A_{,i}R^i_\alpha = 0. \tag{9.6}$$

In particular, the action functional is a group invariant: $S_{,i}R^i_\alpha = 0$. From the latter identity follows that the field equations are replaced by linear combinations of themselves under a group transformation and hence that solutions go into solutions:

$$\delta S_{,i} = S_{,ij}\delta\phi^j = S_{,ij}R^j_\alpha\delta\xi^\alpha = -S_{,j}R^j_{\alpha,i}\delta\xi^\alpha. \tag{9.7}$$

The right hand side of this equation is equal to 0 if the field equations are satisfied; so the operator $S_{,ij}$ is singular in this case and does not possess a unique inverse. This has the consequence that the solutions of the equation of small disturbances are not uniquely determined, but only modulo transformations of the form

$$\delta\phi'^i = \delta\phi^i + R^i_\alpha\delta\xi^\alpha. \tag{9.8}$$

It is convenient to restrict the $\delta\phi^i$ by imposing further conditions known as *supplementary conditions*:

$$R_{i\alpha}\delta\phi^i = 0, \quad R_{i\alpha} \doteq \gamma_{ij}R^j_\alpha, \tag{9.9}$$

where γ_{ij} is an arbitrary matrix used for lowering field indices $i, j, ...$ except for the single requirement that it be such that the operator

$$\hat{F}_{\alpha\beta} \doteq R_{i\alpha}R^i_\alpha \tag{9.10}$$

be nonsingular and have unique advanced and retarded Green's functions $\hat{G}^{\pm\alpha\beta}$ satisfying

$$\hat{F}_{\alpha\gamma}\hat{G}^{\pm\gamma\beta} = -\delta^\beta_\alpha \tag{9.11}$$

and

$$\lim_{\alpha\to\pm\infty}\hat{G}^{\pm\alpha\beta} = \lim_{\beta\to\mp\infty}\hat{G}^{\pm\alpha\beta} = 0, \tag{9.12}$$

where the limit of the index is to be understood as the limit of the point x^μ that is represented by the corresponding index. The supplementary conditions may be made to hold by carrying out a transformation of the form (9.8), with

$$\delta\xi^\alpha = \hat{G}^{\pm\alpha\beta}R_{i\beta}\delta\phi^i, \tag{9.13}$$

which will make the $\delta\phi^i$ unique.

For the purpose of lowering group indices $\alpha, \beta, ...$ another matrix $\tilde{\gamma}_{\alpha\beta}$ is introduced. It is arbitrary except that it is required to be nonsingular and to possess a unique inverse that may be used to raise group indices. Then the matrix

$$F_{ij} \doteq S_{,ij} + R_{i\alpha}\tilde{\gamma}^{-1\alpha\beta}R_{j\beta} \tag{9.14}$$

is nonsingular. When the supplementary conditions are satisfied, $S_{,ij}$ may be replaced by F_{ij} in the equation of small disturbances, eq. (9.4), yielding

$$F_{ij}\delta\phi^j = 0. \tag{9.15}$$

The advanced and retarded Green's functions $G^{\pm ij}$ of the operator F_{ij} satisfy

$$F_{ik}G^{\pm kj} = -\delta_i^j. \tag{9.16}$$

The *commutator function*

$$\tilde{G}^{ij} = G^{+ij} - G^{-ij} \tag{9.17}$$

is a necessary ingredient in the Poisson bracket of two observables,

$$(A, B) = A_{,i}\tilde{G}^{ij}B_{,j}, \tag{9.18}$$

its definition being due to Peierls. The corresponding quantum form reads

$$[\hat{A}, \hat{B}] = i\hat{A}_{,i}\hat{\tilde{G}}^{ij}\hat{B}_{,j}. \tag{9.19}$$

In this and all following equations involving quantum operators a caret on a symbol denotes that the quantity represented by this symbol is a quantum operator.

A useful and important class of invariants can be built out of the *asymptotic fields* defined by

$$\phi^{\pm i} \doteq \phi^i - G_0^{\pm ij}(S_{,j} - S^0_{,jk}\phi^k). \tag{9.20}$$

This definition is based on the formal (WKB-) expansion of the action,

$$S = \frac{1}{2}S^0_{,ij}\phi^i\phi^j + \frac{1}{6}S^0_{,ijk}\phi^i\phi^j\phi^k + ... \tag{9.21}$$

An index 0, in either the upper or lower position, indicates that that quantity to which it is affixed is evaluated at the zero point $\phi^i = 0$, which corresponds to flat empty spacetime in the case of gravity. Notice that the zero point is a solution of the field equations, hence linear terms in the expansion above vanish. The asymptotic fields satisfy the equation

$$S^0_{,ij}\phi^{\pm j} = 0 \tag{9.22}$$

which is independent of the choice of the matrices γ_{ij} and $\tilde{\gamma}_{\alpha\beta}$. A change thereof produces only a gauge transformation in the asymptotic fields having the form

$$\delta\phi^{\pm i} = R^i_{0\alpha}\delta\zeta^{\pm\alpha}. \tag{9.23}$$

The same change is produced by a group transformation with the special form

$$\delta\zeta^{\pm\alpha} = -\hat{G}_0^{\pm\alpha\beta}R^0_{i\beta}R^i_\gamma\delta\xi^\gamma, \tag{9.24}$$

hence also

$$S^0_{,ij}(\phi^{\pm j} + \delta\phi^{\pm j}) = 0 \tag{9.25}$$

holds. Because of this one can construct many group invariants, so-called *asymptotic invariants* out of the asymptotic fields by introducing a set of field-independent quantities $I_{Ai}, I_{Bi}, ...$ which obey $I_{Ai}R^i_{0\alpha} = 0, I_{Bi}R^i_{0\alpha} = 0, ...$ The quantities defined by

$$A^\pm \doteq I_{Ai}\phi^{\pm i},\ B^\pm \doteq I_{Bi}\phi^{\pm i},\ ... \tag{9.26}$$

are then asymptotic invariants. This construction works only when the field equations are satisfied, therefore the latter quantities are *conditional invariants*, fulfilling the slightly relaxed group invariance condition

$$A_{,i}R^i_\alpha = S_{,i}a^i_\alpha. \tag{9.27}$$

Since asymptotic invariants are observables, they have a well-defined Poisson bracket, given by

$$(A^\pm, B^\pm) = I_{Ai}\tilde{G}_0^{ij}I_{Bj}. \tag{9.28}$$

The asymptotic fields $\phi^{\pm i}$ are the solutions of the *linearized theory* obtained from the action functional $\frac{1}{2}S^0_{,ij}\phi^{\pm i}\phi^{\pm j}$. This theory is well-understood; the choice of the most suitable invariants for the gravitational field, their quantization and applications to scattering processes are explicitly given in [DeW67].

9.3 Existence theorem for a suitable gauge

In order to be able to use DeWitt's method that we have presented in the preceding section, we first need a suitable coordinate system with respect to which the shell metric $g_{\mu\nu}$ differs only slightly from a flat (but otherwise arbitrary) metric $f_{\mu\nu}$. By a flat metric we mean one that is obtained by a regular coordinate transformation from the Minkowski

metric $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$. Since there are many flat metrics on $\mathcal{M}$ in any given coordinate system, it is not straightforward to tell whether there exists one, that is close enough to the shell metric in the given coordinates. Also the notion of distance between two metrics needs to be properly defined. These issues are discussed in more detail in the appendix B.

Motivated by the results found in the aforementioned appendix, we impose the following conditions on the pair of metrics $g_{\mu\nu}$ and $f_{\mu\nu}$, where the metric $f_{\mu\nu}$ is flat, while $g_{\mu\nu}$ describes the spacetime of the shell:

- The metrics $g_{\mu\nu}$ and $f_{\mu\nu}$ are regular (i.e. $|\det(g_{\mu\nu}(x))| > 0, \forall x$) and continuous everywhere on the spacetime manifold $\mathcal{M}$,
- The shell metric $g_{\mu\nu}$ can be expanded about the small parameter $s = \frac{2E}{R_M}$, such that $g_{\mu\nu}(s) \approx f_{\mu\nu} + \mathcal{O}(s)$, and the limit $g_{\mu\nu}(s) \to f_{\mu\nu}$ as $s \to 0$ exists.

These conditions strongly restrict the possible coordinates. The first condition ensures that the metrics are regular and continuous at the shell. It excludes, e.g., the singular DNEF coordinates (5.1, 5.8). The second one is the most important one. It ensures that the two metrics differ not too much from each other and that they agree if s (i.e. E) is zero.

Both the regular DNEF (5.45, 5.46) and the CR (cf. section 4 of chapter 6) coordinates do not satisfy the second condition. This is because the $\theta\theta$-components, $-R^2$, of both these metrics diverge at infinity. It it thus very improbable that one finds a suitable flat metric that has a corresponding $\theta\theta$-component that can compensate the divergence of $-R^2$ at infinity. Polar coordinates seem hence not to be suited for DeWitt's method. This issue is discussed in more detail in section 2 of the appendix B.

The *Cartesian form* (6.74, 6.75) of the *central-regular (CR) metric* found in chapter 6 satisfies, however, all of the conditions above. We therefore choose as $g_{\mu\nu}$ the Cartesian CR metric (CCR) and as $f_{\mu\nu}$ the Minkowski metric $\eta_{\mu\nu}$ and thus prove the following existence theorem:

Theorem 9.1 *There exists a coordinate system with respect to which the metric of the shell-mirror spacetime satisfies the conditions (9.3) that are required in order that DeWitt's methods can be applied.*

The explicit proof is a bit lengthy and is therefore written down in the appendix C.

9.4 What is needed from DeWitt's method for the shell-mirror model

Our system consists of the null spherical thin shell, its gravitational field and the mirror, as defined earlier in chapter 3. Our choice for the metric of the spacetime is the Cartesian

central-regular (CCR) metric (C.2, C.3) that has been constructed chapter 6 and the appendix C. In the present case, the background metric is simply the Cartesian Minkowski metric $\eta_{\mu\nu}$. The metric field propagating on this background is the difference of the CCR and the Minkowski metrics,

$$h_{\mu\nu} \doteq g_{\mu\nu} - \eta_{\mu\nu}, \tag{9.29}$$

and will be explicitly written down in the next paragraph. Since $h_{\mu\nu}$ is a small perturbation, $|h_{\mu\nu}| \ll 1$, one can ignore anything that is of higher than first order in this quantity. Therefore, one can use $\eta_{\mu\nu}$ instead of $g_{\mu\nu}$ to raise or lower indices, because the corrections are of higher order in the perturbation.

The trajectory of the shell is represented by a curve on the background described by a set of functions $X^A(\lambda)$ depending on the (arbitrary) parameter λ. Both the metric and the trajectory of the shell depend on the solution-parameters (E, v) of Einstein's equations:

$$h_{\mu\nu} = h_{\mu\nu}(E, v; x^\mu), \quad X^A(\lambda) = X^A(E, v; \lambda), \tag{9.30}$$

where x^μ is the coordinate system for which the metric of the shell spacetime obtains the Cartesian form, i.e. $(\tau, \vec{X})$.

The requirement that $h_{\mu\nu}$ be a small perturbation of the Minkowski metric does not completely specify the coordinates, because there may be other coordinate systems in which the metric can still be written as η plus a small perturbation, but this perturbation will be, of course, different. Thus, the decomposition (9.29) is not unique. So we must allow for small changes of the coordinate system such that the decomposition (9.29) still holds. Such infinitesimal gauge (i.e. coordinate) transformations are given by

$$\delta\xi^\mu = \xi^\mu d\epsilon, \tag{9.31}$$

the arbitrary vector field $\xi(x)$ being the generator and $d\epsilon$ being an infinitesimally small but positive number. The vector field $\xi(x)$ vanishes outside a finite but otherwise arbitrary region of spacetime. This requirement will be, however, replaced by a less restrictive one in section 7. The explicit form of the effects of the gauge transformations on the fields will be also stated later, namely in the last section. Taking the infinitesimal gauge transformations into account, the quantities that describe the model read:

$$h_{\mu\nu} = h_{\mu\nu}(E, v; \xi(x); x), \quad X^A(\lambda) = X^A(E, v; \xi(x); \lambda). \tag{9.32}$$

In the language of the previous paragraph, the fields $h_{\mu\nu}, X^A$ are *asymptotic fields*, the quantities E, v are the *physical observables* that will be used for quantization, and the *group transformations* are represented by the generators ξ^μ. Nothing else will be needed from DeWitt's method, because the classical solutions are already known.

9.5 Classical dynamics in the CCR gauge

In the Cartesian central-regular (CCR) coordinates $x^\mu = (\tau, \vec{X})$, the metric disturbance $h_{\mu\nu}$ reads

$$\begin{aligned} h_{00} &= A - 1, \\ h_{0k} &= 0, \quad k = 1, 2, 3, \\ h_{kk} &= 1 - \frac{R^2}{X^2} + \left(\frac{R^2}{X^2} - A\right)\frac{X_k^2}{X^2}, \\ h_{kl} &= \left(\frac{R^2}{X^2} - A\right)\frac{X_k X_l}{X^2}, \quad k \neq l, \end{aligned} \tag{9.33}$$

where the metric functions $R = R(\tau, X)$, $A = A(\tau, X)$ have been written down in section 6 of chapter 6 and in the appendix C. The above equations are only valid in the parts of the spacetime, where the CR coordinates are double-null. The form of the disturbance in the wedges $K_\pm$ is, however, straightforwardly computed by using the components of the corresponding shell metric. The components of $h_{\mu\nu}$ depend on the solution parameters (E, v) in virtue of their occurrence in the metric functions R and A.

The equation for the trajectory of the shell with respect to the DNEF coordinates (U, V) is given by $V = v$, $U = u$ for the in- and outgoing shells, respectively. From the coordinate transformation $V = \tau + X$, $U = \tau - X$ follows that

$$\tau(\lambda) + X(\lambda) = v, \quad \tau(\lambda) - X(\lambda) = u. \tag{9.34}$$

At the mirror the two trajectories meet in the point $V = v, U = u = v - 2R_M^*$. Hence, expressed in the coordinates (τ, X),

$$\tau(\lambda_0) + X(\lambda_0) = v, \quad \tau(\lambda_0) - X(\lambda_0) = v - 2R_M^*, \tag{9.35}$$

and, therefore,

$$X(\lambda_0) = R_M^*, \quad \tau(\lambda_0) = v - R_M^*. \tag{9.36}$$

This suggests the following ansatz for the shell's trajectory $X^A(\lambda) \doteq (\tau(\lambda), X(\lambda))$:

$$\begin{aligned} \tau(\lambda) &= \lambda - \lambda_0 + v - R_M^*, \\ X(\lambda) &= |\lambda - \lambda_0| + R_M^*, \quad \lambda \in \mathbb{R}. \end{aligned} \tag{9.37}$$

The ansatz obviously satisfies eqs. (9.34) for all λ. By putting $\lambda_0 = 0$, the equations reduce to

$$\begin{aligned} \tau(\lambda) &= \lambda + v - R^*_M, \\ X(\lambda) &= |\lambda| + R^*_M, \quad \lambda \in \mathbb{R}. \end{aligned} \tag{9.38}$$

The sign of λ changes at the mirror from -1 for the ingoing to $+1$ for the outgoing shell. The trajectory of the shell depends on the solution parameters v and, through R^*_M, the shell energy E:

$$X^A(\lambda) = X^A(E, v; \lambda). \tag{9.39}$$

Eqs. (9.38) are the same equations as those that have been obtained in chapter 3, but now they are expressed in the coordinates τ and X.

9.6 Scattering time

9.6.1 Coordinate-based definition

In chapter 4 we have assumed that the operator $\widehat{\Delta s}$ made from the classical proper scattering time Δs given by eq. (3.69) can be identified with the sojourn time. We hence repeat the construction of the proper time measured by an observer at a constant radius ρ between the two times he and the shell meet from the fourth section of chapter 4, using the new coordinates τ and X.

The classical scattering time of the shell under the radius ρ is given by the difference of the two times at which the shell's trajectory is at the radius ρ:

$$\tau_\rho \doteq \tau(\lambda^+_\rho) - \tau(\lambda^-_\rho). \tag{9.40}$$

Care is to be taken when one is talking about radii. Here, by ρ, we are meaning a constant value of the geometric radius defined by the metric, $R(\tau, X) = \rho$. That R is truly the geometric radius can be shown by expressing the metric in the polar coordinates τ, X: $ds^2 = A(d\tau^2 - dX^2) - R^2 d\Omega^2$. The orbits of the rotation group are then given by constant values of the function $R(\tau, X)$. If $\mathcal{S}$ is the surface area of an orbit of the rotation group, then the geometric radius reads

$$r_{\text{geom}} = \sqrt{\frac{\mathcal{S}}{4\pi}} = R(\tau, X). \tag{9.41}$$

It is obvious that this function is an invariant under rotationally symmetric coordinate transformations. The equation $R(\tau, X) = \rho = \textit{constant}$ defines thus a curve in the plane

spanned by the coordinates τ and X. In order to find the explicit form of this curve, one has to solve

$$\rho = R(\tau, X) = const \tag{9.42}$$

for τ (or X) in all of the three regions I, II, III of spacetime. But since the observer residing at the radius ρ is in the region I outside the shell during the measurement of the scattering time, it suffices to find the expression in region I alone. Notice that we have solved this problem already once for the Schwarzschild scattering time in chapter 3. In region I, the solution of eq. (9.42) is given by

$$X = \rho^* \doteq \rho + 2E \ln \left| \frac{\rho}{2E} - 1 \right|, \tag{9.43}$$

$$\tau \in (v - \rho^*, v - 2R_M^* + \rho^*) \doteq (\tau^-, \tau^+). \tag{9.44}$$

The boundary values of the τ-interval follow from the shell's equations (9.35). The equations

$$\tau^- = \tau(\lambda_\rho^-), \quad \tau^+ = \tau(\lambda_\rho^+) \tag{9.45}$$

determine the values of λ looked for:

$$v - \rho^* = \lambda_\rho^- + v - R_M^* \;\Rightarrow\; \lambda_\rho^- = R_M^* - \rho^*, \tag{9.46}$$

$$v - \rho^* - 2R_M^* = \lambda_\rho^+ + v - R_M^* \;\Rightarrow\; \lambda_\rho^+ = -R_M^* + \rho^*. \tag{9.47}$$

Alternatively, one can solve the equation

$$\rho^* = X(\lambda_\rho^\pm), \tag{9.48}$$

which yields the same result. The scattering time is thus given by

$$\tau_\rho = \lambda_\rho^+ - \lambda_\rho^- = 2(\rho^* - R_M^*) = 2(\rho - R_M) + 4E \ln \left| \frac{\rho - 2E}{R_M - 2E} \right|. \tag{9.49}$$

This is the same result as given by eq. (3.67) of chapter 3 indicating that τ is equal to the Schwarzschild time in region I.

The proper time s the observer is measuring is defined by the relation $ds^2 = Ad\tau^2$. Since the metric component A is constant at the constant radius ρ in region I, it holds that $s = \sqrt{A}\tau + const.$ Hence, the difference of two time measurements reads

$$\Delta s = \sqrt{A}\Delta\tau = \sqrt{1 - \frac{2E}{\rho}}\Delta\tau. \tag{9.50}$$

The *proper scattering time* measured by an observer at the constant radius ρ is, therefore,

$$s_\rho = 2\sqrt{1 - \frac{2E}{\rho}}\left(\rho - R_M + 2E\ln\left|\frac{\rho - 2E}{R_M - 2E}\right|\right). \tag{9.51}$$

Notice that this quantity does not depend on the solution-parameter v. We demonstrate later in this chapter that the scattering time defined in this way is not invariant under infinitesimal gauge transformations and, therefore, is not an observable that can be used for quantization.

9.6.2 Gauge-invariant definition

There is a second way to define the residence time of the observer between his two encounters with the shell. The resulting quantity is invariant under infinitesimal gauge transformations as will be shown later in this chapter. The gauge-invariant definition has been written down in section 5 of the fourth chapter. It is the difference t_s of the two proper times $s_\pm$ measured by the ideal clock of an observer at rest at the constant radius ρ. We adopt the construction of t_s from the aforementioned chapter to the CCR coordinates τ and X.

The trajectory of the observer is given by the equation $R(\tau, X) = \rho$. A parametrized curve $\gamma(\mu)$ describing this trajectory in region I reads

$$X(\mu) = \rho^*, \quad \tau(\mu) = \mu, \quad \mu \in (\mu_-, \mu_+). \tag{9.52}$$

The infinitesimal proper time ds measured by the observer can be obtained from the line element of the spacetime in the points of the trajectory,

$$ds = \sqrt{g_{\mu\nu}dx^\mu dx^\nu} = \sqrt{A}\sqrt{d\tau^2 - dX^2}. \tag{9.53}$$

The proper time along the curve γ reads thus

$$t_s = \int_\gamma ds = \int_\gamma \sqrt{A}\sqrt{d\tau^2 - dX^2}. \tag{9.54}$$

From $d\tau = d\mu$, $dX = 0$ follows that the above integral can be written as

$$\int_\gamma ds = \int_{\mu_-}^{\mu_+} d\mu \sqrt{A(X = \rho^*, \tau = \mu)} = \sqrt{1 - \frac{2E}{\rho}} \int_{\mu_-}^{\mu_+} d\mu = \sqrt{1 - \frac{2E}{\rho}} \int_{\tau^-}^{\tau^+} d\tau, \quad (9.55)$$

where the fact that $A = \sqrt{1 - \frac{2E}{\rho}} = constant$ for $R = \rho$ in region I has been used. Obviously the quantity on the right hand side of this equation is the proper scattering time s_ρ defined by eq. (9.51) above. Hence, $t_s(\rho) = s_\rho$, as expected. In section 8 of this chapter we show that the thus defined scattering time is invariant under infinitesimal gauge transformations.

9.7 Infinitesimal gauge transformations

According to DeWitt and the previous subsections, the infinitesimal gauge transformations are given by

$$x^\mu \to x'^\mu = x^\mu - \delta\xi^\mu(x), \quad \delta\xi^\mu(x) \doteq \xi^\mu(x) d\epsilon. \quad (9.56)$$

Since the model is spherically symmetric, it seems to be sensible to restrict the gauge transformations to those that preserve this symmetry. This confines the arbitrariness of the vector fields ξ^μ to rotationally symmetric ones. In coordinates adapted to the spherical symmetry, e.g. (τ, X, θ, ϕ), rotationally symmetric vector fields ξ satisfy

$$\xi^\mu = \xi^\mu(\tau, X) = \left(\xi^0(\tau, X), \xi^X(\tau, X), 0, 0\right). \quad (9.57)$$

In the Cartesian coordinates obtained from the transformation $\tau = \tau$, $x^k = Xn^k$, such vector fields are of the form

$$\xi^\mu_{cart} = (\xi^0, \frac{x^k}{X}\xi^X). \quad (9.58)$$

This restriction will simplify many calculations. Earlier in this chapter we have required the vector field ξ to vanish outside a finite but arbitrary region. We will weaken this requirement when we discuss the asymptotic form of ξ in the subsection after the next.

9.7.1 Order of magnitude of gauge transformations and metric perturbations

When one is comparing the effects of the metric perturbation $h_{\mu\nu}$ and that of infinitesimal gauge transformations given by eq. (9.56), the question arises whether they are of the

same order of magnitude. If this is not the case, then the approximating procedure of linearization may be not self-consistent. In order to show that our methods are self-consistent we will consider the problem in the general case, using DeWitt's notation.

Let ϕ^i be the solution of the variational equation $S_{,i}\phi^i = 0$. Then, a small perturbation of the action $S + \delta S$ will have the solution $\phi^i + \delta\phi^i$. Let this perturbation be due to the coupling to a weak external agent described by a unspecified (the actual form will not be needed) functional A of the fields ϕ^i:

$$\delta S \doteq A d\epsilon. \tag{9.59}$$

The perturbation $\delta\phi^i$ then satisfies the equation of small disturbances

$$S_{,i}[\phi + \delta\phi] \approx S_{,i}[\phi] + S_{,ij}[\phi]\delta\phi^j = S_{,ij}[\phi]\delta\phi^j = -A_{,i}d\epsilon, \tag{9.60}$$

thus the perturbed field is given by

$$\delta\phi^i = -G^{ij}A_{,j}d\epsilon, \tag{9.61}$$

where G^{ij} is the Green's function belonging to the differential operator $S_{,ij}$ or, if necessary, to its invertible extension F_{ij}, respectively. Thus, the perturbed field is of the order of the (infinitesimally) small number $d\epsilon$:

$$\delta\phi^i \propto d\epsilon. \tag{9.62}$$

The disturbances induced by infinitesimal gauge transformations $\phi' = \phi + \delta\phi$ are of the same order of magnitude. This follows clearly from the form of $\delta\phi$,

$$\delta\phi^i = R^i_\alpha \xi^\alpha d\epsilon \propto d\epsilon. \tag{9.63}$$

In the next section we will show that metric perturbations of the flat spacetime can be considered being induced by a diffeomorphism denoted by χ and those induced by gauge transformations by another diffeomorphism, $\psi_{d\epsilon}$. Since the metric perturbation h is of the same order than effects due to gauge transformations, the use of the linearized theory is justified, because the zeroth order in the expansion of the full metric g is the flat Minkowski metric η. First order effects are given by h and the Lie derivative of η along the vector field ξ that are both of the same order of magnitude $d\epsilon$.

9.7.2 Asymptotics of infinitesimal gauge transformations

We have stated earlier in this chapter that the generators ξ of the infinitesimal coordinate transformations must vanish outside an arbitrary compact region of spacetime. This seems

to suffice when considering only the scattering time under a fixed finite radius. But since also quantities that are defined only in the asymptotic region may become important, a trivial action of the gauge group there is undesirable. Therefore, it seems to be necessary to weaken the requirement of compact support. We will hence replace the requirement $\xi^\mu = 0$ in the asymptotic region by the less restrictive condition

$$\xi^\mu(x) = \zeta^\mu = const. \tag{9.64}$$

This choice is motivated by DeWitt's definition of asymptotic invariants and the action of the gauge group thereupon. If an infinitesimal gauge transformation is given by $\delta\phi^i = R^i_\alpha \xi^\alpha d\epsilon$, then its action on asymptotic fields $\phi^{\pm i}$ reads

$$\delta\phi^{\pm i} = R^i_{0\alpha}\zeta^\alpha d\epsilon, \tag{9.65}$$

where ζ is the generator of a member of an infinitesimal Abelian group and $R^i_{0\alpha}$ is the zero-point value of R^i_α, which can be understood as the inhomogeneous part of the general group transformation law given by

$$R^i_\alpha \xi^\alpha d\epsilon = R^i_{0\alpha}\xi^\alpha d\epsilon + R^i_{\alpha,j}\phi^j \xi^\alpha d\epsilon. \tag{9.66}$$

The generators ζ of the asymptotic gauge group must be such that the change induced by them on the metric perturbation remains small in the asymptotic region. Since $h_{\mu\nu} \to 0$ in the asymptotic region, also the disturbance

$$\delta h_{\mu\nu} = (\zeta_{\mu,\nu} + \zeta_{\nu,\mu})d\epsilon \tag{9.67}$$

must vanish. The asymptotic group is also, as stated above, Abelian. Therefore, a good realization of these two requirements seems to be $\zeta^\mu = const$. So, all over the asymptotic region the infinitesimal gauge transformations reduce to rigid infinitesimal translations of the coordinates x^μ and become thus *asymptotic symmetries*. It is easy to show that the vector $\zeta^\mu d\epsilon = const.$ generates an isometry, neglecting corrections of the second order in $d\epsilon$. It has to be pointed out that the above choice is dependent on the choice of coordinates, $\zeta = const$ being the appropriate choice for the Cartesian coordinates (τ, x^k).

The constant vector ζ^μ (still with respect to Cartesian coordinates) can be decomposed into a linear combination of translations along the four coordinate axes:

$$\zeta^\mu = yt^\mu + \sum_{k=1}^{3} \beta_k v^\mu_k, \tag{9.68}$$

where y, β_k are arbitrary infinitesimally small real numbers that do not vanish all at the same time and where

$$t^\mu \doteq (1,0,0,0),\ v^\mu_1 \doteq (0,1,0,0),\ v^\mu_2 \doteq (0,0,1,0),\ v^\mu_3 \doteq (0,0,0,1) \tag{9.69}$$

are the four base vectors. The symmetry can thus be split into a time shift and a spatial translation, both being still infinitesimal. The time shift has a physical effect on the asymptotic retarded (or advanced) time of the shell, as it will be shown explicitly in the next section. The spatial part of the translation, however, does not seem to have any interesting effect on the physical objects other than making the describing spatial coordinates less adapted to the spherical symmetry. Therefore, we will exclude the asymptotic spatial translations in the following and keep only the infinitesimal time shifts; $\delta\zeta^\mu$ is then given by

$$\delta\zeta^\mu \doteq \zeta^\mu d\epsilon = (yd\epsilon, 0, 0, 0). \tag{9.70}$$

Hence, infinitesimal gauge transformations generated by any vector field ξ reduce to *infinitesimal time shifts in the asymptotic region*. On can thus understand such a transformation as the composition of a gauge transformation generated by the vector $\bar{\xi}^\mu$ that converges asymptotically to the zero vector and a constant infinitesimal time shift $\sigma(y)$, whose action on the coordinates reads

$$[\sigma(y)](x^\mu)d\epsilon = (\tau + yd\epsilon, x^k). \tag{9.71}$$

The abstract vector ξ is then given by

$$\xi = \sigma(y) \circ \bar{\xi}. \tag{9.72}$$

For each $y \in \mathbb{R}$ there is thus a whole class of vectors $\bar{\xi}$. We can formulate this result in a more mathematical language: Let G be the group of all (infinitesimal) gauge transformations with generators ξ that converge asymptotically to the Abelian generators ζ. Let N be the normal subgroup of G consisting of all those ξ that converge to the zero vector in the asymptotic region such that the fall-off conditions on the metric are conserved, i.e. of all those ξ for which $\zeta = 0$. Denote such an element by $\bar{\xi}$. Then the factor group of asymptotic symmetries G/N consists of the time shifts $\sigma(y)$ which build a one-dimensional continuous group of translations.

9.8 Effect of the gauge transformations on the fields

We show how the infinitesimal gauge transformations given by eq. (9.56) affect the form of the metric perturbation and the shell trajectory.

9.8.1 Effect on the trajectory of the shell

The effect of the infinitesimal transformation eq. (9.56) on the trajectory of the shell, given by eq. (9.38), is given by

$$X^A(\lambda) \to X'^A(\lambda) = X^A(\lambda) - \delta\xi^A(x), \quad \delta\xi^A(x) = (\delta\xi^0, \delta\xi^X), \tag{9.73}$$

where

$$\delta\xi^X \doteq \sum_{k=1}^{3} \frac{x^k}{X}\delta\xi^k. \tag{9.74}$$

This follows from the transformation law for the radial coordinate X:

$$\begin{aligned} X' &= \sqrt{\sum_{k=1}^{3}(x^k - \delta\xi^k)^2} \\ &= \sqrt{\sum_{k=1}^{3}((x^k)^2 - 2x^k\delta\xi^k + (\delta\xi^k)^2)} \\ &\approx \sqrt{\sum_{k=1}^{3}((x^k)^2 - 2x^k\delta\xi^k)} \\ &\approx X - \sum_{k=1}^{3}\frac{x^k}{X}\delta\xi^k, \end{aligned} \tag{9.75}$$

where quadratic terms in the $\delta\xi^k$ have been neglected.

As mentioned above, the asymptotic time shift symmetry transformations $\sigma(y)d\epsilon$ have an explicit effect on the trajectory of the shell. The disturbance $\delta\xi^0$ is in this case simply given by

$$\delta\xi^0 = yd\epsilon, \tag{9.76}$$

which leads to

$$X'^A(\lambda) = (\lambda + v - R^*_M + yd\epsilon, |\lambda| + R^*_M) \doteq (\lambda + v' - R^*_M, |\lambda| + R^*_M), \tag{9.77}$$

where $v' \doteq v + yd\epsilon$. The trajectory $X'^A(\lambda)$ of the shell with asymptotic retarded time v after the symmetry transformation looks therefore exactly the same as the untransformed trajectory of the shell with the parameter v'. One can thus also understand the asymptotic time shifts as mappings from one solution to another.

9.8.2 Effect on the metric perturbation

We formulate the problem of gauge invariance in a slightly more geometric way, following the treatment in [Car97]. The linearized theory can be thought of as one describing the behaviour of tensor fields on a given flat background. Since one can consider (passive) coordinate transformations between the charts of a manifold as active diffeomorphisms on the manifold, one can introduce a *background spacetime* $\mathcal{M}_B$ equipped with the flat metric $\eta_{\mu\nu}$, a *physical spacetime* $\mathcal{M}_P$ with some metric $g_{\mu\nu}$ satisfying Einstein's equations and a *diffeomorphism*

$$\chi : \mathcal{M}_B \to \mathcal{M}_P. \tag{9.78}$$

The two manifolds are diffeomorphic, thus they are, as manifolds, the same, but equipped with different tensor fields. Since the linearized theory takes place on the flat background spacetime, one is interested in the pull-back $(\chi_* g)_{\mu\nu}$ of the physical metric to the background spacetime $\mathcal{M}_B$. One can thus define the perturbation as the difference between the pulled-back physical metric and the flat one:

$$h_{\mu\nu} = (\chi_* g)_{\mu\nu} - \eta_{\mu\nu}. \tag{9.79}$$

Of course, the diffeomorphism χ is required to be such that $h_{\mu\nu}$ remains small. Then, it follows from the fact that $g_{\mu\nu}$ satisfies Einstein's equations on the physical spacetime that the perturbation obeys the linearized equations on the background. The issue of gauge invariance in this formulation is thus simply the fact that there are many permissible diffeomorphisms between $\mathcal{M}_B$ and $\mathcal{M}_P$. Let such a diffeomorphism be generated by the vector field $\xi^\mu(x)$ from eq. (9.56) and be denoted by $\psi_{d\epsilon}$, $d\epsilon$ being an (infinitesimally) small number. If χ is a permissible diffeomorphism, then so is the composition $(\chi \circ \psi_{d\epsilon})$, although with a different value of the perturbation:

$$h^{(d\epsilon)}_{\mu\nu} = [(\chi \circ \psi_{d\epsilon})_* g]_{\mu\nu} - \eta_{\mu\nu} = [\psi_{d\epsilon *}(\chi_* g)]_{\mu\nu} - \eta_{\mu\nu}. \tag{9.80}$$

Plugging in the decomposition given by eq. (9.79) yields

$$h^{(d\epsilon)}_{\mu\nu} = \psi_{d\epsilon *}(h+\eta)_{\mu\nu} - \eta_{\mu\nu} = \psi_{d\epsilon *}(h_{\mu\nu}) + \psi_{d\epsilon *}(\eta_{\mu\nu}) - \eta_{\mu\nu}. \tag{9.81}$$

Since $d\epsilon$ is small, $\psi_{d\epsilon *}(h_{\mu\nu})$ equals to $h_{\mu\nu}$ in lowest order, while the other two terms define a Lie derivative along the vector field $\xi^\mu(x)$:

$$\begin{aligned} h^{(d\epsilon)}_{\mu\nu} &= \psi_{d\epsilon *}(h_{\mu\nu}) + \left[\frac{\psi_{d\epsilon *}(\eta_{\mu\nu}) - \eta_{\mu\nu}}{d\epsilon}\right] d\epsilon \\ &= h_{\mu\nu} + \mathcal{L}_\xi \eta_{\mu\nu} d\epsilon \\ &= h_{\mu\nu} + (\xi_{\mu,\nu} + \xi_{\nu,\mu}) d\epsilon \\ &\doteq h_{\mu\nu} + \delta h_{\mu\nu}. \end{aligned} \tag{9.82}$$

The equality on the third line follows from the fact that covariant derivatives can be replaced by partial derivatives in approximations to lowest order. Eq. (9.82) tells us what kind of metric perturbations lead to physically equivalent space-times: those related to each other by $(\xi_{\mu,\nu} + \xi_{\nu,\mu})d\epsilon$ for some vector ξ^μ.

For the rotationally symmetric vector fields $\xi^\mu = (\xi^0, \xi^k)$, where $\xi^k = \frac{x^k}{X}\xi^X(\tau, X)$, the explicit expression for $\delta h_{\mu\nu}$ reads:

$$\begin{aligned}
\delta h_{00} &= 2\xi^0_{,\tau} d\epsilon, \\
\delta h_{0k} &= \frac{x^k}{X}\left(\xi^0_{,X} + \xi^X_{,\tau}\right) d\epsilon, \quad k = 1, 2, 3, \\
\delta h_{kk} &= 2\left[\frac{\xi^X}{X} + \frac{(x^k)^2}{X^2}\left(\xi^X_{,X} - \frac{\xi^X}{X}\right)\right] d\epsilon, \\
\delta h_{kl} &= 2\frac{x^k x^l}{X^2}\left(\xi^X_{,X} - \frac{\xi^X}{X}\right) d\epsilon, \quad k \neq l.
\end{aligned} \tag{9.83}$$

These formulae are, again, only true in the regions of spacetime for which the CR coordinates are DN. The corresponding expression for $\delta h_{\mu\nu}$ in the non-DN wedges $K_\pm$ is computed in a similar way.

9.8.3 Effect on the scattering time

The action of infinitesimal coordinate transformations also affects the scattering time τ_ρ (and, in consequence, the proper scattering time s_ρ). In the transformed coordinates τ', X', the trajectory of the observer (in the region I) at $R = \rho$ is given by

$$\begin{aligned}
X'(\mu) &= X(\mu) - \xi^X(\mu)d\epsilon = \rho^* - \xi^X(\mu)d\epsilon, \quad \mu \in (\mu_-, \mu_+), \\
\tau'(\mu) &= \tau(\mu) - \xi^0(\mu)d\epsilon = \mu - \xi^0(\mu)d\epsilon.
\end{aligned} \tag{9.84}$$

That of the shell reads

$$\begin{aligned}
X'(\lambda) &= X(\lambda) - \xi^X(\lambda)d\epsilon = |\lambda| + R^*_M - \xi^X(\lambda)d\epsilon, \quad \lambda \in (\lambda_-, \lambda_+), \\
\tau'(\lambda) &= \tau(\lambda) - \xi^0(\lambda)d\epsilon = \lambda + v - R^*_M - \xi^0(\lambda)d\epsilon.
\end{aligned} \tag{9.85}$$

Here,

$$\mu_\pm = \lambda_\pm + v - R^*_M = \pm(\rho^* - R^*_M) + v - R^*_M. \tag{9.86}$$

The scattering time with respect to the new coordinates τ', X' is thus given by

$$\begin{aligned} \tau'_\rho &= \tau'(\lambda_+) - \tau'(\lambda_-) = \lambda_+ - \lambda_- - [\xi^0(\lambda_+) - \xi^0(\lambda_-)]d\epsilon \\ &= \tau_\rho - [\xi^0(\lambda_+) - \xi^0(\lambda_-)]d\epsilon \end{aligned} \tag{9.87}$$

Even in the case of rotationally symmetric vector fields ξ, the bracket in eq. (9.87) does not vanish in general, so this definition of scattering time is *not gauge invariant* and is, therefore, not a suitable observable for the quantum description. Only in the asymptotic region (i.e. for $\rho \to \infty$) the scattering time is invariant, because there the vector field $\xi^\mu(x)$ converges to a fixed constant vector ζ^μ, such that the terms in the bracket cancel. But, of course, the scattering time under an infinite radius is infinite and makes thus not much sense.

There is, however, a second way to define the scattering time, as we have shown earlier: the proper time measured by the ideal clock of the observer along his trajectory. In the unperturbed case this definition coincided with the first one, but this is not true if a non-vanishing and non-trivial generator ξ^μ is present.

In the coordinates τ, X, the infinitesimal proper time interval ds_ρ along the trajectory of the observer is simply $\sqrt{1 - \frac{2E}{\rho}}d\mu$. We show, that this quantity remains invariant under the infinitesimal gauge transformations in question. The differentials of the new coordinates along the trajectory given by $R = \rho$ read

$$dX' = -\xi^X_{,\tau} d\epsilon d\mu, \quad d\tau' = (1 - \xi^0_{,\tau} d\epsilon)d\mu. \tag{9.88}$$

The line element along this trajectory reads thus

$$ds'^2_\rho = Bd\tau'^2 - CdX'^2 - 2Dd\tau' dX' \approx \left[B(1 - 2\xi^0_{,\tau}d\epsilon) + 2D\xi^X_{,\tau}d\epsilon\right] d\mu^2, \tag{9.89}$$

where terms of second or higher order in $d\epsilon$ have been neglected. The metric components B, C and D can be obtained by comparing the invariant line element ds^2 with respect to the two coordinate systems. Neglecting higher order terms, the components read

$$B = \frac{A}{1 - 2\xi^0_{,\tau}d\epsilon}, \; C = \frac{A}{1 - 2\xi^X_{,X}d\epsilon}, \; D = \frac{A(\xi^X_{,\tau} - \xi^0_{,X})d\epsilon}{1 - (\xi^X_{,X} + \xi^0_{,\tau})d\epsilon} \tag{9.90}$$

Since D is of order $d\epsilon$ the second term in the expression for ds'^2_ρ can be neglected, thus the result reads

$$ds'^2_\rho = A(\mu)d\mu^2 = ds^2_\rho. \tag{9.91}$$

Hence, the infinitesimal proper time ds_ρ of the observer at the constant radius ρ is really a gauge invariant. The integral of ds_ρ along the curve defined by $R = \rho$ in region I is therefore again given by

$$s'_\rho = \int_{\mu_-}^{\mu_+} \sqrt{A(\mu)} d\mu = \sqrt{1 - \frac{2E}{\rho}}(\mu_+ - \mu_-) = 2\sqrt{1 - \frac{2E}{\rho}}(\rho^* - R^*_M) = s_\rho. \qquad (9.92)$$

Hence, the second definition of the scattering time is *gauge invariant* (at least in the DeWitt approximation), in contrast to the first one. The two definitions coincide in the special case of the Cartesian coordinates (τ, x^k) but not in non-trivial infinitesimal transformations thereof.

In chapter 4 we have already once computed the two definitions of the scattering time in the non-perturbative case. Since the first definition of scattering time is not gauge invariant in the semi-classical case, it clearly cannot be in the full theory. The second definition is not only invariant under infinitesimal coordinate transformations but also under finite ones, because $t_s = s_+ - s_-$, where the $s_\pm$ are gauge invariant. This is pointed out in chapter 4.

Chapter 10

Discussion and outlook

We summarize and discuss the main results of this thesis, point out the problems that have occurred and tell what needs to be done yet.

We first have found out that there is no definition of the time delay for our model that can be made finite. Even though the time delay can be, at least mathematically, consistently regularized in the case of the Coulomb potential, this is not feasible for the gravitational one, because there the regularization would be state-dependent. We thus have to renounce to define the scattering time in the asymptotic region and must content ourselves with a quantity measured at a finite distance from the mirror: the proper scattering time measured by an observer at rest between his two encounters with the shell. We then have found that the sojourn time for finite distances, that has been used to define the time delay, seems to be a suitable choice for the quantum analog of the scattering time. But the sojourn time, the way it has been defined, works only for particle scattering in Minkowski spacetime. The generalization to curved spacetimes raised conceptual problems we did not know how to cope with. We have therefore assumed that the sojourn time can be interpreted as the mean value of an operator that represents the classical scattering time in the corresponding quantum theory. Such an operator is, of course, only well-defined if its classical counterpart is a gauge-invariant constant of motion, a so-called Dirac observable. We have constructed such gauge-invariant quantities — the proper times of an observer at rest measured as the shell passes the observer inwards and outwards, respectively. Their difference is equal to the proper scattering time, which is thus also a Dirac observable. In order to prove that the true degrees of freedom of our model are really the shell's energy E and its asymptotic advanced time v, and that they are canonically conjugated to each other, we have performed the canonical reduction program of the Hamiltonian action given by LWF and HKK. The method had to be adapted to incorporate the non-trivial inner boundary of the spacetime due to the mirror, because the standard methods became not applicable there. The reduction could be explicitly performed by making some further assumptions and by constructing a gauge that is regular at the mirror. This central-regular gauge has, in addition, the nice property that it satisfies the conditions on a coordinate system in order that the perturbative methods of quantization invented by

DeWitt can be applied to our model. We thus have proven an existence theorem for such a gauge for our system. This theorem can be probably generalized to incorporate other systems related to ours. We, finally, have set up a well-defined quantum theory where the scattering time is represented by a self-adjoint operator and explicitly computed its expected value, the sojourn time, for a special kind of wave packet. The rather surprising result is that the sojourn time for packets of finite width remains finite at and above the Schwarzschild energy even if the corresponding classical scattering time is infinite! The sojourn time has, so to say, a resonance at the Schwarzschild energy, while the peak remains finite. There is the minor flaw in the computations that the operator $\hat{v}$ is not self-adjoint, such that the classical observable v is not represented by a quantum observable. But since $\hat{v}$ only appears in the two proper times $\hat{s}_\pm$ but not in their difference $\hat{t}_s$, the sojourn time, which is the really interesting quantity, this shortcoming does no seem to affect our results significantly. It seems to be possible to replace our operator by a really self-adjoint $\hat{v}$, but the correct definition involves taking the square root of a differential operator, which makes the computation much more intricate.

Our findings are, if surprising and interesting, of course, far from solving the problem on an astrophysical scale, that would require a model that is much more appropriate than our simple system. But they are first steps on a way to measure scattering times in such systems. Our results give a hint that the quantum collapse is significantly different form its classical analog.

There is a lot of work to be done yet to answer several open questions. First of all, the quantum theory must be adapted such that also v is represented by a self-adjoint operator. It seems that there is a scale invariance in the resulting quantities we have obtained. Every interesting quantity in our model (i.e. the sojourn time, the spreads of the proper times) seems to scale with the radius of the mirror (or, equivalently, with its corresponding Schwarzschild energy). This conjecture has to be substantiated. Then, our methods have to be extended to include the collapse case without the mirror (cf. the unitary quantum theory of a single null thin shell found by Hájíček [Háj01]), since the mirror can not be considered really a physical object. There is work to be done to apply DeWitt's quantization method to our system — we have only constructed the coordinates and the classical dynamics of the fields on the background — in order that one can compare the results with our findings of the exact approach in the low-energy regime. This would be a good way to check the validity of this approach. The scattering times we found are too short for astrophysical objects, according to the present astronomical observations. Of course, our model is hardly appropriate for, e.g., a collapsing star; hence, the calculations could be improved by developing more realistic models. DeWitt's perturbative method seems to be a good starting point for the fulfillment this task.

Appendix A

Schwarzschild sojourn time

A.1 Introduction

In this appendix we give some arguments against the use of a generalization of the standard definition of the sojourn time as written down in eq. (2.39) of chapter 2. For this end we assume that there is a well-defined quantum theory where the energy E and the position r of the shell are represented by a conjugate pair of self-adjoint operators on a Hilbert space $\mathcal{K}$. The eigenfunctions of $\hat{E}$ are denoted by $|p\rangle$, whereas that of $\hat{r}$ read $|r\rangle$. The evolution parameter is the Minkowski time T_M, and $\hat{E}$ is the corresponding Hamilton operator. This quantum theory seems not to be very suitable to describe our system, but it has the advantage that it is simple and that it suffices to demonstrate the problems we want point out.

In the next section we collect some interesting results about the Minkowski sojourn time T_S. Then we define the Schwarzschild sojourn time and argue why this definition is not a good one.

A.2 Minkowski Sojourn Time

We recall some basic notions concerning the standard Minkowski sojourn time T_S defined in eq. (2.39) of chapter 2. We suppose that the standard definition is a good one and that its interpretation as the mean residence time of a particle inside a given region is justified. The notion of sojourn time in the context of particle physics seems to date back to Goldberger and Watson [GW64]. It has been given a precise mathematical foundation by Jauch and Marchand, since it is a main concept used in their definition of time delay in scattering processes. We have presented these ideas in chapter 2.

The sojourn time of a Hilbert state ϕ under the radius R is given by the expression

$$T_S(\phi) = \int_{-\infty}^{\infty} dT_M \, \|\hat{P}_R \phi(r, T_M)\|^2 = \int_{-\infty}^{\infty} dT_M \, \langle \phi_T | \, \hat{P}_R \, | \phi_T \rangle. \tag{A.1}$$

T_M is, as always, Minkowski time, ϕ_T is the time development of the scattering state ϕ and $\hat{P}_R$ is the operator projecting the states of $\mathcal{K}$ into the interval $[0, R]$. In the position representation $\hat{P}_R$ acts as follows:

$$\hat{P}_R \phi(r) = \chi_{[0,R]} \phi(r) = \begin{cases} \phi(r) & : \quad r \leq R \\ 0 & : \quad r > R \end{cases} . \tag{A.2}$$

Since $\hat{P}_R$ is a projector, $\hat{P}_R^2 = \hat{P}_R$ holds.

The interpretation of T_S as the sojourn time is more obvious if we approximate the integral by a sum over small time intervals $[T_{k-1}, T_k]$, $k = 1, ..., K$, $T_0 = -\tau$, $T_K = \tau$, where τ is chosen to be large enough:

$$T_S(\phi) \approx \sum_{k=1}^{K} \|\hat{P}_R e^{-i\hat{E}u_k} \phi\|^2 \, (T_k - T_{k-1}), \quad u_k \doteq \frac{T_{k-1} + T_k}{2}. \tag{A.3}$$

This is the weighted sum of small time intervals, the weight being the probability that the state ϕ is lying inside the radius R during this time interval. The sum is the total time spent there by a particle described by the wave function ϕ, provided the time evolution is given by $\phi_T = e^{-i\hat{E}T_M} \phi$.

It has to be emphasized that T_S is the *mean* residence time of a particle described by the wave packet ϕ_T inside the region with $r \leq R$.

Using the Heisenberg picture one can write the sojourn time in a simpler form by computing the time integral. In the Heisenberg picture the formula for the sojourn time reads:

$$T_S(\phi) = \int_{-\infty}^{\infty} dT_M \, \langle \phi | \, e^{i\hat{E}T_M} \hat{P}_R e^{-i\hat{E}T_M} \, | \phi \rangle. \tag{A.4}$$

By expanding ϕ into energy eigenstates we obtain

$$T_S(\phi) = \int_0^{\infty} dp' \int_0^{\infty} dp \, \tilde{\phi}^*(p') \tilde{\phi}(p) \int_{-\infty}^{\infty} dT_M \, e^{i(p'-p)T_M} \langle p' | \, \hat{P}_R \, | p \rangle. \tag{A.5}$$

The time integration can be done explicitly yielding a δ-function. We can then perform one of the remaining integrations which leads to

$$T_S(\phi) = 2\pi \int_0^{\infty} dp \, \left| \tilde{\phi}(p) \right|^2 \langle p | \, \hat{P}_R \, | p \rangle. \tag{A.6}$$

The matrix elements of the projector $\hat{P}_R$ can be evaluated by expanding the energy eigenstates in terms of eigenstates of the position operator $\hat{r}$. A straightforward calculation, where it is assumed that

$$\langle r|p\rangle = \sqrt{\frac{2}{\pi}}\sin pr, \tag{A.7}$$

yields the result

$$T_S(\phi) = 2R - 2\int_0^\infty dp\,\left|\tilde{\phi}(p)\right|^2\,\frac{\sin pR\cos pR}{p}. \tag{A.8}$$

The first term on the right hand side is clearly the classical sojourn time for a light-like particle inside the region with radius R (we put $R_M = 0$ here for the sake of simplicity). The second term is the correction due to the wave-like nature of the quantum mechanical packet. In eq. (A.8) the interpretation of T_S as the mean residence time seems very obvious.

By reintroducing Planck's constant $\hbar$ into all the formulae and expanding T_S in powers of $\hbar$, which we made up to the fourth order, one obtains

$$T_S(\phi) = 2R + \mathcal{O}(\hbar^4). \tag{A.9}$$

Obviously, the quantum corrections are very small.

A.3 Schwarzschild Sojourn Time

A.3.1 Definition

In chapter 4 we proposed to generalize the Minkowski sojourn time to the case of the curved shell-mirror spacetime by replacing the Minkowski time T_M by the proper time s at the radius R. In the expression for $T_S(\Phi)$ the differential dT_M must thus be replaced by the infinitesimal proper time ds. But ds depends on the energy of the shell which becomes an operator in the quantum theory. Hence, also ds is an operator, $\widehat{ds}$. To simplify the discussion we do not compute the proper sojourn time but only the mean Schwarzschild time, that does not have the square root prefactor, along the trajectory. The problems that occur are the same in both cases. Thus we replace dT_M by the infinitesimal Schwarzschild time $\widehat{dT}$, that, being dependent on $\hat{E}$, is an operator. dT can be written as

$$dT = z dT_M, \quad z \doteq \frac{\partial T}{\partial T_M}, \tag{A.10}$$

where z is the Minkowski-time derivative of T along the shell. It is given by

$$z = \frac{r}{r - 2E}, \tag{A.11}$$

which follows from the equations of motion that have been written down in eqs. (3.26) of chapter 3. Since we have assumed that the evolution parameter in the quantum theory is T_M, this construction seems to be sensible. We further assume that z can be represented by a self-adjoint operator $\hat{z}$ by a suitable factor ordering and that it has a complete set of eigenfunctions in $\mathcal{K}$, the eigenfunctions being denoted by $|z\rangle$. Then, states in $\mathcal{K}$ can be expanded in terms of these eigenfunctions and the expectation value $\langle \hat{z} \rangle = \langle \psi | \, \hat{z} \, | \psi \rangle$ is a well-defined quantity. Note that z is time-independent in the Schrödinger picture. The *Schwarzschild sojourn time* is thus defined as follows:

$$T_z(\phi) \doteq \int_{-\infty}^{\infty} dT_M \, \langle \phi_T | \, \hat{P}_R \hat{z} \hat{P}_R \, | \phi_T \rangle, \tag{A.12}$$

where $\phi \in \mathcal{K}$ and $\phi_T = e^{-iET_M}\phi$. This seems to be the most sensible factor ordering of the operators inside the bracket. If the limit $R \to \infty$ in the brackets is taken, just the well-defined expected value $\langle \hat{z} \rangle$ is obtained. The time integral over it is the total (and of course infinite) Schwarzschild time. Thus, at least in this case, the definition seems to be justified.

A.3.2 Discussion

We explore the physical meaning of the Schwarzschild sojourn time T_z and argue why it can be used only in a very specific case.

We have assumed in the preceding section that z can be made into a self-adjoint operator $\hat{z}$ on the Hilbert space $\mathcal{K}$ and that it has a complete system of normalized eigenfunctions $|z\rangle$. Then any state $|\phi\rangle$ can be expanded into these eigenfunctions. Hence, the normalized eigenfunctions $|r\rangle$ of the position operator $\hat{r}$ can be expanded in terms of the $|z\rangle$:

$$|r\rangle = \sum_z \alpha_z^r |z\rangle. \tag{A.13}$$

Here and in the following formulae we write the expansions of the wave functions as sums over the eigenstates over some range of the eigenvalues. These sums are to be understood as generalized sums, standing for ordinary summations in the case of a discrete spectrum and for integrations if the spectrum of the corresponding operator is continuous. Since we do not know the explicit form of $\hat{z}$, its eigenfunctions and its spectrum, we keep our formulae as general as possible. This is justified because we are interested only in qualitative results, for which the general form is sufficiently well suited.

A (time-dependent) Hilbert state $|\phi_T\rangle$ can be expanded into eigenfunctions of the position operator; its z-expansion is, therefore,

$$|\phi_T\rangle = \sum_{r=0}^{\infty} c_r(T_M)|r\rangle = \sum_{r=0}^{\infty} c_r(T_M) \sum_z \alpha_z^r |z\rangle. \tag{A.14}$$

Its projection by $\hat{P}_R$ reads:

$$\hat{P}_R|\phi_T\rangle = \sum_{r=0}^{R} c_r(T_M) \sum_z \alpha_z^r |z\rangle. \tag{A.15}$$

Hence, the weight in the Minkowski sojourn time, $\langle\phi_T| \hat{P}_R |\phi_T\rangle$, is given by

$$\langle\phi_T| \hat{P}_R |\phi_T\rangle = \sum_{r=0}^{R} |c_r(T_M)|^2 = \sum_{r'=0}^{R} (c_{r'})^*(T_M) \sum_{r=0}^{R} c_r(T_M) \sum_z (\alpha_z^{r'})^* \alpha_z^r, \tag{A.16}$$

since $\sum_z (\alpha_z^{r'})^* \alpha_z^r = \delta_{r'r}$. Using this, we can rewrite eq. (A.3):

$$T_S(\phi) \approx \sum_z \sum_{k=1}^{K} (T_k - T_{k-1}) \sum_{r'=0}^{R} (c_{r'})^*(u_k) \sum_{r=0}^{R} c_r(u_k) (\alpha_z^{r'})^* \alpha_z^r. \tag{A.17}$$

This is the same weighted sum as above, only decomposed into eigenstates of the operator $\hat{z}$.

The action of $\hat{z}$ on a state $|\phi_T\rangle$ is given by

$$\hat{z}|\phi_T\rangle = \sum_{r=0}^{\infty} c_r(T_M) \sum_z z\alpha_z^r |z\rangle. \tag{A.18}$$

Its action on the projected state $\hat{P}_R|\phi_T\rangle$ reads

$$\hat{z}\hat{P}_R|\phi_T\rangle = \sum_{r=0}^{R} c_r(T_M) \sum_z z\alpha_z^r |z\rangle. \tag{A.19}$$

Notice that *the two operators $\hat{z}$ and $\hat{P}_R$ do not commute* in general, because $\sum_z z\alpha_z^r|z\rangle$ can usually not be written in the form $C|r\rangle$, where C is some constant. The expectation value of the operator $\hat{P}_R\hat{z}\hat{P}_R$ is given by

$$\langle\phi_T| \hat{P}_R\hat{z}\hat{P}_R |\phi_T\rangle = \sum_z z \sum_{r'=0}^{R} \left(c_{r'}(T_M)\alpha_z^{r'}\right)^* \sum_{r=0}^{R} c_r(T_M)\alpha_z^r. \tag{A.20}$$

We can now write our definition of the Schwarzschild sojourn time in terms of the expansion (A.20):

$$\begin{aligned} T_z(\phi) &= \int_{-\infty}^{\infty} dT_M \, \langle \phi_T | \, \hat{P}_R \hat{z} \hat{P}_R \, | \phi_T \rangle \\ &= \sum_z \int_{-\infty}^{\infty} z dT_M \sum_{r'=0}^{R} \left(c_{r'}(T_M) \alpha_z^{r'} \right)^* \sum_{r=0}^{R} c_r(T_M) \alpha_z^r. \end{aligned} \tag{A.21}$$

If we perform the same approximation by finite time intervals as in the previous section for the Minkowski sojourn time, we obtain

$$\begin{aligned} T_z(\phi) &\approx \sum_z \sum_{k=1}^{K} z(T_k - T_{k-1}) \sum_{r'=0}^{R} (c_{r'}(u_k))^* \sum_{r=0}^{R} c_r(u_k) \alpha_z^r (\alpha_z^{r'})^* \\ &= \sum_z \sum_{k=1}^{K} (t_k - t_{k-1})_z \sum_{r'=0}^{R} (c_{r'}(u_k))^* \sum_{r=0}^{R} c_r(u_k) \alpha_z^r (\alpha_z^{r'})^*. \end{aligned} \tag{A.22}$$

This is the weighted sum for each z over the Schwarzschild time intervals $[t_{k-1}, t_k]$. The weight is the same as before. The sum over all z gives the total 'time' spent by the state inside the radius R. This interpretation sounds very good, but it, unfortunately, has one serious problem — it is *wrong*: the quantity $z(T_k - T_{k-1})$ could be interpreted as the Schwarzschild time that the particle in the sub-state $|z\rangle$ spends under R if the state was under R; but it *is not*, unless the operators $\hat{P}_R$ and $\hat{z}$ commute, when they have common eigenstates. But this is usually not the case.

We now want to illustrate what went wrong. For this, we consider a simplified (and very unrealistic) system given by

$$|\phi_T\rangle = c_1 |r_1\rangle + c_2 |r_2\rangle + c_3 |r_3\rangle, \tag{A.23}$$

where $r_1, r_2 < R$ and $r_3 > R$. The projector's action on this state reads

$$\hat{P}_R |\phi_T\rangle = c_1 |r_1\rangle + c_2 |r_2\rangle. \tag{A.24}$$

We further assume that $\hat{z}$ has only two different eigenstates and that the position eigenstates can be written as

$$|r_i\rangle = \alpha_i |z_1\rangle + \beta_i |z_2\rangle. \tag{A.25}$$

The expectation value eq. (A.20) in the formula for the Schwarzschild sojourn time for this system reads

$$\begin{aligned}\langle\phi_T|\,\hat{P}_R\hat{z}\hat{P}_R\,|\phi_T\rangle &= z_1\left(|c_1|^2|\alpha_1|^2+|c_2|^2|\alpha_2|^2+\alpha_1^*\alpha_2c_1^*c_2+\alpha_1\alpha_2^*c_1c_2^*\right)\\ &+ z_2\left(|c_1|^2|\beta_1|^2+|c_2|^2|\beta_2|^2+\beta_1^*\beta_2c_1^*c_2+\beta_1\beta_2^*c_1c_2^*\right).\end{aligned} \tag{A.26}$$

The clue is that what is written in the upper bracket is *not* the expected value of $|z_1\rangle$ of being under R. If this was the case, the decomposition of the Schwarzschild sojourn time in z-eigenfunctions would make sense. This stems from the fact that the sub-states $|z_i\rangle$ alone are not under R even although the special linear combinations $|r_l\rangle, l=1,2$, are.

The whole construction fails because the two operators $\hat{P}_R$ and $\hat{z}$ do not commute. What happens if they *do* commute? We assume that the $|\phi_T\rangle$ can be decomposed as follows:

$$|\phi_T\rangle=\sum_z\alpha_z(T_M)|z\rangle=\sum_{z_R}A_{z_R}(T_M)|z_R\rangle+\sum_{z_\infty}B_{z_\infty}(T_M)|z_\infty\rangle, \tag{A.27}$$

where the projector acts on the two kinds of z-eigenfunctions as $\hat{P}_R|z_R\rangle = |z_R\rangle$ and $\hat{P}_R|z_\infty\rangle=0$. The existence of such a decomposition is granted if the commutator $[\hat{z},\hat{P}_R]$ vanishes. In this special case the decomposition of the bracket in (A.12) into a sum over z seems to have a good interpretation:

$$\langle\phi_T|\,\hat{P}_R\hat{z}\hat{P}_R\,|\phi_T\rangle=\sum_{z_R}z_R\,|A_{z_R}(T_M)|^2\,, \tag{A.28}$$

and the Schwarzschild sojourn time thus reads

$$T_z(\phi)=\sum_{z_R}\int_{-\infty}^{\infty}z_R dT_M\;|A_{z_R}(T_M)|^2\,, \tag{A.29}$$

where $|A_{z_R}(T_M)|^2$ is the weight factor of ϕ_T in the sub-state $|z_R\rangle$.

The interpretation of eq. (A.22) seems to make sense in this case, because now the sub-states $|z_R\rangle$ are really under the radius R. But there are more difficulties that have to be taken into account.

It is a well-known fact that measurements of observables in quantum mechanics have a strong influence on the quantum states. For instance, measuring the energy of a wave packet destroys the wave packet — immediately after the measurement the system is in a particular eigenstate of the energy operator. This collapse of the wave function also occurs in our system when we measure the Schwarzschild sojourn time. At each instant T_M we make consecutive measurements of the operators $\hat{P}_R$ and $\hat{z}$. Since the wave packet is not necessarily an eigenstate of one of these two operators and since they usually do not commute, a measurement at the time T_1 destroys the wave function. Hence, at a

later time T_2 not the original packet will be measured, but the time development of the eigenstate of $\hat{z}$ which has been created at the time T_1. Therefore, from the point of view of the standard interpretation of quantum mechanics such a series of repeated measurements does not make sense. It would be much better to let the packet evolve until some final time T_f and only then make the whole time measurement at once. For this reason it seems to be a good idea to look for some way to perform the integration over time first and acting on the state afterwards. One possible method to do this is writing the operator in the expression for the Schwarzschild sojourn time in the *Heisenberg picture*:

$$T_z(\phi) = \langle\phi| \int_\infty^\infty dT_M \, e^{i\hat{E}T_M} \hat{P}_R \hat{z} \hat{P}_R e^{-i\hat{E}T_M} \, |\phi\rangle. \tag{A.30}$$

Here, the the whole expression between the brackets, including the time integral, is one operator, representing the measurement of the entire sojourn time in the state $|\phi\rangle$. Decomposing $|\phi\rangle$ into eigenstates $|p\rangle$ of the energy (Hamilton) operator E and integrating over time yields:

$$\begin{aligned} T_z(\phi) &= \int_\infty^\infty dT_M \int_0^\infty dp' \int_0^\infty dp \, \phi^*(p')\phi(p) \langle p'| \, e^{i\hat{E}T_M} \hat{P}_R \hat{z} \hat{P}_R e^{-i\hat{E}T_M} \, |p\rangle \\ &= \int_0^\infty dp' \int_0^\infty dp \, \phi^*(p')\phi(p) \int_\infty^\infty dT_M \, e^{i(p'-p)T_M} \, \langle p'| \, \hat{P}_R \hat{z} \hat{P}_R \, |p\rangle \\ &= 2\pi \int_0^\infty dp \, |\phi(p)|^2 \, \langle p| \, \hat{P}_R \hat{z} \hat{P}_R \, |p\rangle. \end{aligned} \tag{A.31}$$

To obtain the last line we have used a representation of the δ-function. The decomposition of a state into energy eigenstates is always possible and the matrix elements of the operator $\hat{P}_R \hat{z} \hat{P}_R$ should be, in principle, calculable. Provided the quantity $T_z(\phi)$ can be given a physical meaning as Schwarzschild sojourn time, which seems to be the case when $\hat{z}$ and $\hat{P}_R$ commute, also the last line of eq. (A.31) seems to make sense. It has, moreover, the advantage that T_M is not present anymore, such that the measurement problem mentioned above is removed.

We conclude that the Schwarzschild sojourn time is not well-defined in the general case, and there seem to be deep problems that must be treated carefully. Therefore, we renounce to stick to the definition of sojourn time so closely and choose a different way to define the quantum scattering time.

Appendix B

Distance of metrics as a topological problem

B.1 Introduction

In chapter 9 we have discussed the semi-classical approach to quantization by DeWitt and applied it to our shell-mirror system. In this approach the metric describing the spacetime must be decomposed into a sum of a flat background metric and a small perturbation. We have found a coordinate system satisfying the desired properties for such a decomposition. In this chapter we want to elucidate some of the issues that have appeared in the course of our application of DeWitt's method.

We cast the problem of the distance of two metrics in a topological language in order to exploit the useful properties of this method. The space of all Lorentzian metrics on the manifold $\mathcal{M}$, which we denote by $\mathrm{Lor}(\mathcal{M})$ in the following, has been thoroughly explored by several authors [BE81, HE73, Ler73], there are various topologies, which can be put on this space [Haw71]. First we recall and reformulate the problem and give some possible definitions for the closeness of two metrics. Next we discuss the properties of sequences of metrics in $\mathrm{Lor}(\mathcal{M})$ and observe, that one must be careful when taking limits. In the last section we state some known properties of the space of metrics and sketch some ideas for further investigations.

B.2 Closeness of metrics, topologies on $\mathrm{Lor}(\mathcal{M})$

We give several possible definitions for the closeness of two metrics $g_{\mu\nu}$ and $f_{\mu\nu}$ in a given coordinate system x^μ and show how these definitions are related to the various topologies that can be put on $\mathrm{Lor}(\mathcal{M})$.

B.2.1 Closeness conditions in $\mathrm{Lor}(\mathcal{M})$

Let $\epsilon > 0$ be a small number and $\epsilon(x) > 0$ a small and continuous real function on $\mathcal{M}$, x denoting a point of $\mathcal{M}$. Then one can define the following *closeness conditions* on a pair $f_{\mu\nu}, g_{\mu\nu}$ of metrics on $\mathcal{M}$:

$$|g_{\mu\nu}(x) - f_{\mu\nu}(x)| < \epsilon, \quad \forall x \in \mathcal{M}, \tag{B.1}$$

$$|g_{\mu\nu}(x) - f_{\mu\nu}(x)| < \epsilon(x), \quad \forall x \in \mathcal{M} \tag{B.2}$$

and

$$|g_{\mu\nu}(x) - f_{\mu\nu}(x)| \begin{cases} < \epsilon\,|f_{\mu\nu}(x)| & : \ \mu,\nu \text{ such that } f_{\mu\nu} \neq 0 \\ < \epsilon & : \ \mu,\nu \text{ such that } f_{\mu\nu} = 0 \end{cases}, \ \forall x \in \mathcal{M}. \tag{B.3}$$

The condition (B.3) seems to be the most suitable one, if metrics are compared, for which some of the components diverge as specific regions on the manifold $\mathcal{M}$ are reached. This is, for example, the case for the shell metric expressed in the regular DNEF coordinates (U, V, θ, ϕ), defined by eqs. (5.45, 5.46),

$$ds^2 = A(U,V)dUdV - R^2(U,V)d\Omega^2, \tag{B.4}$$

where the $\theta\theta$-component, $-R^2$, tends to $-\infty$ as $V \to \infty$ or $U \to -\infty$. If this component, for example, is compared with the $\theta\theta$-component of the double-null Minkowski metric given by

$$ds^2 = dUdV - \frac{(V-U)^2}{4}d\Omega^2, \tag{B.5}$$

i.e. $-R_0^2 \doteq -\frac{(V-U)^2}{4}$, then the condition (B.3) reads

$$|-R^2 + R_0^2| < \epsilon\,|R_0^2|. \tag{B.6}$$

This inequality is equivalent to

$$\left|1 - \frac{R^2}{R_0^2}\right| < \epsilon. \tag{B.7}$$

Whenever for the functions R and R_0 holds everywhere on the manifold that

$$\frac{R^2}{R_0^2} = 1 + \epsilon K(\rho), \quad |K| < 1, \tag{B.8}$$

where ρ is a radial coordinate on the manifold, then the condition (B.7) is satisfied. eq. (B.8) implies that

$$\frac{R^2}{R_0^2} \approx 1 + \mathcal{O}(\rho^{-\zeta}), \quad \zeta > 0, \tag{B.9}$$

as $\rho \to \infty$. That may, however, not be sufficient for the condition (B.1), that in this example reads

$$|-R^2 + R_0^2| = \epsilon R_0^2 K(\rho) < \epsilon. \tag{B.10}$$

This is due to the fact that even if the quotient $\frac{R}{R_0}$ remains small everywhere, the difference $|-R^2+R_0^2|$ can attain arbitrarily large values at infinity, because the fall-off of $K(\rho)$ may be too slow to compensate the diverging factor R_0^2 at infinity. For this reason condition (B.3) is more suitable than (B.1), whenever the metrics are expressed in coordinates, that are singular at infinity, like the UV-type or the polar coordinates, cf. section 5 of chapter 6. If $f_{\mu\nu}$ is the Cartesian Minkowski metric $\eta_{\mu\nu}$, then the two conditions of course agree.

In condition (B.2) one usually chooses the function $\epsilon(x)$ such that it goes to 0 at infinity, which we assume too in the following. So, at infinity, the condition (B.2) is more restrictive than (B.1). Condition (B.3) is not a special case of (B.2), because in (B.3) the form of the right hand side depends on the component of the metric, whereas in (B.2) it is the same for all components. In this sense, the condition (B.3) is less beautiful than (B.2), and we will observe in the next paragraph that there is no obvious relation of (B.3) to a topology on Lor($\mathcal{M}$).

If one does not allow coordinates for which any of the metric components of $f_{\mu\nu}$ and $g_{\mu\nu}$ diverge, then the conditions (B.1) or (B.2) are very well suited to compare two metrics. Our choice for the DeWitt-compatible metrics in chapter 9 satisfies this requirement.

B.2.2 Topologies on Lor($\mathcal{M}$)

One can relate the neighbourhoods defined by the conditions from the last paragraph to the neighbourhoods defining the various topologies that can be put on Lor($\mathcal{M}$) [Haw71]. The neighbourhoods defined by (B.1) are just those of the C^0 *open topology* on Lor($\mathcal{M}$). Those defined by (B.2) belong to the class of *(Whitney) fine* C^0 *topologies*. The latter is, as indicated by its name, finer than the former, corresponding to our observation above, that the second condition is more severe than the first. There seems to be no obvious topology related to the third condition (B.3), however. The detailed description of the topologies can be found in [Haw71].

B.3 Limits of metrics

In chapter 9 we have required that the shell metric $g_{\mu\nu}$ can be written as a flat metric $f_{\mu\nu}$ plus a small perturbation depending on the parameter $s = \frac{2E}{R_M}$. Hence, in the limit $E \to 0$, $g_{\mu\nu}$ must converge to $f_{\mu\nu}$ in a well-defined way. In this section we point out that taking limits of this kind on $\mathrm{Lor}(\mathcal{M})$ is an intricate business.

The limit $E \to 0$ is a limit of a family of spacetimes with a common parameter, i.e. the shell energy E in our case, on the space of all Lorentz metrics $\mathrm{Lor}(\mathcal{M})$. Limits on $\mathrm{Lor}(\mathcal{M})$ have been explored by Geroch [Ger69], who observed that a one-parameter family of metrics, $g(\lambda)$, λ being the parameter, can attain different limiting metrics as λ goes to 0 or ∞ in different coordinate systems. His example was the Schwarzschild spacetime, where he considered three coordinate systems with respect to which the limiting metric was the Minkowski, a divergent, and a Kasner metric, respectively. It seems thus, that, under coordinate transformations, this kind of sequences of metrics in $\mathrm{Lor}(\mathcal{M})$ are not mapped to sequences in a way, that the limiting metric is mapped on a metric that are related by a regular coordinate transformation. This is because the coordinate transformation depends on the parameter λ. In the limit $\lambda \to 0$ (or ∞), the coordinate transformation may become singular and thus not usable to transform the limit metric in the original coordinates into the new ones.

Even though the limits are not necessarily related by coordinate transformations, Geroch found some properties which are hereditary to the limit spacetime. For example, the Petrov type can only be the same or a more degenerate type in the limit. So, for example, a sequence of type D metrics allows only type D, N or O limits. Paiva et al. [PRM93, PRHM97] found a way to explicitly compute the various possible limits for a given metric and also stated which Segré types can be inherited by a limit metric.

B.4 Some properties of $\mathrm{Lor}(\mathcal{M})$

This is a summary of a few collected properties of $\mathrm{Lor}(\mathcal{M})$.

$\mathrm{Lor}(\mathcal{M})$ is the space of all Lorentzian metrics on a fixed, non-compact, four-dimensional manifold $\mathcal{M}$ without boundary. One can put various topologies on $\mathrm{Lor}(\mathcal{M})$ [Haw71], amongst them are the *compact-open*, *open* and *Whitney fine topologies*. The two latter have been defined earlier. The neighbourhoods of the compact-open topologies are defined in the same way as those for the open topologies, but the condition needs to hold only on a compact subset of $\mathcal{M}$. All of these topologies can be generalized to the order k in the derivatives of the metrics, resulting in the so-called C^k topologies. Two metrics that are close to each other in a C^0 topology have light-cones that are near.

One can put an equivalence relation $\sim$ on $\mathrm{Lor}(\mathcal{M})$ (see for example Beem and Ehrlich [BE81]) by defining two metrics $g_1, g_2 \in \mathrm{Lor}(\mathcal{M})$ to be equivalent if there exists a smooth conformal factor $\Omega : \mathcal{M} \to (0, \infty)$ such that $g_1 = \Omega^2 g_2$. We denote the equivalence class

of $g \in \mathrm{Lor}(\mathcal{M})$ by $[g]$. The quotient space $\mathrm{Lor}(\mathcal{M})/\sim$ of equivalence classes is denoted by $\mathrm{Con}(\mathcal{M})$. There is a natural projection map $\pi : \mathrm{Lor}(\mathcal{M}) \to \mathrm{Con}(\mathcal{M})$ given by $\pi(g) = [g]$. The fine C^0 topology on $\mathrm{Lor}(\mathcal{M})$ induces a quotient topology on $\mathrm{Con}(\mathcal{M})$ in the usual way: a subset A of $\mathrm{Con}(\mathcal{M})$ is defined to be open in this topology if the inverse image $\pi^{-1}(A)$ is open in the fine C^0 topology on $\mathrm{Lor}(\mathcal{M})$. $\mathrm{Con}(\mathcal{M})$ can also be equipped with the so-called interval topology, but we do not go into details here. It is known that the quotient and interval topologies agree on $\mathrm{Con}(\mathcal{M})$. Thus, two conformal classes $[g_1]$ and $[g_2]$ are close in either of these topologies on $\mathrm{Con}(\mathcal{M})$, if and only if at all points p of $\mathcal{M}$, the metrics g_1 and g_2 have light cones which are close in the tangent space $T_p\mathcal{M}$. The properties of $\mathrm{Con}(\mathcal{M})$ have been used to make statements about so-called stable properties on $\mathrm{Lor}(\mathcal{M})$ [BE81, HE73].

The quotient space $\mathrm{Con}(\mathcal{M})$ seems not to be very useful for our needs, however, because it does not take the gauge invariance into account. Hence, there are flat metrics in several equivalence classes from $\mathrm{Con}(\mathcal{M})$, and so are shell metrics.

In order to be able to treat the problem in a more gauge invariant way, we have tried to make a similar construction for the equivalence classes of metrics that are related by regular coordinate transformations. The quotient space $\mathrm{Phys}(\mathcal{M})$ of these equivalence classes also inherits the C^0 topology from $\mathrm{Lor}(\mathcal{M})$. Now, all flat metrics are collected in one equivalence class, say $\boldsymbol{\eta}$, and all shell metrics belonging to a specific solution given by the parameters (E, v) are represented by $\mathbf{g}(E, v)$. Consider a one-parameter sequence of equivalence classes of shell metrics with parameter E, $\{\mathbf{g}(E, v)\}$ (the other parameter v being constant), that converges to $\boldsymbol{\eta}$. It would be good to know, how sequences of representatives of the equivalence classes in $\mathrm{Lor}(\mathcal{M})$ behave in this case. Especially interesting is the question, whether there exists at all a convergent sequence in $\mathrm{Lor}(\mathcal{M})$ if there is one in the quotient space. It is clear that the opposite holds: if a sequence of metrics converges in $\mathrm{Lor}(\mathcal{M})$ then also the sequence of its equivalence classes in $\mathrm{Phys}(\mathcal{M})$. We are still working on the answer of this question.

Appendix C

Proof of the existence theorem

C.1 The Cartesian central-regular metric

In this appendix we show that the Cartesian central-regular (CCR) metric satisfies the conditions for DeWitt's approximation (cf. chapter 9) if a certain parameter is small enough. This parameter is

$$s \doteq \frac{2E}{R_M}. \tag{C.1}$$

s is the Schwarzschild radius of the Schwarzschild spacetime with mass E divided by the radius of the mirror. By this we prove the existence theorem 9.1 for a suitable gauge that has been stated in chapter 9.

The CCR metric is the central-regular (CR) metric, that has been computed in section 6 of chapter 6, expressed with respect to the Cartesian coordinates $\tau, \vec{X}$ (cf. section 4 of chapter 6). The line element for the regions where the CR coordinates are DN (i.e. I, $M_\pm$, $Z_\pm$, cf. figure (6.1)) reads:

$$ds^2 = Ad\tau^2 - \frac{R^2}{X^2}d\vec{X}^2 + \left(\frac{R^2}{X^2} - A\right)\left(\frac{\vec{X}\cdot d\vec{X}}{X}\right)^2, \tag{C.2}$$

whereas that of the non-DN regions ($K_\pm$) is given by

$$ds^2 = Qd\tau^2 + \frac{J}{X}d\tau\vec{X}\cdot d\vec{X} - \frac{R^2}{X^2}d\vec{X}^2 + \left(\frac{R^2}{X^2} + W\right)\left(\frac{\vec{X}\cdot d\vec{X}}{X}\right)^2. \tag{C.3}$$

The explicit form of the metric components is given in section 3 of this appendix.

Cartesian coordinates have the advantage over polar (6.59) or UV-type (6.57) coordinates that the metric is much better suited for DeWitt's theory. The reason is that the radial

function R runs to infinity as spatial infinity is approached. If two such metrics with different radial functions are compared (e.g. our DN metric and the DNM metric) then the difference $R^2 - R_0^2$ of the $\theta\theta$-components for instance, albeit probably very small in finite regions, may become unboundedly large at spatial infinity. Therefore, UV-type and polar coordinates are not good choices, even if they are the most adapted to spherical symmetry. In Cartesian coordinates the function R appears only in the quotient $\frac{R}{X}$ which has a much better behaviour as one goes near to spatial infinity. The difference between the corresponding components of two metrics is then suppressed by a X^2 in the denominator which also tends to infinity. It is thus much more probable that two Cartesian type metrics are only slightly different at spatial infinity. This issue has been discussed in more detail in the previous appendix.

In the following sections we explicitly write down the metric components and show that for a small parameter s, the CCR metric approaches the Minkowski metric sufficiently well in each of the seven parts of the spacetime (cf. fig. (6.1)).

C.2 Expansion of some functions in the parameter s

First we need the expansion about small s of some of the functions appearing in the CCR coordinates. We first define the abbreviations

$$\gamma_+ \doteq \frac{\tilde{V} - \tilde{u}}{2R_M}, \tag{C.4}$$

$$\eta_+ \doteq \frac{\tilde{V} - \tilde{v}}{2R_M} \tag{C.5}$$

in region II and

$$\gamma_- \doteq \frac{\tilde{v} - \tilde{U}}{2R_M}, \tag{C.6}$$

$$\eta_- \doteq \frac{\tilde{u} - \tilde{U}}{2R_M} \tag{C.7}$$

in region III, respectively. An easy calculation using eqs. (6.39) or (6.55) shows that

$$\eta_\pm = \gamma_\pm - 1. \tag{C.8}$$

The following developments in the parameter s hold:

$$L = 2R_M \ln\left|\frac{\gamma_+}{s} - 1\right| s \approx -2R_M s \ln s + 2R_M \ln(\gamma_+)s + \mathcal{O}(s^2), \tag{C.9}$$

$$L - c = 2R_M \ln\left|\frac{\eta_+}{1-s} + 1\right| s \approx 2R_M \ln(\gamma_+)s + \mathcal{O}(s^2), \tag{C.10}$$

$$L_{,\tilde{V}} = \frac{4E}{\tilde{V} - \tilde{u} - 4E} = \frac{s}{\gamma_+ - s} \approx \frac{s}{\gamma_+} + \mathcal{O}(s^2), \tag{C.11}$$

$$c = c' = 2R_M \ln\left|\frac{1}{s} - 1\right| s \approx -2R_M s \ln s + \mathcal{O}(s^2), \tag{C.12}$$

$$L' - c' = 2R_M \ln\left|\frac{\eta_-}{1-s} + 1\right| s \approx 2R_M \ln(\gamma_-)s + \mathcal{O}(s^2), \tag{C.13}$$

$$L'_{,\tilde{U}} = \frac{4E}{\tilde{v} - \tilde{U} - 4E} = \frac{s}{\gamma_- - s} \approx \frac{s}{\gamma_-} + \mathcal{O}(s^2). \tag{C.14}$$

These expansions will be needed to determine the behaviour of the metric functions in the regions $K_\pm$ for small s.

C.3 CCR metric components and their expansion

We compute the CCR metric components in the seven regions separately and show that they can be expanded in the small parameter s. Notice that even if the CCR metric is defined with respect to the Cartesian coordinates τ, $\vec{X}$, we write down the metric components as functions of the corresponding DN coordinates U and V, just for the sake of simplicity. The transformation between the two coordinate systems is given by the equations (6.59) and (6.64).

C.3.1 Region I

The functions A, R and X appearing in the metric (C.2) in region I read

$$A = 1 - \frac{2E}{R}, \tag{C.15}$$

$$R = 2E\kappa\left(e^{\frac{V-U}{4E}}\right) \tag{C.16}$$

and

$$X = \frac{V-U}{2}. \tag{C.17}$$

Using the results about the asymptotic expansion of the Kruskal function κ from the appendix E, we find the expansions of the metric components in I about small values of s. We do this as follows: First, we express R and A in terms of s:

$$R = R_M s \kappa\left(e^{\frac{\gamma}{s}}\right), \quad \gamma \doteq \frac{V-U}{2R_M}, \tag{C.18}$$

$$A = 1 - \frac{1}{\kappa\left(e^{\frac{\gamma}{s}}\right)}. \tag{C.19}$$

From eq. (E.10) (putting $y = \gamma/s$) follows the expansion of R about s,

$$R \approx \frac{V-U}{2} + R_M s \ln s - R_M \ln(\gamma) s + \mathcal{O}(s^2) \xrightarrow{s\to 0} \frac{V-U}{2} \tag{C.20}$$

and that of A,

$$A \approx 1 - \frac{s}{\gamma} \xrightarrow{s\to 0} 1. \tag{C.21}$$

Finally, the expansion of the metric component $\frac{R^2}{X^2}$ is given by

$$\frac{R^2}{X^2} \approx 1 + \frac{2}{\gamma} s \ln s - \frac{2}{\gamma} \ln(\gamma) s + \mathcal{O}(s^2) \xrightarrow{s\to 0} 1. \tag{C.22}$$

The two last equations show that for small s the CCR metric components of (C.2) in region I differ from the Minkowski metric components only by terms proportional to $s \ln s$ or higher orders of s. By choosing s small enough one can thus control the difference between the two metrics. The CCR metric does therefore satisfy the DeWitt condition in the region I.

C.3.2 Region II, Z_+

The functions appearing in the metric components read

$$R = \frac{u-U}{2} + 2E\kappa\left(e^{\frac{V-u}{4E}}\right), \tag{C.23}$$

$$A = 1 - \frac{1}{\kappa\left(e^{\frac{V-u}{4E}}\right)} \tag{C.24}$$

and

$$X = \frac{V-U}{2} \tag{C.25}$$

The asymptotic expansions of R, A and $\frac{R^2}{X^2}$ for small s read (using eq. (E.10))

$$R \approx \frac{V-U}{2} + R_M s \ln s - R_M \ln\left(\frac{V-u}{2R_M}\right) s + \mathcal{O}(s^2) \overset{s\to 0}{\longrightarrow} \frac{V-U}{2}, \tag{C.26}$$

$$A \approx 1 - \frac{2R_M}{V-u} s \overset{s\to 0}{\longrightarrow} 1 \tag{C.27}$$

and

$$\frac{R^2}{X^2} \approx 1 + 4\frac{R_M}{V-U} s \ln s - 4\frac{R_M}{V-U} \ln\left(\frac{V-u}{2R_M}\right) s + \mathcal{O}(s^2) \overset{s\to 0}{\longrightarrow} 1. \tag{C.28}$$

The last two approximations show that the CCR metric (C.2) in region Z_+ is a perturbation of the Minkowski metric of the order of $s \ln s$. The conditions for the applicability of DeWitt's approximation are thus satisfied in region Z_+ if s is chosen appropriately.

C.3.3 Region II, M_+

In the wedge M_+ the metric functions are given by the components

$$X = \frac{V-U}{2}, \quad R = \frac{V-U-c}{2}, \quad A = 1, \quad \frac{R^2}{X^2} = 1 - \frac{2c}{V-U} + \frac{c^2}{V-U}. \tag{C.29}$$

These are almost the components of the Minkowski metric, the only difference being the occurrence of the constant c. Since $c = 4E \ln\left|\frac{R_M}{2E} - 1\right|$, it can be expanded in powers of s. Hence, according to eq. (C.12), it holds that $c \approx -2R_M s \ln s + \mathcal{O}(s^2)$. Thus, the components of the CCR metric in M_+ are only a small perturbation of the Minkowski metric and approach it when $s \to 0$. DeWitt's condition is thus satisfied in M_+.

C.3.4 Region II, K_+

In order to find the metric components in this wedge we need the line element (6.58) from chapter 6. By comparison of this line element with the components of the flat DN reference metric, the functions F, G and H are found to be

$$F = -\frac{V_{,\tilde{U}}}{V_{,\tilde{V}}}, \quad G = \frac{1}{2V_{,\tilde{V}}}, \quad H = 0. \tag{C.30}$$

By inserting these quantities into eq. (C.3), the metric components of the Cartesian form are obtained:

$$Q = \frac{1 - V_{,\tilde{U}}}{V_{,\tilde{V}}}, \quad W = -\frac{1 + V_{,\tilde{U}}}{V_{,\tilde{V}}}, \quad J = 2\frac{V_{,\tilde{U}}}{V_{,\tilde{V}}}, \quad R = \frac{\tilde{V}(V,U) - U - \tilde{u} + \bar{u}}{2}. \tag{C.31}$$

Using the expansions from the second section and the estimates from appendix D, we can discuss the behaviour of the metric functions for small s. We first need to know the expressions for the derivatives of V, (V being given by eq. (6.36)), with respect to the reference coordinates,

$$V_{,\tilde{U}} = (L-c)q_{+,\tilde{U}} = -(L-c)\frac{\psi(x=\tilde{U}-z;a,z)}{K}, \tag{C.32}$$

$$\begin{aligned} V_{,\tilde{V}} &= 1+(1+q_+)L_{,\tilde{V}}+(L-c)q_{+,\tilde{V}} \\ &= 1+(1+q_+)L_{,\tilde{V}}+\frac{(L-c)\psi}{2K}\left[(\alpha-\beta)\frac{\tilde{U}-z}{a}+\alpha+\beta\right], \end{aligned} \tag{C.33}$$

where eqs. (D.7) and (D.33) from the appendix D have been used. Recalling that in K_+ the quantity a is given by

$$a = \frac{\alpha-\beta}{2}(\tilde{V}-\tilde{v}) = (\alpha-\beta)R_M\eta_+, \tag{C.34}$$

the expansion of $L-c$, eq. (C.10), and the estimate (D.21), we find that

$$|V_{,\tilde{U}}| < \frac{(L-c)e^{\frac{1}{3}}}{(\alpha-\beta)R_M\eta_+} \approx \frac{2e^{\frac{1}{3}}}{(\alpha-\beta)}\frac{\ln(\gamma_+)}{(\gamma_+-1)}s+\mathcal{O}(s^2) \overset{s\to 0}{\longrightarrow} 0, \tag{C.35}$$

hence $V_{,\tilde{U}}$ is controlled by the behaviour of s. For $V_{,\tilde{V}}$, the estimate reads:

$$|V_{,\tilde{V}}| < 1+L_{,\tilde{V}}+\frac{3(L-c)e^{\frac{1}{3}}}{2(\alpha-\beta)R_M\eta_+} \approx 1+\left[\frac{1}{\gamma_+}+\frac{3e^{\frac{1}{3}}}{(\alpha-\beta)}\frac{\ln(\gamma_+)}{(\gamma_+-1)}\right]s+\mathcal{O}(s^2) \overset{s\to 0}{\longrightarrow} 1, \tag{C.36}$$

where we have used eqs. (C.10, C.11 and D.20). Also $V_{,\tilde{V}}$ can be expanded in the parameter s. Therefore, V is a perturbation of $\tilde{V}$ of the order of magnitude of s.

Since both derivatives of V are controlled by s, this is also true for the metric components. For Q, W and J we thus find that

$$Q \approx 1+\mathcal{O}(s), \quad W \approx -1+\mathcal{O}(s), \quad J \approx 0+\mathcal{O}(s). \tag{C.37}$$

R is slightly more tricky because we do not know the explicit form of the function $\tilde{V}(U,V)$. But we can expand V in terms of s by using eqs. (C.9) and (C.10):

$$V \approx \tilde{V}-\tilde{u}+\bar{u}-2R_M s\ln s+2R_M(q_++1)\ln(\gamma_+)s+\mathcal{O}(s^2). \tag{C.38}$$

For small s we can approximately solve this equation for $\tilde{V}$:

$$\tilde{V} \approx V + \tilde{u} - \bar{u} + \mathcal{O}(s \ln s), \tag{C.39}$$

hence, according to eq. (C.31),

$$R \approx \frac{V-U}{2} + \mathcal{O}(s \ln s), \tag{C.40}$$

which leads to

$$\frac{R^2}{X^2} \approx 1 + \mathcal{O}(s \ln s). \tag{C.41}$$

Since $s \ln s$ is small for small s, all Cartesian metric components are controlled by the parameter s and satisfy, therefore, the DeWitt condition.

We have already shown in section 6 of chapter 6 that the components of the CR metric are smooth at the boundaries of the wedge. Since the transformation from the UV to the Cartesian coordinates is smooth, this property is inherited by the components of the CCR metric.

C.3.5 Region III, Z_-

Similarly to the treatment in region Z_+, we obtain the metric functions $X = \frac{V-U}{2}$,

$$R = \frac{V-v}{2} + 2E\kappa\left(e^{\frac{v-U}{4E}}\right) \tag{C.42}$$

and

$$A = 1 - \frac{1}{\kappa\left(e^{\frac{v-U}{4E}}\right)}. \tag{C.43}$$

The components of the Cartesian metric in (C.2) can thus be easily computed. Their asymptotic expansion for small s is, similar to that of the components in the region Z_+, given by:

$$A \approx 1 - \frac{2R_M}{v-U}s \overset{s\to 0}{\longrightarrow} 1 \tag{C.44}$$

and

$$\frac{R^2}{X^2} \approx 1 + 4\frac{R_M}{V-U}s\ln s + 4\frac{R_M}{V-U}\ln\left(\frac{v-U}{2R_M}\right)s + \mathcal{O}(s^2) \overset{s\to 0}{\longrightarrow} 1. \tag{C.45}$$

Hence, the Cartesian metric in Z_- does satisfy DeWitt's condition in the region Z_-.

C.3.6 Region III, M_-

As in the region M_+ we obtain the c-shifted Minkowski line element that satisfies DeWitt's condition for appropriate s.

C.3.7 Region III, K_-

The components of the CCR line element (C.3) read

$$Q = \frac{1 - U_{,\tilde{V}}}{U_{,\tilde{U}}}, \quad W = -\frac{1 + U_{,\tilde{V}}}{U_{,\tilde{U}}}, \quad J = -2\frac{U_{,\tilde{V}}}{U_{,\tilde{U}}}, \quad R = \frac{V + \tilde{v} - \bar{v} - \tilde{U}(U,V)}{2}. \tag{C.46}$$

In these equations, the coordinate U is given by (6.52). Its derivatives read (using eqs. (D.13, D.35 and D.36) from the appendix D):

$$U_{,\tilde{V}} = -(L' - c')\frac{\psi(\tilde{V} - y)}{K} \tag{C.47}$$

and

$$U_{,\tilde{U}} = 1 + L'_{,\tilde{U}} q_- + \frac{(L' - c')\psi(\tilde{V} - y)}{2K}\left[(\alpha - \beta)\frac{\tilde{V} - y}{a} + \alpha + \beta\right]. \tag{C.48}$$

They have the estimates

$$|U_{,\tilde{V}}| < \frac{(L' - c')e^{\frac{1}{3}}}{(\alpha - \beta)R_M\eta_-} \approx \frac{2e^{\frac{1}{3}}}{(\alpha - \beta)}\frac{\ln(\gamma_-)}{(\gamma_- - 1)}s + \mathcal{O}(s^2) \overset{s\to 0}{\longrightarrow} 0, \tag{C.49}$$

$$|U_{,\tilde{U}}| < 1 + L'_{,\tilde{U}} + \frac{3(L' - c')e^{\frac{1}{3}}}{2(\alpha - \beta)R_M\eta_-} \approx 1 + \left[\frac{1}{\gamma_-} + \frac{3e^{\frac{1}{3}}}{(\alpha - \beta)}\frac{\ln(\gamma_-)}{(\gamma_- - 1)}\right]s + \mathcal{O}(s^2) \overset{s\to 0}{\longrightarrow} 1, \tag{C.50}$$

where we have used the eqs. (C.13, C.14), the estimates from appendix D, eqs. (D.37, D.38), and that

$$a = \frac{\alpha - \beta}{2}(\tilde{U} - \tilde{u}) = -(\alpha - \beta)R_M\eta_- \tag{C.51}$$

in the wedge K_-. Thus, the derivatives have a behaviour corresponding to those of region K_+. The consequence is that the metric components in K_- are all perturbations of the Minkowski metric components controlled by the parameter s and hence satisfy the DeWitt condition.

C.4 Conclusion

We conclude that the CCR metric is everywhere a small perturbation of the Minkowski metric, if the parameter s is small enough, and that in the limit $s \to 0$ it converges to $\eta_{\mu\nu}$. Since the metric is also continuous at the shell and smooth everywhere else, it satisfies the conditions for DeWitt's method. We thus have proven the existence theorem (9.1) by constructing an explicit example.

Appendix D

The interpolation function

D.1 Definition

Let $\omega(x)$ be a function on the interval $[a, b]$, with $a < b$. We call ω a C^∞-*interpolation* on $[a, b]$ if the following properties hold:

- ω is C^∞ on $[a, b]$,
- $\omega(a) = A$, $\omega(b) = B$, for some A and B.

If one considers the change of a metric under a regular coordinate transformation, then one observes that the transformed metric components involve the first derivatives of the new coordinates with respect to the old ones. Hence, in order that the metric satisfy the conditions for being suitable for DeWitt's semi-classical theory, the first derivatives must fulfill certain conditions. Roughly, these requirements boil down to the condition, that the first derivatives do not have too large values for a given small parameter, i.e. $s = \frac{2E}{R_M}$ in our case. But in the interpolating regions $K_\pm$, derivatives of the new coordinates involve derivatives of an C^∞-interpolation ω, so we need a means to control the behaviour of $\omega_{,\tilde{U}}$ and $\omega_{,\tilde{V}}$. Control of the derivative of ω is available if

$$\max_{[a,b]} |\omega'(x)| < k\frac{|B - A|}{b - a} \tag{D.1}$$

for some positive constant k with order of magnitude 1. In the next paragraphs we construct the interpolating functions $\omega_\pm$ with this property for the coordinates in the wedges $K_\pm$.

D.2 The function q

We make use of the fact that the function

$$\psi(x) \doteq \begin{cases} e^{\frac{a^2}{(x-a)(x+a)}} & : \quad x \in (-a, a), \\ 0 & : \quad x \leq -a \quad \text{or} \quad x \geq a \end{cases} \tag{D.2}$$

is C^∞. This bell-shaped function is symmetric with respect to the ψ-axis and has its maximum at $x = 0$:

$$\max_{[-a,a]} \psi(x) = \psi(x = 0) = e^{-1}. \tag{D.3}$$

Its integral is also C^∞ and strictly monotonously increases from 0 at $x = -a$ to some finite maximal value K at $x = a$. For $x > a$, the integral has the constant value K. We denote this integral by

$$\phi(x) \doteq \int_{-a}^{x} dy\, \psi(y). \tag{D.4}$$

K, the area under the curve $\psi(x)$ on the interval $[-a, a]$, is given by

$$K \doteq \int_{-a}^{a} dy\, \psi(y) = \phi(a). \tag{D.5}$$

Then,

$$q \doteq -\frac{\phi(x)}{K} \tag{D.6}$$

is a C^∞-interpolation on $[-a, a]$, where the quotient K normalizes q to -1. The values of q at $x = \pm a$ are given by $q(-a) = 0$ and $q(a) = -1$, hence q interpolates between $A = 0$ and $B = -1$. The function q will we needed to construct the interpolating functions $\omega_\pm$ in the next subsections.

The derivative of q with respect to x reads

$$q' = -\frac{\psi(x)}{K}. \tag{D.7}$$

The normalization factor K is bounded from both sides: an upper bound is clearly given by the area of the rectangle $[-a, a] \times [0, e^{-1}]$. A lower bound can be found by computing the value of ψ at the points $x = \pm\frac{a}{2}$:

$$\psi(x = \pm\frac{a}{2}) = e^{-\frac{4}{3}}, \tag{D.8}$$

It follows that the rectangle $[-\frac{a}{2}, \frac{a}{2}] \times [0, e^{-\frac{4}{3}}]$ in the $(x, \psi(x))$-plane lies under the curve $\psi(x)$ for all $x \in [-a, a]$, its area is hence a lower bound for K. Bounds for K are, therefore, given by

$$ae^{-\frac{4}{3}} < K < 2ae^{-1}. \tag{D.9}$$

They are needed to estimate quantities involving $\frac{1}{K}$.

D.3 Interpolating function ω_+ for K_+

ω_+ is to be a function of the two reference coordinates $\tilde{U}, \tilde{V}$, interpolating on the interval $\tilde{U} \in [\tilde{U}_{l_+}, \tilde{U}_{k_+}]$ between $\omega_+|_{l_+} = 0$ and $\omega_+|_{k_+} = -(L - c)$. With these properties of ω_+, the function V in K_+ given by eq. (6.36) interpolates between the desired values of V, i.e. $V|_{l_+} = \tilde{V} - \tilde{u} + \bar{u} + L$ and $V|_{k_+} = \tilde{V} - \tilde{u} + \bar{u} + c$.

We take the function q defined in eq. (D.6) of the last section and replace the parameters x and a by

$$x = \tilde{U} - z, \quad z \doteq \frac{\tilde{U}_{k_+} + \tilde{U}_{l_+}}{2}, \quad a = \frac{\tilde{U}_{k_+} - \tilde{U}_{l_+}}{2}. \tag{D.10}$$

By these replacements, the centre of the interpolation function is shifted to the right place, and the correct interval, on which ω_+ is defined, is taken. q interpolates between 0 and -1, but we want it to interpolate between 0 and $-(L - c)$. Hence, it has to be multiplied with $L - c$ to yield the desired ω_+:

$$\omega_+ \doteq (L - c)q_+ = (L - c)\, q(x = \tilde{U} - z; a(\tilde{V}), z(\tilde{V})). \tag{D.11}$$

D.4 Interpolating function ω_- for K_-

The construction of ω_- is completely analogous to that of ω_+. The coordinate U in region K_- is defined by eq. (6.52), where $\omega_- = (L' - c')q_-$. We again take the function q and replace the parameters x and a by

$$x = \tilde{V} - y, \quad y \doteq \frac{\tilde{V}_{l_-} + \tilde{V}_{k_-}}{2}, \quad a = \frac{\tilde{V}_{l_-} - \tilde{V}_{k_-}}{2}, \tag{D.12}$$

where $\tilde{V}_{l_-}$ and $\tilde{V}_{k_-}$ are defined by eqs. (6.44) and (6.49), respectively. This q_- interpolates again between 0 and -1 on the interval $[\tilde{V}_{k_-}, \tilde{V}_{l_-}]$, but we want it to interpolate between 0 and $-(L' - c')$. Hence, it has to be multiplied with $L' - c'$ to yield the ω_- looked for:

$$\omega_- \doteq (L' - c')q_- = (L' - c')\, q(x = \tilde{V} - y; a(\tilde{U}), y(\tilde{U})). \tag{D.13}$$

D.5 Some useful estimates

The following estimates will be useful in the next sections: From (D.9) follows that

$$\frac{1}{K} < \frac{e^{\frac{4}{3}}}{a}. \tag{D.14}$$

The function ψ is of course bounded by its maximum, (D.3):

$$|\psi| \leq \frac{1}{e}. \tag{D.15}$$

The definition of the two numbers α and β has been given in eqs. (6.17) and (6.18), hence the estimates

$$\alpha + \beta < 2, \quad 0 < \alpha - \beta < 1 \tag{D.16}$$

are surely valid. In the wedge K_+ it holds that

$$|\tilde{U} - z| < a, \tag{D.17}$$

whereas in region K_- the following estimate is satisfied:

$$|\tilde{V} - y| < a. \tag{D.18}$$

From (D.14) and (D.15) one obtains that

$$|q'| < \frac{e^{\frac{1}{3}}}{a}. \tag{D.19}$$

Useful bounds for q are given by

$$0 \leq 1 + q \leq 1. \tag{D.20}$$

D.6 Derivatives of q_+, estimates

In order to estimate the derivatives of ω_+ with respect to the reference coordinates $\tilde{U}$ and $\tilde{V}$ in K_+, we need estimates for the derivatives of q_+. The derivative with respect to $\tilde{U}$ is simply q' with the replacements from eq. (D.10) and is hence, using eq. (D.19), easily estimated:

$$|q_{+,\tilde{U}}| < \frac{e^{\frac{1}{3}}}{a} \tag{D.21}$$

The derivative with respect to $\tilde{V}$ is more intricate. We first observe that q_+ depends on $\tilde{V}$ only through $a(\tilde{V})$ and $z(\tilde{V})$, hence,

$$q_{+,\tilde{V}} = q_{+,a} a_{,\tilde{V}} + q_{+,z} z_{,\tilde{V}}. \tag{D.22}$$

The derivatives of a and z are straightforwardly found to be

$$a_{,\tilde{V}} = \frac{\alpha - \beta}{2}, \quad z_{,\tilde{V}} = \frac{\alpha + \beta}{2}. \tag{D.23}$$

This follows from eq. (D.10) and the definition of $\tilde{U}_{k_+}$ and $\tilde{U}_{l_+}$. Their estimates are found using eq. (D.16):

$$|a_{,\tilde{V}}| < \frac{1}{2}, \quad |z_{,\tilde{V}}| < 1. \tag{D.24}$$

The derivative of q_+ with respect to z reads

$$q_{+,z} = \frac{\psi(\tilde{U} - z)}{K}. \tag{D.25}$$

It can be estimated using (D.14) and (D.15):

$$|q_{+,z}| < \frac{e^{\frac{1}{3}}}{a}. \tag{D.26}$$

The derivative with respect to a is given by

$$q_{+,a} = -\frac{1}{K} \int_{-a}^{\tilde{U}-z} d\xi \, \frac{\partial \psi}{\partial a} + \frac{\phi(\tilde{U} - z)}{K^2} \int_{-a}^{a} d\xi \, \frac{\partial \psi}{\partial a}. \tag{D.27}$$

We observe that

$$\frac{\partial \psi}{\partial a} = \frac{2a^2 \xi \psi}{(\xi^2 - a^2)^2} = -\frac{\xi}{a} \frac{\partial \psi}{\partial \xi}, \tag{D.28}$$

so the integrals in eq. (D.27) can be evaluated by integration by parts, yielding

$$\int_{-a}^{a} d\xi \, \frac{\partial \psi}{\partial a} = \frac{K}{a} \tag{D.29}$$

and

$$\int_{-a}^{\tilde{U}-z} d\xi \, \frac{\partial \psi}{\partial a} = \frac{\phi(\tilde{U} - z)}{a} - \frac{\tilde{U} - z}{a} \psi(\tilde{U} - z). \tag{D.30}$$

Hence,

$$q_{+,a} = \frac{\tilde{U} - z}{aK} \psi(\tilde{U} - z). \tag{D.31}$$

Using eqs. (D.14, D.15 and D.17) its estimate is found to be again

$$|q_{+,a}| < \frac{e^{\frac{1}{3}}}{a}. \tag{D.32}$$

Collecting these results, we obtain for $q_{+,\tilde{V}}$ the formula

$$q_{+,\tilde{V}} = \frac{\psi(\tilde{U} - z)}{2K} \left((\alpha - \beta) \frac{\tilde{U} - z}{a} + \alpha + \beta \right), \tag{D.33}$$

while an estimate thereof is given by

$$|q_{+,\tilde{V}}| < \frac{3e^{\frac{1}{3}}}{2a}. \tag{D.34}$$

We observe that both $|q_{+,\tilde{U}}|$ and $|q_{+,\tilde{V}}|$ are bounded by $\frac{1}{a}$ times a constant of order of magnitude 1.

D.7 Derivatives of q_-, estimates

The analogous treatment of q_- leads to the final result, that

$$q_{-,\tilde{V}} = -\frac{\psi(\tilde{V} - y)}{K} \tag{D.35}$$

and

$$q_{-,\tilde{U}} = \frac{\psi(\tilde{V} - y)}{2K} \left((\alpha - \beta) \frac{\tilde{V} - y}{a} + \alpha + \beta \right). \tag{D.36}$$

Hence, the estimates for the derivatives of q_- read

$$|q_{-,\tilde{V}}| < \frac{e^{\frac{1}{3}}}{a} \tag{D.37}$$

and

$$|q_{-,\tilde{U}}| < \frac{3e^{\frac{1}{3}}}{2a}. \tag{D.38}$$

Appendix E

The Kruskal function

We collect some useful results on the Kruskal function denoted by κ. $\kappa(x)$ is defined implicitly by its inverse:

$$x = \kappa^{-1}(z) = (z-1)e^z. \tag{E.1}$$

For $x = e^y$, it holds that $z = \kappa(e^y)$, so the following equation holds:

$$y = z + \ln|z-1|. \tag{E.2}$$

κ is strictly monotonously growing on the interval $x \in [-1, \infty)$, taking the following values at special points:

$$\kappa(-1) = 0, \quad \kappa(0) = 1, \quad \kappa(\infty) = \infty. \tag{E.3}$$

The derivative of κ can be determined by the derivative of its inverse and using the inverse function theorem:

$$(\kappa^{-1})'(z) = ze^z = \frac{1}{\kappa'(x)} \Longrightarrow \kappa'(x) = \frac{1}{\kappa(x)e^{\kappa(x)}}. \tag{E.4}$$

If $x = e^y$,

$$\kappa_{,y}(e^y) = 1 - \frac{1}{z}, \quad z = \kappa(e^y). \tag{E.5}$$

The Kruskal function is closely related to Lambert's W function $W(q)$ that is defined implicitly by:

$$W(q)e^{W(q)} = q, \tag{E.6}$$

the relation being

$$\kappa(x) = 1 + W(xe^{-1}). \tag{E.7}$$

There is an abundant literature about the Lambert W function. We are interested in the asymptotic expansion of κ for large x. The asymptotic expansion of W can be found in the literature (e.g. [CGH+96]), the three first terms being

$$W(q) \approx \ln q - \ln(\ln q) + \frac{\ln(\ln q)}{\ln q}. \tag{E.8}$$

The asymptotic expansion of κ follows thus from eq. (E.7):

$$\kappa(x) \approx \ln x - \ln(\ln(x) - 1) + \frac{\ln(\ln(x) - 1)}{\ln(x) - 1}. \tag{E.9}$$

Most interesting is the case $x = e^y$. The first two terms of the expansion then read

$$\kappa(e^y) \approx y - \ln(y - 1). \tag{E.10}$$

Appendix F

Self-adjoint extensions of symmetric operators

We briefly review a very useful method to check whether a symmetric operator has a self-adjoint extension, and if, how many there are. We introduce the concept of *deficiency indices* and state *von Neumann's theorem*, cf. e.g. [RS75, BFV01]. We then demonstrate that the operator $\hat{v}$ used in chapter 8 has no self-adjoint extension, but that its square is self-adjoint. The first three sections closely follow the treatment in [BFV01].

F.1 Deficiency indices and von Neumann's theorem

The deficiency indices of an operator $\hat{P}$ are an ordered pair of positive integers (n_+, n_-). Its knowledge, upon the use of von Neumann's theorem, immediately gives the answer to the question how many self-adjoint extensions the operator $\hat{P}$ does admit. Let $\mathcal{H}$ be a Hilbert space. An operator $(\hat{P}, \mathcal{D}(\hat{P}))$, where $\mathcal{D}(\hat{P})$ is the domain of the operator $\hat{P}$ in $\mathcal{H}$, is said to be

- *densely defined* if the subset $\mathcal{D}(\hat{P})$ is dense in $\mathcal{H}$,
- *closed* if ϕ_n is a sequence in $\mathcal{D}(\hat{P})$ such that $\lim_{n\to\infty} \phi_n = \phi$, $\lim_{n\to\infty} \hat{P}\phi_n = \psi$ then $\phi \in \mathcal{D}(\hat{P})$ and $\hat{P}\phi = \psi$,
- *symmetric* if $\forall \phi, \psi \in \mathcal{D}(\hat{P})$ it holds that $(\hat{P}\phi, \psi) = (\phi, \hat{P}\psi)$.
- *self-adjoint* if it is densely defined, symmetric and it holds that $\mathcal{D}(\hat{P}^\dagger) = \mathcal{D}(\hat{P})$ and $\hat{P}^\dagger = \hat{P}$, where $\hat{P}^\dagger$ is the *adjoint* of $\hat{P}$.

Let us assume that the operator $(\hat{P}, \mathcal{D}(\hat{P}))$ is densely defined, closed, symmetric and let $(\hat{P}^\dagger, \mathcal{D}(\hat{P}^\dagger))$ be its adjoint.

One defines the *deficiency subspaces* $N_\pm$ by

$$\begin{aligned} N_+ &\doteq \{\psi \in \mathcal{D}(\hat{P}^\dagger), \quad \hat{P}^\dagger\psi = z_+\psi \quad \Im(z_+) > 0\}, \\ N_- &\doteq \{\psi \in \mathcal{D}(\hat{P}^\dagger), \quad \hat{P}^\dagger\psi = z_-\psi \quad \Im(z_-) < 0\}, \end{aligned} \tag{F.1}$$

with the respective dimensions n_+, n_- that are the *deficiency indices*. The crucial point is that the dimension is independent on the choice of the complex number $z_\pm$ as long as it lies in the upper/lower half-plane. The determination of the deficiency indices thus boils down to the counting of how many solutions of the equation $\hat{P}^\dagger\psi = z\psi$ have finite norm. The following theorem by von Neumann [vN29] determines the number of self-adjoint extensions from the knowledge of the deficiency indices.

Theorem F.1 (von Neumann, 1929) *For an operator $\hat{P}$ with the deficiency indices (n_+, n_-) there are three possibilities:*

1. *If $n_+ = n_- = 0$, then $\hat{P}$ is self-adjoint.*
2. *If $n_+ = n_- = n \geq 1$, then $\hat{P}$ has infinitely many self-adjoint extensions, parametrized by a unitary $n \times n$ matrix.*
3. *If $n_+ \neq n_-$, then $\hat{P}$ has no self-adjoint extension.*

A self-adjoint operator has only real eigenvalues. Thus, if the equation $\hat{P}^\dagger\psi = z\psi$ has solutions in the Hilbert state for which z has a non-vanishing imaginary part, then $\hat{P}$ is clearly not self-adjoint. But if the dimensions of the two deficiency subspaces are equal, then the operator can be made self-adjoint by imposing suitable additional boundary conditions on the Hilbert states.

F.2 The momentum operator

We now apply this analysis to the momentum operator $\hat{k} = -i\partial_x$ on the Hilbert space $\mathcal{H} = \mathcal{L}^2(a, b)$, where $a, b \in \mathbb{R} \cup \{-\infty, \infty\}$. To use von Neumann's theorem we have to find the functions $\psi_\pm(x)$ that are the solutions of the equations

$$\hat{k}^\dagger\psi_\pm(x) = -i\partial_x\psi_\pm(x) = \pm\frac{i}{\lambda}\psi_\pm(x), \tag{F.2}$$

where $\lambda > 0$ has the dimension of a length. The solutions of the eqs. (F.2) are of course

$$\psi_\pm(x) = C_\pm e^{\mp\frac{x}{\lambda}}. \tag{F.3}$$

Now we have to discuss the different intervals $[a, b]$.

A. The momentum operator on $\mathbb{R}$: None of the functions (F.3) belongs to the Hilbert space $\mathcal{L}^2(\mathbb{R})$. Thus, the deficiency indices are $(0,0)$, as $\hat{k}$ possesses no non-real eigenvalues. According to the theorem the momentum operator is hence self-adjoint.

B. The momentum operator on the positive half-axis: Only ψ_+ belongs to the Hilbert space $\mathcal{L}^2(0,\infty)$, hence the deficiency indices are $(1,0)$. Thus, by von Neumann's theorem, there is no self-adjoint extension of $\hat{k}$ on the interval $[0,\infty)$. This implies that the momentum is not a measurable quantity in this situation.

C. The momentum operator on a finite interval: Both of the solutions (F.3) are in the corresponding Hilbert space, hence the deficiency indices are $(1,1)$. Thus, there is a one-parameter family of self-adjoint extensions.

F.3 The Laplacian

We make the same analysis for the Laplacian $\Delta = -\partial_x^2$ on the Hilbert space $\mathcal{L}^2(a,b)$. To compute the deficiency indices one has to solve the equation

$$-\partial_x^2 \phi_\pm(x) = \pm k_0^2 \phi_\pm(x), \quad k_0 > 0, \tag{F.4}$$

the solution being

$$\phi_\pm(x) = a_\pm e^{k_\pm x} + b_\pm e^{-k_\pm x}, \quad k_\pm \doteq \frac{1 \mp i}{\sqrt{2}} k_0. \tag{F.5}$$

A. The Laplacian on $\mathbb{R}$: The deficiency indices are $(0,0)$, hence the Laplacian is self-adjoint and uniquely defined on $\mathcal{L}^2(\mathbb{R})$, as expected.

B. The Laplacian on the positive half-axis: The deficiency indices are $(1,1)$, hence there is a one-parameter family of self-adjoint extensions. The corresponding boundary conditions can be expressed as

$$\phi(0) = \nu \phi'(0), \quad \nu = -\tan\left(\frac{\alpha}{2}\right), \quad \alpha \in [0, 2\pi]. \tag{F.6}$$

C. The Laplacian on a finite interval: In this case the deficiency indices are $(2,2)$, hence the self-adjoint extensions are parametrized by a $U(2)$ matrix.

F.4 The operator $\hat{v}$

In our system the observable v plays the role of the conjugate momentum of the energy E. It is thus straightforward to represent v by the momentum operator $-i\partial_p$, where $\hat{E}\phi = p\phi$ in the energy representation, in the corresponding quantum theory. But in chapter 8 we have observed that there are problems with this interpretation because the domain of E is restricted to the positive semi-axis. According to the results of section 2 of this appendix, $\hat{v}$ thus has no self-adjoint extension on the Hilbert space $\mathcal{L}^2(0,\infty)$. We hence, in chapter 8, took a different pair of observables to construct a well-defined quantum theory, the energy E and the dilatation $D = Ev$. The appropriate scalar product for this choice is

$$(\phi,\psi) = \int_0^\infty \frac{dp}{p}\bar{\phi}(p)\psi(p). \tag{F.7}$$

The corresponding Hilbert space $\mathcal{K}$ consists of the complex functions on the interval $[0,\infty)$ for that the norm derived from this scalar product is finite. The dilatation operator $\hat{D}$ is self-adjoint on this Hilbert space, as can be easily checked. However, it has no self-adjoint extension if the standard scalar product is taken.

We then tried to represent the observable v by the symmetric operator

$$\hat{v} = \frac{1}{\hat{E}^{\frac{1}{2}}}\hat{D}\frac{1}{\hat{E}^{\frac{1}{2}}} = \frac{i}{2p} - i\partial_p. \tag{F.8}$$

We now show that this operator also fails to have a self-adjoint extension. For this end we solve the equations

$$\hat{v}^\dagger\psi_\pm(p) = \pm\frac{i}{a}\psi_\pm(p), \quad a > 0. \tag{F.9}$$

The solutions read:

$$\psi_\pm(p) = C_\pm\sqrt{p}e^{\mp\frac{p}{a}}. \tag{F.10}$$

Of course the solution ψ_+ has finite norm with respect to the scalar product (F.7), but this is not true for ψ_- which diverges at infinity. Hence, the deficiency indices are $(1,0)$ which means that the operator $\hat{v}$ has no self-adjoint extension.

We now demonstrate that the square of the operator given by eq. (F.8), however, is uniquely self-adjoint on the Hilbert space $\mathcal{K}$. We denote this operator by $\hat{Q}$. In the energy representation it is given by the differential operator

$$\hat{Q} = -\frac{3}{4p^2} + \frac{\partial_p}{p} - \partial_p^2. \tag{F.11}$$

In order to find the deficiency indices we have to solve the differential equation

$$\hat{Q}\psi_{\pm}(p) = -\frac{3}{4p^2}\psi_{\pm}(p) + \frac{1}{p}\partial_p\psi_{\pm}(p) - \partial_p^2\psi_{\pm}(p) = \pm ia^2\psi_{\pm}(p), \quad a > 0. \tag{F.12}$$

The solutions read

$$\psi_+(p) = C_{1+}\sqrt{p}\sin\left(\frac{1+i}{\sqrt{2}}ap\right) + C_{2+}\sqrt{p}\cos\left(\frac{1+i}{\sqrt{2}}ap\right) \tag{F.13}$$

and

$$\psi_-(p) = C_{1-}\sqrt{p}\sinh\left(\frac{1+i}{\sqrt{2}}ap\right) + C_{2-}\sqrt{p}\cosh\left(\frac{1+i}{\sqrt{2}}ap\right). \tag{F.14}$$

One then can show that both of these functions have infinite norm, such that the deficiency indices are $(0,0)$. Hence, $\hat{Q}$ is self-adjoint.

By applying the spectral theorem one could hence define a self-adjoint operator $\hat{v}$ by taking the square root of $\hat{Q}$. But one must be careful with the signs, because the spectrum of $\hat{v}$ must be the whole real line. The solution of these problems is postponed to a later time.

Appendix G

Invariant classification

G.1 Introduction

This appendix is devoted to the invariant classification of the shell-mirror spacetime. It is shown that in the case of our null-shell-spacetime there are not enough non-vanishing independent invariants such that one could express the metric in terms of them. The Petrov and Segré types in all regions of spacetime are determined and the maximal possible subset of invariants in each of the regions according to Zakhary and McIntosh [ZM97] is found. It is demonstrated that even at the shell there is only one non-vanishing independent algebraic invariant, in contrast to the expectation of at least two. The last sections are devoted to the invariant classification due to Takeno, that, however, breaks down in the point where the shell is reflected by the mirror. Some useful results about derivatives at surface layers are written down.

G.2 Petrov type

We first determine the Petrov type of our model using the methods presented in e.g. [KSMH80] and [Ste77], cf. also the extensive literature cited therein.

G.2.1 Weyl tensor

The *Petrov classification* is the discrimination of Riemannian spaces with respect to the algebraic properties of the *Weyl conformal tensor*

$$\begin{aligned} C_{\mu\nu\rho\sigma} \doteq\ & R_{\mu\nu\rho\sigma} - \frac{1}{2}\left(g_{\mu\rho}R_{\nu\sigma} - g_{\nu\rho}R_{\mu\sigma} + g_{\nu\sigma}R_{\mu\rho} - g_{\mu\sigma}R_{\nu\rho}\right) \\ & + \frac{\mathcal{R}}{6}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right). \end{aligned} \tag{G.1}$$

The Weyl tensor is completely traceless, i.e. the contraction with respect to each pair of indices vanishes. It thus has 10 independent components. A spacetime with zero Weyl tensor is called conformally flat. $C_{\mu\nu\rho\sigma}$ has the same symmetries as the Riemann tensor.

G.2.2 Tetrad components

We introduce the *complex null tetrad* consisting of the two real null vectors k^μ, l^μ and the two complex conjugate null vectors m^μ, $\bar{m}^\mu$. Its components are defined by the following decomposition of the metric tensor:

$$g_{\mu\nu} = k_\mu l_\nu + k_\nu l_\mu - m_\mu \bar{m}_\nu - m_\nu \bar{m}_\mu \tag{G.2}$$

and by the requirement that the scalar products of the tetrad vectors vanish apart from

$$k^\mu l_\mu = 1, \quad m^\mu \bar{m}_\mu = -1. \tag{G.3}$$

With the help of this tetrad we can define the five complex coefficients in the expansion of the Weyl tensor (see [KSMH80] for details):

$$\begin{aligned}
\Psi_0 &\doteq C_{\mu\nu\rho\sigma} k^\mu m^\nu k^\rho m^\sigma, \\
\Psi_1 &\doteq C_{\mu\nu\rho\sigma} k^\mu l^\nu k^\rho m^\sigma, \\
\Psi_2 &\doteq \frac{1}{2} C_{\mu\nu\rho\sigma} k^\mu l^\nu (k^\rho l^\sigma - m^\rho \bar{m}^\sigma), \\
\Psi_3 &\doteq C_{\mu\nu\rho\sigma} l^\mu k^\nu l^\rho \bar{m}^\sigma, \\
\Psi_4 &\doteq C_{\mu\nu\rho\sigma} l^\mu \bar{m}^\nu l^\rho \bar{m}^\sigma.
\end{aligned} \tag{G.4}$$

The *traceless part of the Ricci-tensor* is defined by

$$S_{\mu\nu} \doteq R_{\mu\nu} - \frac{\mathcal{R}}{4} g_{\mu\nu}. \tag{G.5}$$

We also define the following complex quantities:

$$
\begin{aligned}
\Phi_{00} = \bar{\Phi}_{00} &\doteq \frac{1}{2} S_{\mu\nu} k^\mu k^\nu, \\
\Phi_{01} = \bar{\Phi}_{10} &\doteq \frac{1}{2} S_{\mu\nu} k^\mu m^\nu, \\
\Phi_{02} = \bar{\Phi}_{20} &\doteq \frac{1}{2} S_{\mu\nu} m^\mu m^\nu, \\
\Phi_{11} = \bar{\Phi}_{11} &\doteq \frac{1}{4} S_{\mu\nu} (k^\mu l^\nu - m^\mu \bar{m}^\nu), \\
\Phi_{12} = \bar{\Phi}_{21} &\doteq \frac{1}{2} S_{\mu\nu} l^\mu m^\nu, \\
\Phi_{22} = \bar{\Phi}_{22} &\doteq \frac{1}{2} S_{\mu\nu} l^\mu l^\nu.
\end{aligned} \tag{G.6}
$$

They are the tetrad components of $S_{\mu\nu}$. The set $\{\Psi_a, \Phi_{AB}\}$, $a = 0, ..., 4$, $A, B = 0, ..., 2$ is very useful for the classification of spacetimes and for building invariant scalars. It has also an important meaning in the Newman-Penrose formalism of general relativity based on complex null tetrads and spinors.

G.2.3 Determination of the Petrov type

The Petrov type at a point p of a given spacetime can be determined by calculating the roots of the quartic algebraic equation

$$\Psi_0 - 4c\Psi_1 + 6c^2\Psi_2 - 4c^3\Psi_3 + c^4\Psi_4 = 0 \tag{G.7}$$

where the coefficients Ψ_a are given with respect to an arbitrary complex null tetrad at p. The Petrov type of the spacetime in the point p is found by inspection of the multiplicity of the roots c. The possible cases are written down in the table (G.1):

Petrov type	**Multiplicities of the roots c**
I	(1,1,1,1)
II	(2,1,1)
III	(3,1)
D	(2,2)
N	(4)
O	No roots - Weyl tensor vanishes identically

Table G.1: *Petrov classification of spacetimes.*

Notice that the Petrov type is a local property which can change to a different type at a different point for certain spacetimes, including the shell spacetime. One can also use the algorithm by d'Inverno and Russell-Clark (see p.64 of [KSMH80]) to determine the Petrov type.

G.2.4 Petrov type of the regular DNEF metric

We apply the methods of the preceding paragraphs to find the Petrov type of our model in the various regions. The shell-mirror spacetime is described by the regular DNEF metric defined in chapter 5 by the eqs. (5.47, 5.48).

Region I

Since this region is a part of Schwarzschild spacetime we expect it to have the same Petrov type, i.e. type D. This can be easily verified by computing the non-vanishing independent components of the Weyl tensor,

$$\begin{aligned} C_{0101} &= \frac{E(R-2E)^2}{2R^5}, \\ C_{0212} &= -\frac{E(R-2E)}{2R^2}, \\ C_{0313} &= -\frac{E(R-2E)}{2R^2}\sin^2\theta, \\ C_{2323} &= -2ER\sin^2\theta. \end{aligned} \tag{G.8}$$

We need the complex null tetrad corresponding to the metric in DNEF coordinates. The contravariant form of the tetrad vectors reads

$$\begin{aligned} k^\mu &= (0, \sqrt{\frac{2}{A}}, 0, 0), \\ l^\mu &= (\sqrt{\frac{2}{A}}, 0, 0, 0), \\ m^\mu &= (0, 0, -\frac{1}{R}, -\frac{i}{R}\sin\theta), \\ \bar{m}^\mu &= (0, 0, -\frac{1}{R}, \frac{i}{R}\sin\theta). \end{aligned} \tag{G.9}$$

Now we are able to calculate the coefficients Ψ_a. They read:

$$\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0, \quad \Psi_2 = \frac{E}{R^3}. \tag{G.10}$$

Hence, the algorithm of d'Inverno and Russell-Clarke tells us that the DNEF metric in region I belongs to a spacetime of *Petrov type D*.

Regions II and III

The spacetime in these regions is flat, the metric being a regular transformation of the Minkowski metric. The Weyl tensor vanishes identically, and there are no non-zero coefficients. Therefore, these parts of the shell spacetime are of *Petrov type O*.

At the shell

At the outgoing shell $U = u$ the non-zero independent components of the Weyl tensor read

$$\begin{aligned} C_{0101} &= \frac{E(R-2E)^2}{4R^5}, \\ C_{2020} &= -3E\delta(0), \\ C_{3030} &= -3E\delta(0)\sin^2\theta, \\ C_{0212} &= -\frac{E(R-2E)}{4R^2}, \\ C_{0313} &= -\frac{E(R-2E)}{4R^2}\sin^2\theta, \\ C_{2323} &= -ER\sin^2\theta. \end{aligned} \tag{G.11}$$

They are found by using the methods presented in section 5 of this appendix. The coefficients Ψ_a vanish apart from

$$\Psi_2 = \frac{E}{2R^2}. \tag{G.12}$$

Although there are, compared to the region I, two new non-vanishing components of the Weyl tensor, there are no new non-vanishing tetrad components. This is because the terms proportional to the distribution $\delta(0)$ are canceled out because they are multiplied by zero terms in the Ψ_a's. Hence, also *at the shell* the spacetime is of *Petrov type D*. We observe that the coefficient Ψ_2 at the shell is the mean of the interior and exterior values. The same results hold, of course, also at the ingoing shell $V = v$.

G.3 Segré type

We determine the *Segré type* of the DNEF metric in the various regions. The Segré (or Plebański) classification is the analogue of the Petrov one for the *trace-free Ricci tensor* $S_{\mu\nu}$, eq. (G.5). We do not go into details but refer to the treatment in [KSMH80].

Since the Ricci scalar $\mathcal{R}$ vanishes everywhere for our model, as it is shown in the section 6 of this appendix, the Ricci tensor and its trace-free part coincide. In regions I, II and III the Ricci tensor is zero since the space is empty on both sides of the shell. This corresponds to the Segré type $[(111,1)]$, according to the notation of [KSMH80]. At the shell, however, there is one non-vanishing component of the Ricci tensor that is computed using the methods of section 5, i.e. eq. (G.32):

$$R_{VV}(r) = \frac{A(r)-1}{R(r)}\delta(0) \tag{G.13}$$

for the ingoing and

$$R_{UU}(r) = \frac{A(r)-1}{R(r)}\delta(0) \tag{G.14}$$

for the outgoing shell. Here, a quantity Q evaluated at the shell is denoted by $Q(r)$. Hence, by virtue of Einstein's equations, the energy momentum tensor is of the form of a pure radiation field:

$$T_{UU} = cR_{UU} = \Phi^2 k_U^2, \quad k_\mu k^\mu = 0 \tag{G.15}$$

in the outgoing and

$$T_{VV} = cR_{VV} = \Phi^2 k_V^2 \tag{G.16}$$

in the ingoing case. This tensor has the algebraic type $[(11,2)]$, see [KSMH80]. Of course, the algebraic types of $T_{\mu\nu}$ and $R_{\mu\nu}$ coincide for zero cosmological constant and zero scalar curvature.

We summarize the Petrov and Segré types for the various regions of the shell metric in the table (G.2):

Region	**Petrov type**	**Segré type**
I	D	[(111,1)]
II	O	[(111,1)]
III	O	[(111,1)]
$U = u$	D	[(11,2)]
$V = v$	D	[(11,2)]

Table G.2: *Petrov and Segré types of the regions of the shell spacetime.*

This result will be very useful in the next section.

G.4 A complete set of curvature invariants

A set of invariant scalars that uniquely characterize the spacetime geometry would be extremely useful. Having such a set of invariants in hand, it is possible to obtain useful information about the geometry without using explicit coordinates.

For a general spacetime, there are at most 14 independent algebraic invariants made from the Riemann tensor and its covariant derivatives. Since the computation of covariant derivatives can be very tedious, it is sensible to look for invariants of low degree in the derivatives of the Riemann tensor. Amongst many authors who have claimed to have found such independent sets, the set of Zakhary and McIntosh (ZM) [ZM97] seems to be the most precise. Their set contains of 17 invariants of lowest possible degree and is complete in the sense that for each Petrov and Segré type it contains a maximal subset of independent invariants. Their result is a generalization of the (non-complete) set by Carminati and McLenaghan [CML91] which is used in the computer algebra system MAPLE.

G.4.1 General features

The set of ZM consists of three types of Riemann invariants: 1) *Weyl invariants* made from the Weyl tensor. There are in general 4 independent real Weyl invariants, usually represented by 2 complex quantities. 2) *Ricci invariants*: There are in general 4 independent Ricci invariants, one of which is the Ricci scalar. The other 3 are made from the trace-free part of $R_{\mu\nu}$. 3) *Mixed invariants*: There are at most 6 independent real mixed invariants for any particular case.

ZM found the maximum possible number of independent invariants for each of the $6 \times 15 = 90$ Petrov and Segré types. For each type they gave the subset of independent invariants, and for most of the types they also stated the relations between dependent and independent invariants, the so-called syzygies.

G.4.2 The complete set of invariants

The set consists of the 4 real Weyl invariants

$$\begin{aligned} I_1 &\doteq \Re(I), \\ I_2 &\doteq \Im(I), \\ I_3 &\doteq \Re(J), \\ I_4 &\doteq \Im(J), \end{aligned} \tag{G.17}$$

where

$$I \doteq \frac{1}{3}(\Psi_0\Psi_4 - 4\Psi_1\Psi_3 + 3\Psi_2^2) \tag{G.18}$$

and

$$J \doteq \Psi_0\Psi_2\Psi_4 + 2\Psi_1\Psi_2\Psi_3 - \Psi_0\Psi_3^2 - \Psi_1^2\Psi_4 - \Psi_2^3, \tag{G.19}$$

the 4 real Ricci invariants

$$\begin{aligned} I_5 &\doteq \mathcal{R}, \\ I_6 &\doteq \frac{2}{3}(\Phi_{00}\Phi_{22} - 2\Phi_{01}\Phi_{21} + \Phi_{02}\Phi_{20} - 2\Phi_{10}\Phi_{12} + 2\Phi_{11}^2), \\ I_7 &\doteq 2(\Phi_{00}\Phi_{11}\Phi_{22} + \Phi_{01}\Phi_{12}\Phi_{20} + \Phi_{02}\Phi_{10}\Phi_{21} \\ &\quad -\Phi_{00}\Phi_{12}\Phi_{21} - \Phi_{01}\Phi_{10}\Phi_{22} - \Phi_{02}\Phi_{11}\Phi_{20}), \\ I_8 &\doteq \text{complicated expression in the } \Phi_{AB} \end{aligned} \tag{G.20}$$

and 9 real mixed invariants $I_9, ..., I_{17}$ given by even lengthier expressions (see [ZM97] for the explicit expressions).

G.4.3 The subsets for the case of the shell metric

The maximal possible subsets for the various regions, according to their Petrov and Segré types (G.2), are written down in the table (G.3):

Region	**Independent invariants**
I	$\mathcal{R}$, I
II and III	$\mathcal{R}$
$U = u$ and $V = v$	$\mathcal{R}$, I, M_1

Table G.3: *Possible non-vanishing independent invariants in the various regions of the shell spacetime according to their Petrov and Segré types.*

The explicit expression for the mixed invariant $M_1 \doteq I_{15}$ is written down in [ZM97]. The Ricci scalar $\mathcal{R}$ vanishes everywhere in our case, so the remaining potentially non-zero invariants are I and M_1. They read

$$I = \begin{cases} 0 & : \text{ in regions II and III,} \\ \frac{E^2}{2R^6(r)} & : \text{ at the shell,} \\ \frac{E^2}{R^6} & : \text{ in region I,} \end{cases} \tag{G.21}$$

and

$$M_1 = 0. \tag{G.22}$$

We found these results by analyzing the explicit dependence of I and M_1 on the Ψ_a's and the Φ_{AB}'s given in [ZM97] and checked them by calculating the quantities using the tensor package of MAPLE. In a general spacetime of this Petrov and Segré type the quantity M_1 does not vanish, in our case, however, it is zero everywhere. Thus, there is only one surviving non-zero invariant, namely I. I is really a single invariant because in our case it is real. It is discontinuous at the shell, the value there being exactly the mean of the interior and the exterior. This nicely fits into the framework of [BI91] concerning discontinuous quantities at a surface layer, that is presented in the next section.

The conclusion of this section is that *curvature invariants alone do not contain enough information about the shell spacetime*. We expected two different scalars, because the spacetime is time-dependent and spherically symmetric, which means that its metric components are functions of two coordinates. We have, however, found only one non-vanishing scalar, the other one remains to be found, if there is one at all. But what we can say, that this other scalar can not be one made of the Riemann tensor and its covariant derivatives, because the set of ZM for this type is complete. It is probably because of the special nature of light-like objects that the information contained in the invariants is not sufficient for our ends.

There are other examples where the information contained in the curvature scalars is insufficient to characterize a spacetime: For the so-called pp (plane-fronted parallel) gravitational waves (see e.g. [EK62]) *all* invariants made from the Riemann tensor vanish. This is also true for flat spacetime, but the two spacetimes are obviously different! In our case the information about the time dependence of the metric is somehow contained in the condition for the inside and outside regions of the shell spacetime. The energy E of the shell is contained in the invariant I, its asymptotic advanced time v is given in the shell's equation $V = v$ which defines the boundary between the inside and outside regions.

G.5 Derivatives at surface layers

Let Σ be a hypersurface in a four-dimensional Riemannian spacetime. According to Israel [Isr66], Σ is a *boundary surface*, if the metric tensor and its first derivatives with respect to some special class of coordinates are continuous across the hypersurface. It is called a *surface layer*, if the metric functions are only continuous at Σ. Since our light-like thin shell belongs clearly to the second case, we must find a means to define derivatives of the metric functions, in order to obtain meaningful expressions for quantities like the curvature at the shell. In their paper about null shells, [BI91], Barrabès and Israel give an

elegant definition viewing thin shells as distributions. We state only the most important formulae.

Let the equation of the shell Σ in the coordinates x^μ be given by $\phi(x^\mu) = 0$. The *jump* and the *hybrid function* of a quantity F discontinuous at Σ are, respectively, defined by

$$[F] \doteq (F^+(x) - F^-(x))|_\Sigma, \tag{G.23}$$

$$\tilde{F}(x) \doteq F^+(x)\Theta(x) + F^-(x)\Theta(-x), \tag{G.24}$$

where $F^+(F^-)$ are the functions in the outside (inside) regions, respectively, and the step function is given by

$$\Theta(x) \doteq \begin{cases} 1 & : \quad x > 0 \\ \frac{1}{2} & : \quad x = 0 \\ 0 & : \quad x < 0 \end{cases}. \tag{G.25}$$

From these definitions it follows that

$$\partial_\mu \tilde{F} = \widetilde{\partial_\mu F} - [F]\delta(\Phi)\Phi_{,\mu}, \tag{G.26}$$

$$\tilde{F}\tilde{G} = \widetilde{FG} - [F][G]\Theta(\Phi)\Theta(-\Phi). \tag{G.27}$$

δ is the Dirac Delta-function. At the shell, $\Phi(x) = 0$, the hybrid function reduces to the expression

$$\tilde{F}|_\Sigma = \frac{1}{2}\left(F^+ + F^-\right)|_\Sigma. \tag{G.28}$$

Hence,

$$(\partial_\mu \tilde{F})|_\Sigma = \frac{1}{2}\left(F^+_{,\mu} + F^-_{,\mu}\right)|_\Sigma - [F]\,\Phi_{,\mu}|_\Sigma\,\delta(0). \tag{G.29}$$

Since the metric is continuous, it holds that $g_{\alpha\beta} = \tilde{g}_{\alpha\beta}$, $g^{\alpha\beta} = \tilde{g}^{\alpha\beta}$. From the definition of the Christoffel symbols and from eq. (G.24) it follows that at the shell

$$\Gamma^\lambda_{\mu\nu} = \tilde{\Gamma}^\lambda_{\mu\nu} = \frac{1}{2}(\Gamma^{+\lambda}_{\mu\nu} + \Gamma^{-\lambda}_{\mu\nu})|_\Sigma, \tag{G.30}$$

which can be easily verified using that the jump of the metric is zero. The Riemann and Ricci tensors at the shell read thus, respectively,

$$R^\kappa_{\lambda\mu\nu} = \tilde{R}^\kappa_{\lambda\mu\nu} - 2\delta(0)[\Gamma^\kappa_{\lambda[\nu}]\Phi_{,\mu]} - \frac{1}{2}[\Gamma^\rho_{\lambda[\nu}][\Gamma^\kappa_{\rho\mu]}] \tag{G.31}$$

and

$$R_{\lambda\nu} = \tilde{R}_{\lambda\nu} - 2\delta(0)[\Gamma^{\mu}_{\lambda[\nu}]\Phi_{,\mu]} - \frac{1}{2}[\Gamma^{\rho}_{\lambda[\nu}][\Gamma^{\mu}_{\rho\mu]}]. \tag{G.32}$$

The Ricci scalar is, therefore, given by

$$\mathcal{R} = \tilde{\mathcal{R}} - 2\delta(0)g^{\lambda\nu}[\Gamma^{\mu}_{\lambda[\nu}]\Phi_{,\mu]} - \frac{1}{2}g^{\lambda\nu}[\Gamma^{\rho}_{\lambda[\nu}][\Gamma^{\mu}_{\rho\mu]}]. \tag{G.33}$$

G.6 General properties of spherically symmetric spacetimes

We briefly summarize some general results about spherically symmetric spacetimes given by Takeno in his book [Tak66]. We slightly change his notation to match it to our conventions.

G.6.1 Basic equations

The most general spherically symmetric line element in coordinates (t, r, θ, ϕ) adapted to the spherical symmetry reads

$$ds^2 = Cdt^2 + 2Ddtdr - Adr^2 - Bd\Omega^2, \tag{G.34}$$

where A, B, C and D are functions of t, r. They satisfy

$$m \doteq AC + D^2 > 0. \tag{G.35}$$

The inverse metric has non-vanishing components

$$\begin{aligned} g^{00} &= \frac{A}{m}, \\ g^{01} &= \frac{D}{m}, \\ g^{11} &= -\frac{C}{m}, \\ g^{22} &= -\frac{1}{B}, \\ g^{33} &= -\frac{1}{B}\sin^2\theta. \end{aligned} \tag{G.36}$$

The non-zero independent components of the Riemann tensor read

$$f_1 \doteq R_{1212} = \frac{R_{1313}}{\sin^2\theta}, \tag{G.37}$$

$$f_2 \doteq R_{1010}, \tag{G.38}$$

$$f_3 \doteq \frac{R_{2323}}{\sin^2\theta}, \tag{G.39}$$

$$f_4 \doteq R_{2020} = \frac{R_{3030}}{\sin^2\theta}, \tag{G.40}$$

$$f_5 \doteq R_{1220} = \frac{R_{1330}}{\sin^2\theta}. \tag{G.41}$$

The quantities $f_1, ..., f_5$ are lengthy functions of t, r given by eqs. (4.6) of [Tak66]. We will state them explicitly later for special cases. The components of the Ricci tensor are given by

$$R_{00} = \frac{Cf_2}{m} + 2\frac{f_4}{B}, \tag{G.42}$$

$$R_{01} = -2\frac{f_5}{B} + \frac{Df_2}{m}, \tag{G.43}$$

$$R_{11} = 2\frac{f_1}{B} - \frac{Af_2}{m}, \tag{G.44}$$

$$R_{22} = \frac{R_{33}}{\sin^2\theta} = \frac{f_3}{B} + \frac{Cf_1 + 2Df_5 - Af_4}{m}. \tag{G.45}$$

The Ricci scalar hence reads

$$\mathcal{R} = \frac{4}{mB}\left(Af_4 - Cf_1 - 2Df_5\right) + \frac{2f_2}{m} - \frac{f_3}{B^2}. \tag{G.46}$$

Using the non-vanishing mixed (2,2) components of the Riemann tensor, $R^{\rho\sigma}_{\mu\nu}$, one can define the following useful quantities:

$$\alpha \doteq R^{12}_{12} = R^{13}_{13}, \tag{G.47}$$

$$\beta \doteq R^{20}_{20} = R^{30}_{30}, \tag{G.48}$$

$$\gamma \doteq R^{20}_{12} = R^{30}_{13}, \tag{G.49}$$

$$\delta \doteq R^{12}_{20} = R^{13}_{30}, \tag{G.50}$$

$$\xi \doteq R^{10}_{10}, \tag{G.51}$$

$$\eta \doteq R^{23}_{23}. \tag{G.52}$$

The function ξ coincides with the Gaussian curvature of the surfaces of constant B. The relations between the quantities $\alpha, \beta, ...$ and f_a are given by

$$\alpha = \frac{Cf_1 + Df_5}{mB}, \tag{G.53}$$

$$\beta = \frac{-Af_4 + Df_5}{mB}, \tag{G.54}$$

$$\gamma = \frac{Df_1 - Af_5}{mB}, \tag{G.55}$$

$$\delta = \frac{Cf_5 + Df_4}{mB}, \tag{G.56}$$

$$\xi = -\frac{f_2}{m}, \tag{G.57}$$

$$\eta = \frac{f_3}{B^2}. \tag{G.58}$$

In some cases the quantities $\alpha, ..., \eta$ are more useful than the expressions f_a. The Ricci tensor and the scalar curvature can be expressed in terms of the $\alpha, \beta, ...$:

$$R^0_0 = -(2\beta + \xi), \tag{G.59}$$

$$R^1_1 = -(2\alpha + \xi), \tag{G.60}$$

$$R^0_1 = 2\gamma, \tag{G.61}$$

$$R_0^1 = 2\delta, \tag{G.62}$$

$$R_2^2 = R_3^3 = -(\alpha + \beta + \eta), \tag{G.63}$$

$$\mathcal{R} = -2(2\alpha + 2\beta + \xi + \eta). \tag{G.64}$$

Any spherically symmetric line element can be brought into the form

$$ds^2 = Cdt^2 - Adr^2 - Bd\Omega^2, \tag{G.65}$$

by a suitable coordinate transformation. Such a coordinate system is called a *spherically symmetric coordinate system (sscs) in a narrow sense*, whereas the most general case (with $D \neq 0$) is called a *sscs in a wide sense*. Any spherically symmetric line element in a narrow sense can be brought into the form of a so-called *D-type coordinate system*.

G.6.2 Spherically symmetric spacetimes in D-type coordinates

We summarize some results about spherically symmetric line elements in the form

$$ds^2 = 2Ddrdt - Bd\Omega^2 \tag{G.66}$$

from [Tak66]. D and B are in general functions of the two coordinates r, t. We are interested in this kind of line elements, because the metric with respect to the regular DNEF coordinates belongs to this class. In chapter 6 we have referred to these coordinates as of the UV-type.

The relation between the f_a and the $\alpha, ..., \eta$ and their explicit expression in terms of the components of the metric reads in this case:

$$\begin{aligned}
\alpha = \beta &= \frac{f_5}{BD} = \frac{1}{4BD}\left(-2\dot{B}' + \frac{B'\dot{B}}{B}\right), \\
\gamma &= \frac{f_1}{BD} = \frac{1}{4BD}\left(2B'' - \frac{B'^2}{B} - \frac{2B'D'}{D}\right), \\
\delta &= \frac{f_4}{BD} = \frac{1}{4BD}\left(2\ddot{B} - \frac{\dot{B}^2}{B} - \frac{2\dot{B}\dot{D}}{D}\right), \\
\xi &= -\frac{f_2}{D^2} = \frac{1}{D^2}\left(-\dot{D}' + \frac{D'\dot{D}}{D}\right), \\
\eta &= \frac{f_3}{B^2} = -\frac{1}{2B^2}\left(2B + \frac{B'\dot{B}}{D}\right).
\end{aligned} \tag{G.67}$$

Here, an overdot (prime) represents the derivative with respect to t (r).

G.6.3 Invariant scalars

The following quantities are invariant under coordinate transformations:

$$X \doteq \alpha + \beta, \tag{G.68}$$

$$Y \doteq \alpha\beta - \gamma\delta, \tag{G.69}$$

$$L \doteq X^2 - 4Y, \tag{G.70}$$

$$S_1 \doteq \xi + \eta, \tag{G.71}$$

$$S_2 \doteq \xi\eta. \tag{G.72}$$

This holds for any spherically symmetric spacetime. One can express the scalars constructed algebraically from $g_{\mu\nu}$ and $R_{\mu\nu\rho\sigma}$ in these quantities. For example, the Ricci scalar is given by

$$\mathcal{R} = -4X - 2(\xi + \eta) = -4X - 2S_1, \tag{G.73}$$

and is, therefore, an invariant. The equations

$$L = 0 \tag{G.74}$$

and

$$\alpha - \beta = \gamma = \delta = 0 \tag{G.75}$$

are also coordinate invariant. One can use the properties given by eqs. (G.74) and (G.75) to invariantly classify the spherically symmetric spacetimes. In the following we will use ξ and η instead of S_1 and S_2. Only in a few quite pathological cases ξ are η are not scalars. Details can be found in [Tak66].

Coordinate transformations that leave the angular coordinates θ and ϕ invariant are called *T-transformations* [Tak66]. A quantity Q that is invariant under T-transformations is called a *T-scalar*. For example $-B$ and the function R appearing in the components of the regular DNEF metric are such T-scalars, but they are not scalars under more general coordinate transformations. However, any scalar is of course also a T-scalar.

G.6.4 Invariant classification according to Takeno

We briefly state some results of chapter II of [Tak66]. A spherically symmetric spacetime, denoted by S_0, belongs either to the category S_a, when both $L = 0$ and $\alpha - \beta = \gamma = \delta = 0$ hold, to S_b, when $L = 0$ while $\alpha - \beta = \gamma = \delta = 0$ does not hold or to S_c, when $L \neq 0$.

If one excludes the pathologic cases with $B = const.$ (Takeno denotes them by S_{II}), one can also classify any remaining spherically symmetric spacetime denoted by S_I into two types. The property in question is whether the line element of the spacetime can be brought into the form

$$ds^2 = -Adr^2 - r^2 d\Omega^2 + Cdt^2 \tag{G.76}$$

of a so-called *canonical coordinate system (ccs)* or not, but we do not want to go into details and refer to the aforementioned book. The necessary and sufficient condition for a D-type line element to allow a ccs is

$$B'\dot{B} \neq 0. \tag{G.77}$$

We denote, still following [Tak66], the spacetimes satisfying (G.77) by ◯ and the others by △. One finally obtains the invariant classification scheme given by the table (G.4):

S_0					
S_a			S_b	S_c	
S_{II}	S_{Ia}		S_{Ib}	S_{Ic}	
	◯	△	◯	◯	△

Table G.4: *Classification of spacetimes according to Takeno.*

For example, Schwarzschild's exterior solution belongs to type $S_{Ia}(◯)$. We will observe, that in the case of our shell-mirror spacetime, this method of classification becomes impractical because of the discontinuity of the (first) derivatives of the metric at the shell.

G.7 Invariants for the shell spacetime in regular DNEF coordinates

We compute the Takeno invariants for the different regions of the spacetime of our model in the regular DNEF gauge using the methods from the two previous sections. We make the following replacements:

$$t = U, \quad r = V, \quad D = \frac{A(U,V)}{2}, \quad B = R^2(U,V). \tag{G.78}$$

using these replacements we can express the derivatives of D and B with respect to r an t using the derivatives of A and R with respect to U and V:

$$\dot{B} = 2RR_{,U}, \quad B' = 2RR_{,V}, \quad \ddot{B} = 2R_{,U}^2 + 2RR_{,UU}, \quad B'' = 2R_{,V}^2 + 2RR_{,VV},$$

$$\dot{B}' = 2R_{,U}R_{,V} + 2RR_{,UV}, \quad \dot{D} = \frac{A_{,U}}{2}, \quad D' = \frac{A_{,V}}{2}, \quad \dot{D}' = \frac{A_{,UV}}{2}. \tag{G.79}$$

G.7.1 Region I

In the region I exterior to the shell the derivatives of R and A are given by

$$R_{,U} = -\frac{A}{2} = -R_{,V}, \quad A_{,U} = -\frac{EA}{R^2} = -A_{,V}. \tag{G.80}$$

We are now able to calculate the Takeno invariants. They read

$$\alpha = \beta = -\frac{\xi}{2} = -\frac{\eta}{2} = \frac{E}{R^3}, \quad \gamma = \delta = L = 0,$$

$$\mathcal{R} = 0, \quad X = 2\alpha, \quad Y = \alpha^2. \tag{G.81}$$

Hence, there is only one independent scalar in region I, depending on the T-scalar R and the shell energy E. It is no surprise that this scalar is closely related to the single non-vanishing scalar that has been found in section 4 of this appendix, $I = \frac{E^2}{R^6}$.

G.7.2 Regions II and III

In region II the derivatives of R and A read

$$R_{,U} = -\frac{1}{2}, \quad R_{,V} = \frac{A}{2}, \quad A_{,U} = 0, \quad A_{,V} = \frac{EA}{(R - \frac{u-U}{2})^2}, \tag{G.82}$$

whereas in region III they are given by

$$R_{,U} = -\frac{A}{2}, \quad R_{,V} = \frac{1}{2}, \quad A_{,V} = 0, \quad A_{,U} = \frac{EA}{(R - \frac{V-v}{2})^2}. \tag{G.83}$$

Inserting these quantities into the definition of $\alpha, ...$ yields:

$$\alpha = \beta = \gamma = \delta = \xi = \eta = L = 0,$$

$$\mathcal{R} = X = Y = 0, \tag{G.84}$$

as expected.

G.7.3 At the shell

Things become more interesting if we consider these quantities at the shell. Since the metric is at most continuous there, we expect discontinuities in quantities, which depend on its (first) derivatives. We consider the outgoing shell given by the equation $U = u$. The ingoing case is treated similarly. For the outgoing shell the function Φ reads $\Phi(U, V) = U - u$. We exclude the special point $(U, V) = (u, v)$ where the shell is reflected from the mirror for the moment.

Let square brackets denote the jump of a quantity at the shell: $[Q] \doteq Q^+ - Q^-$, and let $P(r)$ denote a quantity P evaluated at the shell. The jumps in the derivatives of R and A are easily computed:

$$[R] = [A] = [R_{,V}] = [A_{,V}] = [R_{,VV}] = [A_{,VV}] = 0,$$

$$[R_{,U}] = \frac{1 - A(r)}{2}, \quad [A_{,U}] = -\frac{EA(r)}{R^2(r)},$$

$$[R_{,UU}] = \frac{EA(r)}{2R^2(r)}, \quad [R_{,UV}] = -\frac{EA(r)}{2R^2(r)}, \quad [A_{,UU}] = \frac{E^2A(r)}{R^4(r)} - \frac{EA^2(r)}{R^3(r)}. \tag{G.85}$$

Using the methods of section 5 we obtain for the hybrid quantities at the shell (in order to keep the notation clear the (r) and the tilde are not written in the following):

$$\dot{B} = -\frac{R}{2}(A - 1), \quad B' = AR, \quad \dot{D} = -\frac{EA}{4R^2}, \quad D' = \frac{EA}{2R^2},$$

$$\ddot{B} = \frac{EA}{2R^2} + R(A - 1)\delta(0) + \frac{A^2 + 1}{4}, \quad B'' = \frac{A^2}{2} + \frac{EA}{R}, \quad \dot{B}' = -\frac{A(A + 1)}{4} - \frac{EA}{2R},$$

$$\dot{D}' = -\frac{E^2A}{4R^4} + \frac{EA^2}{4R^3}. \tag{G.86}$$

Equipped with these equations we are now ready to calculate the curvature quantities $\alpha, ..., \eta$ at the shell. They read

$$\alpha = \beta = -\frac{\xi}{2} = -\frac{\eta}{2} = \frac{E}{2R^3}, \quad \delta = \frac{A - 1}{AR}\delta(0), \quad \gamma = L = 0,$$

$$\mathcal{R} = 0, \quad X = 2\alpha, \quad Y = \alpha^2. \tag{G.87}$$

We observe that the distributional part $\delta(0)$ has entered one of the quantities only, namely δ. Since it appears only in the invariant Y in the product $\gamma\delta$, it has no effect, since $\gamma = 0$.

Various scalars are discontinuous at the shell. The Ricci scalar $\mathcal{R}$ is, however, zero everywhere, even at the shell. L is still equal to zero at the shell, but $\delta = \gamma = 0$ does not hold anymore. This means that, while the regions I, II and III belong to the class $S_{Ia}(\bigcirc)$, the metric at the shell lies in the class S_b. Even though the Ricci scalar vanishes, the Ricci tensor has non-zero components at the shell. The information about the matter of the shell is contained in the equations

$$R_{VV} = \frac{A-1}{R}\delta(0) \tag{G.88}$$

for the ingoing and

$$R_{UU} = \frac{A-1}{R}\delta(0) \tag{G.89}$$

for the outgoing case. This corresponds to a distributional matter distribution in the energy momentum tensor, $T_{VV} \propto \delta(0)$ ($T_{UU} \propto \delta(0)$). This is sensible, because an infinitesimally thin null shell can be considered as a limit of a thick Vaidya shell with finite energy-momentum tensor.

G.7.4 At the point $(U, V) = (u, v)$

We get into big troubles at the singular point $(U, V) = (u, v)$, where the trajectories of the in- and outgoing shells meet at the mirror. Although we can still define the function Φ by putting $\Phi(U, V) = U - V - (u - v)$, it is not clear what is the 'inside' and what the 'outside', because the three regions I, II and III meet at this particular point. There actually is, to be exact, no interior of the shell. So we can not define jumps and hybrid functions in the usual way, and thus the construction of the Takeno invariants fails.

A possible way out of this problem could be found as follows. Instead of considering an infinitesimally thin shell from the beginning, one should take a thin but finite patch filled with radial null radiation commonly described by the Vaidya metric between the exterior Schwarzschild and the interior Minkowski spacetimes. In this case, the metric and its derivatives are less singular, because the surface layer is transformed into two distinct boundary surfaces, and the interesting quantities can be defined everywhere. The actual construction of coordinates, with respect to which the shell is continuous and has continuous first derivatives is quite involved, although one can proceed as in the thin shell case. Double null coordinates for the Vaidya spacetime have been found by (e.g.) [WL86] and [SV00]. But since the construction is tedious, and since there are problems at the

mirror where there is a region (a triangle in the Penrose diagram) where the two thick shells overlap, we will not pursue this path in this work.

Appendix H

Asymptotic properties

H.1 Asymptotic simplicity

We want to find out if the shell spacetime is asymptotically simple and flat. There is no special motivation other than making the classical discussion of our model more complete.

A spacetime $(\mathcal{M}, g)$ is called *asymptotically simple* [HE73] if it satisfies the following conditions: There exists another (smooth) Lorentz manifold $(\tilde{\mathcal{M}}, \tilde{g})$, such that

- $\mathcal{M}$ is an open submanifold of $\tilde{\mathcal{M}}$ with smooth boundary $\partial\mathcal{M} = \mathcal{I} \doteq \mathcal{I}^+ \cup \mathcal{I}^-$;
- there exists a positive, real C^3 function Λ on $\tilde{\mathcal{M}}$, such that $\tilde{g}_{\mu\nu} = \Lambda^2 g_{\mu\nu}$ on $\mathcal{M}$, and such that $\Lambda = 0$, $d\Lambda \neq 0$ on $\mathcal{I}$;
- every null geodesic in $\mathcal{M}$ acquires a future and past endpoint on $\mathcal{I}$.

We now demonstrate that the spacetime of the shell is asymptotically simple. For this end we express the spacetime with respect to the regular DNEF coordinates. We first compactify it by using the coordinate transformation

$$V = \tan p, \quad U = \tan q, \tag{H.1}$$

which yields the line element

$$ds^2 = \frac{1}{\cos^2 p}\frac{1}{\cos^2 q}\left(A\, dpdq - R^2 \cos^2 p \cos^2 q\, d\Omega^2\right). \tag{H.2}$$

Hence, the conformal factor Λ is given by

$$\Lambda = \cos p \cos q. \tag{H.3}$$

The domains of the new null coordinates p, q are now finite and given by

$$p \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \quad \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \quad p > q^*, \tag{H.4}$$

where $q^* > q$. The number q^* is defined by the boundary condition on V and U imposed by the mirror at $R = R_M$, cf. eqs. (5.37, 5.43) of chapter 5. The boundary $\partial\mathcal{M}$ is given by $p = \frac{\pi}{2}$, which defines the null surface of *future light-like infinity* $\mathcal{I}^+$ and $q = -\frac{\pi}{2}$, defining *past light-like infinity* $\mathcal{I}^-$. The points $(p, q) = (\frac{\pi}{2}, \frac{\pi}{2}) \doteq i^+$ (*future time-like infinity*), $(p, q) = (-\frac{\pi}{2}, -\frac{\pi}{2}) \doteq i^-$ (*past time-like infinity*) and $(p, q) = (\frac{\pi}{2}, -\frac{\pi}{2}) \doteq i^0$ (*space-like infinity*) do not belong to $\partial\mathcal{M}$. One can easily convince oneself that the function Λ is smooth, positive and that it satisfies $\Lambda = 0$, $d\Lambda \neq 0$ at $\partial\mathcal{M}$. Hence, the conditions listed above are all fulfilled meaning that the *shell spacetime is asymptotically simple*.

H.2 Asymptotic flatness

An asymptotically simple spacetime is called *asymptotically flat* (or empty) [HE73] if it furthermore satisfies that the Ricci tensor $R_{\mu\nu}$ vanishes in an open neighbourhood of $\mathcal{I}$. The Ricci tensor for the shell spacetime (expressed in the regular DNEF coordinates) has two non-vanishing components given by

$$R_{UU} = \frac{A-1}{R}\,\delta(U-u), \quad R_{VV} = \frac{A-1}{R}\,\delta(v-V). \tag{H.5}$$

They vanish away from the shell but are distributional at the shell. The components have been computed using the methods by Barrabès and Israel, cf. the appendix G. It does not seem that the Ricci tensor is zero in a whole neighbourhood of $\mathcal{I}$. But one can relax the above condition to one on the fall-off behaviour of the metric in the vicinity of a (spatial) hypersurface as $R \to \infty$: the metric is to satisfy

$$g_{\mu\nu} \sim \eta_{\mu\nu} + \frac{\alpha_{\mu\nu}\left(\frac{x^\sigma}{r}, t\right)}{r} + \mathcal{O}\left(\frac{1}{r^{1+\epsilon}}\right), \tag{H.6}$$

where $\epsilon > 0$, $x^\sigma = (t, x^k)$ is a flat coordinate system at space-like infinity, $\alpha_{\mu\nu}$ is an arbitrary symmetric tensor and $r = \sqrt{\vec{x}^2}$. The Cartesian transform of the central-regular metric introduced in chapter 6 satisfies this condition at space-like infinity. There, the metric component $\frac{R^2}{\vec{X}^2}$ is of the order of 1, hence all components of the metric behave as required. Thus, if one allows the less restrictive definition of asymptotic flatness, then the *shell spacetime* is indeed *asymptotically flat*, even though its Ricci tensor has a distributional part.

Bibliography

[AJS77] W. O. Amrein, J. M. Jauch, and K. B. Sinha. *Scattering theory in quantum mechanics*. Reading, 1977. 7, 9, 17, 19, 22, 26, 27

[BE81] J. K. Beem and P. E. Ehrlich. *Global Lorentzian geometry*. Dekker, 1981. 159, 162, 163

[BFV01] Guy Bonneau, Jacques Faraut, and Galliano Valent. Self-adjoint extensions of operators and the teaching of quantum mechanics. *Am. J. Phys.*, 69:322, 2001. 104, 183

[BGG83] D. Bollé, F. Gesztesy, and H. Grosse. Time delay for long range interactions. *J. Math. Phys.*, 24:1529, 1983. 29

[BI91] C. Barrabès and W. Israel. Thin shells in general relativity and cosmology: The lightlike limit. *Phys. Rev.*, D43:1129, 1991. 38, 197

[BKT87] V. A. Berezin, V. A. Kuzmin, and I. I. Tkachev. Dynamics of bubbles in general relativity. *Phys. Rev.*, D36:2919–2944, 1987. 33

[BO75] D. Bollé and T. A. Osborn. The equivalence between time dependent and time independent formulations of time delay. *Phys. Rev.*, D11:3417, 1975. 16, 20

[Bol81] D. Bollé. On classical time delay. *Acta Phys. Aust. Suppl.*, 23:587, 1981. 22

[BSP84] Tom Banks, Leonard Susskind, and Michael E. Peskin. Difficulties for the evolution of pure states into mixed states. *Nucl. Phys.*, B244:125, 1984. 1, 2

[Car97] Sean M. Carroll. Lecture notes on general relativity. *eprint: gr-qc/9712019*, 1997. 144

[CGH+96] R. M. Corless, G. Gonnet, D. Hare, D. Jeffrey, and D. Knuth. On the Lambert W function. *Advances in Computational Mathematics*, 5:329–359, 1996. 182

[CGHS92] Curtis G. Callan Jr., Steven B. Giddings, Jeffrey A. Harvey, and Andrew Strominger. Evanescent black holes. *Phys. Rev.*, D45:1005–1009, 1992. 2

[CML91] J. Carminati and R. G. Mc Lenaghan. Algebraic invariants of the Riemann tensor in a four-dimensional Lorentzian space. *J. Math. Phys*, 32:3135–3140, 1991. 195

[DeW63] B. S. DeWitt. Dynamical theory of groups and fields. In C. DeWitt, editor, *Relativity, groups and topology*, pages 586–820. Gordon & Breach, 1963. 4, 5, 49, 127, 128

[DeW67] B. S. DeWitt. Quantum theory of gravity. II. The manifestly covariant theory. *Phys. Rev.*, 162:1195–1239, 1967. 128, 132

[Dol64] J. D. Dollard. Asymptotic convergence and the Coulomb interaction. *J. Math. Phys.*, 5:729, 1964. 26

[Eis48] L. Eisenbud. Dissertation. Princeton University, unpublished, 1948. 15

[EK62] J. Ehlers and W. Kundt. *Gravitation: an introduction to current research*, chapter 2. Wiley, 1962. 197

[Far73] J. E. G. Farina. *Quantum theory of scattering processes*. Pergamon Press, 1973. 9, 11, 23

[FLWH97] John L. Friedman, Jorma Louko, and Stephen N. Winters-Hilt. Reduced phase space formalism for spherically symmetric geometry with a massive dust shell. *Phys. Rev.*, D56:7674–7691, 1997. 85

[Gas74] J. Gasser. *On the infrared problem in a model of scalar, massless bosons coupled to classical C-number currents*. PhD thesis, Universität Bern, 1974. 25

[Ger69] R. Geroch. Limits of spacetimes. *Commun. Math. Phys.*, 13:180–193, 1969. 162

[Gid92] Steven B. Giddings. Black holes and massive remnants. *Phys. Rev.*, D46:1347–1352, 1992. 2

[Gid95] Steven B. Giddings. The black hole information paradox. *eprint: hep-th/9508151*, 1995. 2

[GW64] M. L. Goldberger and K. M. Watson. *Collision theory*. Wiley, 1964. 9, 16, 151

[Háj92] Petr Hájíček. Quantum mechanics of gravitational collapse. *Commun. Math. Phys.*, 150:545–559, 1992. 37

[Háj00a] Petr Hájíček. Choice of gauge in quantum gravity. *Nucl. Phys. B (Proc. Suppl.)*, 80, 2000. 3, 49, 60

[Háj00b] Petr Hájíček. What simplified models say about unitarity and gravitational collapse. *Nucl. Phys. Proc. Suppl.*, 88:114–123, 2000. 33

[Háj01] Petr Hájíček. Unitary dynamics of spherical null gravitating shells. *Nucl. Phys.*, B603:555–577, 2001. 2, 150

[Háj03] Petr Hájíček. Quantum theory of gravitational collapse (lecture notes on quantum conchology). *Lect. Notes Phys.*, 631:255–299, 2003. 3, 83

[Haw67] S. W. Hawking. The occurrence of singularities in cosmology. III. Causality and singularities. *Proc. Roy. Soc. Lond.*, A300:187–201, 1967. 1

[Haw71] S. W. Hawking. Stable and generic properties in general relativity. *Gen. Rel. Grav.*, 1:393–400, 1971. 159, 161, 162

[Haw75] S. W. Hawking. Particle creation by black holes. *Commun. Math. Phys.*, 43:199–220, 1975. 1

[Haw76] S. W. Hawking. Breakdown of predictability in gravitational collapse. *Phys. Rev.*, D14:2460–2473, 1976. 1

[Haw82] S. W. Hawking. The unpredictability of quantum gravity. *Commun. Math. Phys.*, 87:395, 1982. 1

[HB97] P. Hájíček and J. Bičák. Gauge invariant Hamiltonian formalism for spherically symmetric gravitating shells. *Phys. Rev.*, D56:4706–4719, 1997. 38, 39

[HE73] S. W. Hawking and G. F. R. Ellis. *The large scale structure of space-time.* Cambridge, 1973. 1, 159, 163, 209, 210

[HK00] Petr Hájíček and Jerzy Kijowski. Covariant gauge fixing and Kuchař decomposition. *Phys. Rev.*, D61:024037, 2000. 3, 60, 87

[HK01] Petr Hájíček and Claus Kiefer. Embedding variables in the canonical theory of gravitating shells. *Nucl. Phys.*, B603:531–554, 2001. 2, 4, 55, 60, 83, 87, 88

[HK02a] P. Hájíček and I. Kouletsis. Pair of null gravitating shells I. Space of solutions and its symmetries. *Class. Quant. Grav.*, 19:2529–2550, 2002. 3, 55, 56, 60, 61

[HK02b] P. Hájíček and I. Kouletsis. Pair of null gravitating shells II. Canonical theory and embedding variables. *Class. Quant. Grav.*, 19:2551–2566, 2002. 3, 4, 83, 87, 88, 89

[HP70] S. W. Hawking and R. Penrose. The singularities of gravitational collapse and cosmology. *Proc. Roy. Soc. Lond.*, A314:529–548, 1970. 1

[Isr66] W. Israel. Singular hypersurfaces and thin shells in general relativity. *Il Nuovo Cimento*, 44B:1, 1966. 33, 197

[Jen81] A. Jensen. Time delay in potential scattering theory. Some 'geometric' results. *Commun. Math. Phys.*, 82:435–456, 1981. 22

[Jen82] A. Jensen. On Lavine's formula for time delay. *Math. Scand.*, 54:253–261, 1982. 21

[JM65] J. M. Jauch and B. N. Misra. The spectral representation. *Helv. Phys. Acta*, 38:30, 1965. 9

[JM66] J. M. Jauch and J.-P. Marchand. The unitarity of the S matrix for multichannel scattering. *Helv. Phys. Acta*, 39:325, 1966. 9, 19

[JM67] J. M. Jauch and J.-P. Marchand. The delay time operator for simple scattering systems. *Helv. Phys. Acta*, 40:217, 1967. 9, 16

[JSM72] J. M. Jauch, K. B. Sinha, and B. N. Misra. Time delay in scattering processes. *Helv. Phys. Acta*, 45:398, 1972. 16, 19

[KH02] I. Kouletsis and P. Hájíček. Pair of null gravitating shells III. Algebra of Dirac's observables. *Class. Quant. Grav.*, 19:2567–2586, 2002. 3, 4, 83, 87, 91

[KSMH80] D. Kramer, H. Stephani, M. MacCallum, and E. Herlt. *Exact solutions of Einstein's field equations*. VEB Deutscher Verlag der Wissenschaften, Berlin, 1980. 189, 190, 192, 193, 194

[Kuc71] K. Kuchař. Canonical quantization of cylindrical gravitational waves. *Phys. Rev.*, D4:955–986, 1971. 3

[Kuc94] Karel V. Kuchař. Geometrodynamics of Schwarzschild black holes. *Phys. Rev.*, D50:3961–3981, 1994. 85, 86

[Lav73] R. Lavine. Commutators and local decay. In J. A. La Vita and J.-P. Marchand, editors, *Scattering theory in mathematical physics*, page 141. D. Reidel, 1973. 20

[Ler73] D. E. Lerner. The space of Lorentz metrics. *Commun. Math. Phys.*, 32:19–38, 1973. 159

[LWF98] Jorma Louko, Bernard F. Whiting, and John L. Friedman. Hamiltonian spacetime dynamics with a spherical null-dust shell. *Phys. Rev.*, D57:2279–2298, 1998. 4, 83, 84, 97

[Mar76] P. A. Martin. On the time delay of simple scattering systems. *Commun. Math. Phys.*, 47:221–227, 1976. 16

[Mar81] P. A. Martin. Time delay of quantum scattering processes. *Acta Phys. Aust. Suppl.*, 23:157, 1981. 7, 14, 16, 18, 19, 20, 22

[MZ70] D. Mulherin and I. I. Zinnes. Coulomb scattering. I. Single channel. *J. Math. Phys.*, 11:1402, 1970. 28

[Nar80] H. Narnhofer. Another definition for time delay. *Phys. Rev.*, D22:2387–2390, 1980. 22

[Nar84] H. Narnhofer. Time delay and dilation properties in scattering theory. *J. Math. Phys.*, 25:987, 1984. 21

[New66] R. Newton. *Scattering theory of waves and particles.* McGraw Hill, 1966. 9

[Pen65] Roger Penrose. Gravitational collapse and space-time singularities. *Phys. Rev. Lett.*, 14:57–59, 1965. 1

[Pre92] John Preskill. Do black holes destroy information? *eprint: hep-th/9209058*, 1992. 2

[PRHM97] F. M. Paiva, M. J. Rebouças, G. S. Hall, and M. A. H. MacCallum. Limits of the energy-momentum tensor in general relativity. *eprint: gr-qc/9708045*, 1997. 162

[PRM93] F. M. Paiva, M. J. Rebouças, and M. A. H. MacCallum. On limits of space-times – a coordinate-free approach. *eprint: gr-qc/9302005*, 1993. 162

[RS75] M. Reed and B. Simon. *Methods of mathematical physics. Vol. 2: Fourier analysis, self-adjointness.* Academic, 1975. 104, 183

[RS79] M. Reed and B. Simon. *Methods of mathematical physics. Vol. 3: Scattering theory.* Academic, 1979. 7, 28

[RT74] Tullio Regge and Claudio Teitelboim. Role of surface integrals in the Hamiltonian formulation of general relativity. *Ann. Phys.*, 88:286, 1974. 86

[Smi60] F. T. Smith. Lifetime matrix in collision theory. *Phys. Rev.*, 118:349, 1960. 16, 19

[Ste77] H. Stephani. *Allgemeine Relativitaetstheorie.* VEB Deutscher Verlag der Wissenschaften, Berlin, 1977. 189

[SV00] T. P. Singh and C. Vaz. Radiation flux and spectrum in the Vaidya collapse model. *Phys.Lett.*, B481:74–78, 2000. 207

[Tak66] H. Takeno. *The theory of spherically symmetric space-times (revised edition).* Scientific Reports of the Research Institute for Theoretical Physics, Hiroshima University, 1966. 199, 200, 202, 203, 204

[tH90] Gerard 't Hooft. The black hole interpretation of string theory. *Nucl. Phys.*, B335:138–154, 1990. 2

[vN29] J. von Neumann. Allgemeine Eigenwerttheorie Hermitescher Funktionaloperatoren. *Math. Ann.*, 102:49–131, 1929. 184

[Wig55] E. P. Wigner. Lower limit for the energy derivative of the scattering phase shift. *Phys. Rev.*, 98:145, 1955. 15

[WL86] B. Waugh and K. Lake. Double-null coordinates for the Vaidya metric. *Phys. Rev.*, D34:2978–2984, 1986. 207

[ZM97] E. Zakhary and C. B. G. McIntosh. A complete set of Riemann invariants. *Gen. Rel. Grav.*, 29:539–581, 1997. 189, 195, 196, 197

List of Figures

List of Tables

Acknowledgements

In the first place my thanks go to my thesis advisor Prof. Petr Hájíček for suggesting me to work on this interesting topic and for his competent advice and continuous support.

Further, I would like to thank Prof. Claus Kiefer for kindly agreeing to coreferee the present work and Prof. Jürg Gasser for providing a copy of his PhD thesis.

I also thank Prof. Karel Kuchař, Ioannis Kouletsis and Eric Minassian for many interesting and helpful discussions.

I am very grateful to my family and my friends at the institute, in Berne and all around the world for their support in many ways or just for being there.

Last but not least I will not miss the opportunity to thank Franziska Steiger for her comprehension.