On the zero set of semi-invariants for $\mathbb{D}_n$-quivers

Inauguraldissertation
der Philosophisch-naturwissenschaftlichen Fakultät
der Universität Bern

vorgelegt von

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Leiterin der Arbeit:
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Part I
Preface

The goal of this preface is to serve as a small introductory guide to certain aspects of the theory of representations of quivers, which are implicitly assumed or only noted very briefly in the following two parts of this document. However, this guide is not intended to be complete or self-contained. In particular, no proofs of the presented results are included and only references are given instead.

1 Modules vs. Representations of quivers

At the beginning of any studies in mathematics, one of the first topics to get in touch with is linear algebra – the theory of vector spaces over a field. Later on, when moving from linear algebra to modules over a ring, at first sight one might not expect all too dramatic changes in theory, since the axioms for a module and those for a vector space look “pretty much the same”. But in fact things are completely different, and the concept of a module over a ring turns out to be too general than to bring forth an interesting theory. However, narrowing the scope to certain subclasses of rings enhances the richness of properties of the corresponding categories of modules.

One such subclass of rings, where a lot of investigation has been done and is still going on, are finite dimensional algebras over an algebraically closed field $\mathbb{K}$. Nowadays, a commonly used concept for studying modules over finite dimensional $\mathbb{K}$-algebras are quivers and representations of quivers over $\mathbb{K}$. Originally invented by P. Gabriel in [2], the language of quivers and their representations is more appealing to intuition than the one of modules over finite dimensional $\mathbb{K}$-algebras. This is particularly true when it comes to explicit computations.

A quiver Q is an oriented graph, i.e. a set Q_0 of vertices possibly linked by a set Q_1 of arrows. Figure 1 shows some examples. Unless anything else is specified explicitly, a quiver Q is always considered to be finite, i.e. the sets Q_0 and Q_1 are both finite.

We denote the tail and the head of an arrow $\alpha\colon i \to j$ by $t\alpha = i$ and $h\alpha = j$, respectively. A path $\sigma\colon r \rightsquigarrow s$ of length l in Q is a sequence of consecutive arrows $\alpha_l \cdots \alpha_1$ such that $t\alpha_1 = r$, $h\alpha_l = s$, and $h\alpha_i = t\alpha_{i+1}$, for $i = 1, \ldots, l-1$. For each vertex $i \in Q_0$ there is a path $\varepsilon_i\colon i \rightsquigarrow i$ of length 0. Moreover, each arrow of Q is a path of length 1.

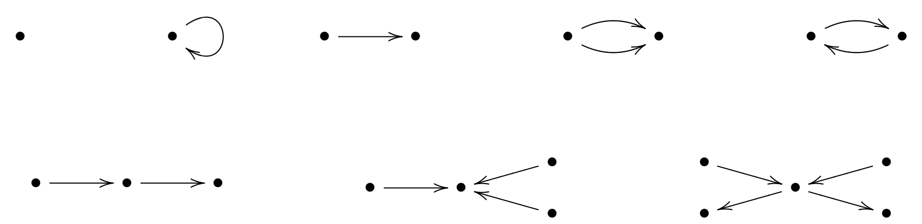


Figure 1: Some examples of quivers

A representation X of a quiver Q over a field $\mathbb{K}$ consists of a family

$$\{X(i)\ ;\ i \in Q_0\}$$

of finite dimensional $\mathbb{K}$-vector spaces together with a family

$$\{X(\alpha)\colon X(i) \to X(j)\ ;\ \alpha\colon i \to j \text{ in } Q_1\}$$

of $\mathbb{K}$-linear maps. Figure 2 gives an example.

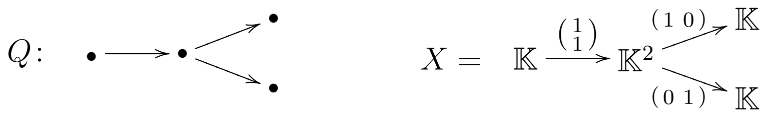


Figure 2: A representation X of a quiver Q

A morphism $f\colon X \to Y$ of representations of Q is a family

$$\{f(i)\colon X(i) \to Y(i)\ ;\ i \in Q_0\}$$

of $\mathbb{K}$-linear maps such that the diagram

$$\begin{tikzcd} X(i) \arrow[r, "f(i)"] \arrow[d, "X(\alpha)"] & Y(i) \arrow[d, "Y(\alpha)"] \\ X(j) \arrow[r, "f(j)"] & Y(j) \end{tikzcd}$$

commutes for each arrow $\alpha\colon i \to j$ in Q_1. We denote by $\mathrm{Hom}_Q(X,Y)$ the set of all morphisms from the representation X to Y. This set is seen to be a finite dimensional $\mathbb{K}$-vector space. Moreover, the composition of two morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ given by

$$(g \circ f)(i) = g(i) \circ f(i),$$

for all $i \in Q_0$, is a morphism from X to Z. Since in addition the composition of morphisms is $\mathbb{K}$-bilinear, $\operatorname{rep}(Q)$ carries the structure of a $\mathbb{K}$-category.

Given a path $\sigma = \alpha_l \cdots \alpha_1$ in Q and a representation X of Q, we define the $\mathbb{K}$-linear map $X(\sigma) = X(\alpha_l) \cdots X(\alpha_1)$. Moreover, we denote by $\mathbb{K}\sigma$ the one dimensional $\mathbb{K}$-vector space having as basis the path σ. With this, a quiver Q gives rise to a $\mathbb{K}$-algebra

$$\mathbb{K}Q = \bigoplus_{\substack{\sigma\colon r \rightsquigarrow s,\\ \text{path in } Q}} \mathbb{K}\sigma.$$

The multiplication of elements of $\mathbb{K}Q$ is defined by setting

$$(\sigma\colon r \rightsquigarrow s) \cdot (\tau\colon u \rightsquigarrow v) = \begin{cases} \sigma \cdot \tau\colon u \rightsquigarrow s & \text{if } v = r, \\ 0 & \text{if } v \neq r, \end{cases}$$

for all paths σ and τ in Q. Similarly, an arbitrary representation X of Q gives rise to a left $\mathbb{K}Q$-module

$$M_X = \bigoplus_{i \in Q_0} X(i).$$

The exterior product for M_X is defined by setting

$$(\sigma\colon r \rightsquigarrow s) \cdot x = \begin{cases} X(\sigma) \cdot x & \text{if } x \in X(r), \\ 0 & \text{if } x \in X(i),\ i \neq r, \end{cases}$$

for all paths σ in Q and for all vectors $x \in X(i)$, $i \in Q_0$. With these constructions we get an equivalence between the category $\operatorname{rep}(Q)$ of all representations of Q and the category $\operatorname{mod} \mathbb{K}Q$ of all left modules over $\mathbb{K}Q$. Note that $\mathbb{K}Q$ is a finite dimensional $\mathbb{K}$-algebra if and only if the quiver Q is finite and contains no oriented cycles, i.e. contains no non-trivial paths with identical starting and ending point.

There is a generalization to the above constructions which we only want to sketch very briefly: Let I be an ideal of $\mathbb{K}Q$ defined by a system

$$R = \left\{ r_i = \sum_{j=1}^{l_i} a_{ij}\sigma_{ij} \ ;\ i = 1, \ldots, k,\ a_{ij} \in \mathbb{K},\ \sigma_{ij} \text{ a path in } Q \right\},$$

of $\mathbb{K}$-linear combinations of paths in Q, where R satisfies some additional conditions which are required for the technical details of the construction and which are omitted here. A representation X of Q is said to be bound by I if the linear map

$$X(r_i) = \sum_{j=1}^{l_i} a_{ij} X(\sigma_{ij})$$

vanishes for all $r_i \in R$. We call the pair (Q, I) a bounded quiver and denote by $\operatorname{rep}(Q, I)$ the full subcategory of $\operatorname{rep}(Q)$ formed by all representations of Q bound by I. There is a categorial equivalence between $\operatorname{rep}(Q, I)$ and $\operatorname{mod} \mathbb{K}Q/I$. And with this generalized construction, every module category $\operatorname{mod} \Lambda$, where Λ is a finite dimensional $\mathbb{K}$-algebra, is equivalent to the category $\operatorname{rep}(Q, I)$ for an appropriate bounded quiver (Q, I). The keyword here is Morita-equivalence. For a complete although rather condensed description of the above concepts, see [1, §4].

2 Some facts on representations

The reason for not going further into details on bounded quivers in the previous section is that we will further narrow our scope and from now on only focus on finite quivers without relations and without oriented cycles. In the language of modules this corresponds to restricting to finite dimensional hereditary $\mathbb{K}$-algebras and their module categories. We will come back to the definition of a hereditary algebra later in this section. For the entire section we fix a quiver Q with properties as above once for all. Any representations mentioned are with reference to Q if nothing else is specified.

2.1 Subrepresentations and quotients

Let Y be a representation, and suppose that

$$\{X(i) \subseteq Y(i)\ ;\ i \in Q_0\}$$

is a family of subspaces such that

$$Y(\alpha)(X(i)) \subseteq X(j),$$

for all arrows $\alpha\colon i \to j$ in Q_1. Setting $X(\alpha) = Y(\alpha)|X(i)$ for each arrow, we obtain a representation X which we call a subrepresentation of Y. Moreover, given a subrepresentation X of Y, the quotient representation Y/X is defined by setting

$$(Y/X)(i) = Y(i)/X(i),$$

for each vertex $i \in Q_0$, and by taking the map

$$(Y/X)(\alpha)\colon Y(i)/X(i) \longrightarrow Y(j)/X(j),$$

induced by $Y(\alpha)$, for every arrow $\alpha\colon i \to j$ in Q_1.

Example 2.1.

(*i*) For any representation X, the zero representation as well as X itself are subrepresentations of X. We call them the trivial subrepresentations of X.

(*ii*) Suppose $f\colon X \to Y$ is a morphism of representations. Then $\ker f$ and $\operatorname{im} f$ are subrepresentations of X and of Y, respectively. Moreover, f induces an isomorphism $\overline{f}\colon X/\ker f \to \operatorname{im} f$.

(*iii*) Suppose Q, X and Y are as follows:

$$Q\colon \quad \bullet \longrightarrow \bullet \qquad\qquad X = \quad 0 \xrightarrow{0} \mathbb{K} \qquad\qquad Y = \quad \mathbb{K} \xrightarrow{1} \mathbb{K}$$

Then clearly, X is a subrepresentation of Y, and the quotient Y/X is given by

$$Y/X = \quad \mathbb{K} \xrightarrow{0} 0\,.$$

Note that Y/X is not a subrepresentation of Y. In contrast to the situation for vector spaces, a quotient representation need not necessarily be a subrepresentation, and conversely a subrepresentation need not necessarily be a quotient.

Whenever X is a subrepresentation of Y then the inclusion $X \hookrightarrow Y$ and the projection $Y \twoheadrightarrow Y/X$ are both morphisms of representations.

2.2 Direct sums and indecomposable representations

For representations X and Y we define the direct sum $X \oplus Y$, by setting

$$(X \oplus Y)(i) = X(i) \oplus Y(i),$$

for each vertex $i \in Q_0$, and by setting

$$(X \oplus Y)(\alpha) = \begin{pmatrix} X(\alpha) & 0 \\ 0 & Y(\alpha) \end{pmatrix},$$

for every arrow $\alpha\colon i \to j$ in Q_0. With this, a representation Z is called indecomposable if $Z \neq 0$ and if Z is not isomorphic to the direct sum of two non-trivial subrepresentations X and Y of Z. Otherwise Z is called decomposable.

Example 2.2. Suppose Q, X, Y and Z are as follows:

$$Q\colon \quad \bullet \longrightarrow \bullet \qquad X = \quad 0 \xrightarrow{0} \mathbb{K} \qquad Y = \quad \mathbb{K} \xrightarrow{1} \mathbb{K} \qquad Z = \quad \mathbb{K}^2 \xrightarrow{(1\ 1)} \mathbb{K}$$

Then X is indecomposable, since it is one dimensional over $\mathbb{K}$. As we have seen in part (iii) of example 2.1, the quotient representation

$$Y/X = \quad \mathbb{K} \xrightarrow{0} 0$$

is not a subrepresentation of Y. Hence X is the only non-trivial subrepresentation of Y up to isomorphism, and so Y must be indecomposable. Finally, Z is decomposable:

$$Z \simeq \quad (\,\mathbb{K} \xrightarrow{1} \mathbb{K}\,) \oplus (\,\mathbb{K} \xrightarrow{0} 0\,).$$

With the notion of decomposition of a representation, we are in the position to state a first important result:

Theorem 2.3 (Krull, Schmidt)**.** *Let $Z \neq 0$ be a representation.*

- (i) *There is a decomposition of Z as a direct sum of indecomposable representations $Z = Z_1 \oplus \cdots \oplus Z_k$.*
- (ii) *If $Z = Z'_1 \oplus \cdots \oplus Z'_l$ is another decomposition of Z into indecomposable representations then $k = l$, and up to renumbering $Z_i \simeq Z'_i$, for all i.*

The above theorem holds in a more general context than that of representations of quivers. For a proof see for instance [3, §3.4].

2.3 Quivers of finite/infinite representation type

With the notion of decomposition of representations, it appears natural to ask about the number of indecomposable representations of a given quiver. We say Q is of finite or infinite representation type, depending on whether there are finitely or infinitely many indecomposable representations up to isomorphism, respectively. Note that the disjoint union of two quivers K and L is of finite representation type if and only if both K and L have this property. So concerning the question of whether Q is of finite or infinite representation type, we may assume Q to be a connected quiver, i.e. in any decomposition of Q as a disjoint union of two full subquivers, one of these subquivers must be empty. Now there is a famous classification:

Theorem 2.4 (Gabriel)**.** *A connected quiver Q is of finite representation type if and only if the underlying non-oriented graph $|Q|$ is one of the following Dynkin diagrams:*

$$\mathbb{A}_n: \quad 1 - 2 - \cdots - n-1 - n \qquad (n \geq 1)$$

$$\mathbb{D}_n: \quad 1 - 2 - \cdots - n-2 \begin{smallmatrix} \diagup\, n-1 \\ \diagdown\, n \end{smallmatrix} \qquad (n \geq 4)$$

$$\mathbb{E}_6: \quad \begin{array}{ccccccccc} & & & & 4 & & & & \\ & & & & | & & & & \\ 1 & - & 2 & - & 3 & - & 5 & - & 6 \end{array}$$

$$\mathbb{E}_7: \quad \begin{array}{ccccccccccc} & & & & 4 & & & & & & \\ & & & & | & & & & & & \\ 1 & - & 2 & - & 3 & - & 5 & - & 6 & - & 7 \end{array}$$

$$\mathbb{E}_8: \quad \begin{array}{ccccccccccccc} & & & & 4 & & & & & & & & \\ & & & & | & & & & & & & & \\ 1 & - & 2 & - & 3 & - & 5 & - & 6 & - & 7 & - & 8 \end{array}$$

The proof of this theorem relies to large parts on the theory of root systems and can be found in [2]. Based on the theorem we will frequently call a quiver of finite representation type a Dynkin quiver, meaning that the underlying non-oriented graph of the quiver is a disjoint union of Dynkin diagrams $\mathbb{A}_n$, $\mathbb{D}_n$, $\mathbb{E}_6$, $\mathbb{E}_7$ and $\mathbb{E}_8$.

2.4 Some special kinds of representations

Here we want to describe some representations featuring special properties. To begin with, a representation is called simple if its only subrepresetations are the trivial ones. For a fixed vertex $i \in Q_0$, we denote by S_i the representation defined by setting

$$S_i(j) = \begin{cases} \mathbb{K} & \text{if } j = i, \\ 0 & \text{if } j \neq i, \end{cases}$$

for every vertex $j \in Q_0$, and $S_i(\alpha) = 0$ for each arrow $\alpha \in Q_1$. Since all S_i are one dimensional, they must be simple representations. Conversely, if S is simple then it is isomorphic to some S_i. Indeed, suppose not. Then up to replacing Q with a full subquiver if necessary, we may assume that S is sincere, i.e. $S(i) \neq 0$ for all $i \in Q_0$. As there exists a sink $k \in Q_0$, i.e. a vertex which may be the head of some arrows but the tail of none, clearly S_k is a non-trivial subrepresentation of S. Now this gives a contradiction.

Next we turn to projective representations. A representation is called projective if any morphism $g\colon P \to Y$ factors through any other morphism $f\colon X \to Y$, whenever f is surjective. Expressed in terms of diagrams this means the following:

$$\begin{tikzcd} X \arrow[r, two heads, "\forall f"] & Y \\ & P \arrow[u, "\forall g"'] \arrow[ul, dashed, "\exists h"] \end{tikzcd} \tag{1}$$

For any surjective morphism $f\colon X \to Y$ and any morphism $g\colon P \to Y$ there exists a morphism $h\colon P \to X$ such that the diagram shown in (1) commutes.

We intend to construct all projective indecomposable representations up to isomorphism. For a fixed vertex $i \in Q_0$, we define a representation P_i, by setting

$$P_i(j) = \bigoplus_{\substack{\sigma\colon i \rightsquigarrow j,\\ i,\, j \text{ fixed}}} \mathbb{K}\sigma,$$

for every $j \in Q_0$, and by defining

$$P_i(\alpha)(\sigma) = \alpha \cdot \sigma,$$

for any path $\sigma\colon i \rightsquigarrow j$ in Q starting at i, and for any arrow $\alpha\colon j \to k$ in Q_1.

Example 2.5. Suppose the quiver Q is given by

$$Q\colon \quad 1 \xrightarrow{\alpha} 2 \xleftarrow{\beta} 3\,.$$

Then the representations P_i are as follows:

$$\begin{aligned}
P_1 &= \quad \mathbb{K}\varepsilon_1 \xrightarrow{\alpha} \mathbb{K}\alpha \xleftarrow{0} 0 &&\simeq \quad \mathbb{K} \xrightarrow{1} \mathbb{K} \xleftarrow{0} 0\\
P_2 &= \quad 0 \xrightarrow{0} \mathbb{K}\varepsilon_2 \xleftarrow{0} 0 &&\simeq \quad 0 \xrightarrow{0} \mathbb{K} \xleftarrow{0} 0\\
P_3 &= \quad 0 \xrightarrow{0} \mathbb{K}\beta \xleftarrow{\beta} \mathbb{K}\varepsilon_3 &&\simeq \quad 0 \xrightarrow{0} \mathbb{K} \xleftarrow{1} \mathbb{K}
\end{aligned}$$

In order to show that the representations P_i are projective, we need the following result:

Lemma 2.6 (Yoneda). *For any vertex $i \in Q_0$ and any representation X, the map*

$$\Phi\colon \operatorname{Hom}_Q(P_i, X) \longrightarrow X(i),$$

sending $f \in \operatorname{Hom}_Q(P_i, X)$ to $f(i)(\varepsilon_i)$ is an isomorphism of $\mathbb{K}$-vector spaces.

The above lemma holds in the general context of category theory. For a proof see for instance [7]. The translation of our situation to the language of categories is as follows: The quiver Q may be seen as a $\mathbb{K}$-category with objects the vertices of Q, and with morphism spaces

$$\operatorname{Mor}(i, j) = \bigoplus_{\substack{\sigma\colon i \rightsquigarrow j,\\ i,\, j \text{ fixed}}} \mathbb{K}\sigma,$$

for all i, $j \in Q_0$. Then a representation X is a functor

$$X\colon Q \longrightarrow \operatorname{vect}_{\mathbb{K}}$$

from Q to the category $\operatorname{vect}_{\mathbb{K}}$ of finite dimensional $\mathbb{K}$-vector spaces, sending a vertex $i \in Q_0$ to $X(i)$ and an arrow $\alpha\colon i \to j$ in Q_1 to the $\mathbb{K}$-linear map $X(\alpha)\colon X(i) \to X(j)$. Particularly, the representation P_i turns out to be the functor $\operatorname{Mor}(i, ?)$, for each $i \in Q_0$.

Now we show that the P_i are projective. By assumption, $f\colon X \to Y$ in diagram (1) is surjective. So we conclude that $g\colon P_i \to Y$ satisfies

$$g(i)(\varepsilon_i) \in Y(i) \subseteq f(i)(X(i)).$$

Hence by applying the Yoneda lemma, a morphism $h\colon P_i \to X$ completing the commutative diagram (1) for $P = P_i$ may be constructed.

From the definition of P_i, it is easy to see that $P_i(i) \simeq \mathbb{K}$ for all $i \in Q_0$. This implies that the ring of endomorphisms of P_i

$$\operatorname{End}_Q(P_i) = \operatorname{Hom}_Q(P_i, P_i) \simeq \mathbb{K},$$

and this in turn means that P_i is indecomposable. Up to isomorphism the P_i are the only projective indecomposable representations. In order to verify this, we need the following construction of a new representation from a given representation X and a finite dimensional $\mathbb{K}$-vector space V. We define this new representation $X \otimes V$ by setting

$$(X \otimes V)(i) = X(i) \otimes_{\mathbb{K}} V,$$

for each vertex $i \in Q_0$ and

$$(X \otimes V)(\alpha) = X(\alpha) \otimes_{\mathbb{K}} \mathrm{id}_V\colon X(i) \otimes_{\mathbb{K}} V \longrightarrow X(j) \otimes_{\mathbb{K}} V,$$

for every arrow $\alpha\colon i \to j$ in Q_1. For a fixed representation Z and a fixed vertex $i \in Q_0$ we define a morphism

$$p_{Z,i}\colon P_i \otimes Z(i) \longrightarrow Z,$$

by setting

$$p_{Z,i}((\sigma\colon i \rightsquigarrow j) \otimes z) = Z(\sigma)(z) \in Z(j),$$

for any path σ starting in i, and any $z \in Z(i)$. Using this, we set

$$p_Z = (\ldots, (p_{Z,i}), \ldots)\colon \bigoplus_{i \in Q_0} P_i \otimes Z(i) \longrightarrow Z.$$

Note that by construction the morphism p_Z is surjective.

Now suppose P is a projective indecomposable representation. As

$$p_P\colon \bigoplus_{i \in Q_0} P_i \otimes P(i) \longrightarrow P$$

is surjective and P is projective, the identity morphism id_P factors through p_P. Thus P must be an indecomposable direct summand of $\bigoplus P_i \otimes P(i)$ and hence must be isomorphic to some P_i, by theorem 2.3.

The above constructions also give way to defining a canonical projective resolution for an arbitrary representation Z. Given such a Z, there is a short exact sequence

$$0 \longrightarrow \bigoplus_{\alpha \in Q_1} P_{h\alpha} \otimes Z(t\alpha) \longrightarrow \bigoplus_{i \in Q_0} P_i \otimes Z(i) \longrightarrow Z \longrightarrow 0. \tag{2}$$

For details, particularly concerning the maps occurring in (2), see [1, §5.1]. The existence of a projective resolution of length at most 1 for arbitrary representations $Z \in \mathrm{rep}(Q)$ as shown in (2) is equivalent to any of the following properties:

(i) Any subrepresentation of a projective representation of Q is projective.

(ii) The extension groups $\mathrm{Ext}^n_Q(X, Y)$ of order $n \geq 2$ vanish for arbitrary representations X and Y of Q.

An algebra yielding a module category which has one and hence all of the above properties is called hereditary. As already mentioned earlier in this section, finite dimensional hereditary $\mathbb{K}$-algebras over an algebraically closed field $\mathbb{K}$ correspond exactly to quiver algebras for finite quivers without relations and without oriented cycles.

Before turning to so called injective representations we want to establish a duality construction for representations, based on the duality construction for vector spaces. The reason for this will become clear later on. Denote by

$$D\colon \operatorname{vect}_{\mathbb{K}} \longrightarrow \operatorname{vect}_{\mathbb{K}}$$

the functor assigning to a $\mathbb{K}$-vector space its dual. From linear algebra it is well known that D is a contravariant equivalence of categories and that D^2 is isomorphic to the identity functor. Also note that D translates surjective $\mathbb{K}$-linear maps to injective ones and vice versa. This induces a contravariant equivalence called duality of representations and again denoted by

$$D\colon \operatorname{rep}(Q) \longrightarrow \operatorname{rep}(Q^{op}),$$

where Q^{op} is the quiver arising from Q by keeping the set of vertices of Q but replacing each arrow $\alpha\colon i \to j$ with its reversed $\alpha^*\colon j \to i$. Using the same symbol D for both the duality of vector spaces and the duality of representations will cause no confusion, as it will always be clear from the context which functor is meant. For any representation $X \in \operatorname{rep}(Q)$ its dual $DX \in \operatorname{rep}(Q^{op})$ is defined by setting

$$(DX)(i) = D(X(i)),$$

for each vertex $i \in Q_0$, and

$$(DX)(\alpha^*) = D(X(\alpha)),$$

for every arrow $\alpha \in Q_1$. Similarly, the dual $Df \in \operatorname{Hom}_{Q^{op}}(DY, DX)$ of an arbitrary morphism $f \in \operatorname{Hom}_Q(X, Y)$ is given by defining

$$(Df)(i) = D(f(i)),$$

for each $i \in Q_0$. Note that as for vector spaces the dual of a surjective morphism of representations is injective and vice versa, and again the square of the duality functor for representations is isomorphic to the identity.

A representation I is called injective if any morphism $g\colon X \to I$ factors through any other morphism $f\colon X \to Y$, whenever f is injective. In terms of diagrams this reads as follows:

$$\begin{tikzcd} X \arrow[r, hook, "\forall f"] \arrow[d, "\forall g"'] & Y \arrow[dl, dashed, "\exists h"] \\ I & \end{tikzcd} \tag{3}$$

For any injective morphism $f\colon X \to Y$ and any morphism $g\colon X \to I$ there exists a morphism $h\colon Y \to I$ making the diagram in (3) commutative.

Using the duality construction above, it turns out that a representation $I \in \mathrm{rep}(Q)$ is injective if and only if its dual $DI \in \mathrm{rep}(Q^{op})$ is projective. In particular, the only injective indecomposable representations of Q are $I_i = DP_i^{op}$, where P_i^{op} is the projective indecomposable representation of Q^{op} associated with the vertex $i \in Q_0$.

Example 2.7. Suppose the quiver Q and its opposite Q^{op} are given by

$$Q\colon \quad 1 \xrightarrow{\alpha} 2 \xleftarrow{\beta} 3 \qquad\qquad Q^{op}\colon \quad 1 \xleftarrow{\alpha^*} 2 \xrightarrow{\beta^*} 3$$

Then the representations P_i^{op} are as follows:

$$\begin{aligned}
P_1^{op} &= \quad \mathbb{K}\varepsilon_1^* \xleftarrow{0} 0 \xrightarrow{0} 0 && \simeq \quad \mathbb{K} \xleftarrow{0} 0 \xrightarrow{0} 0 \\
P_2^{op} &= \quad \mathbb{K}\alpha^* \xleftarrow{\alpha^*} \mathbb{K}\varepsilon_2^* \xrightarrow{\beta^*} \mathbb{K}\beta^* && \simeq \quad \mathbb{K} \xleftarrow{1} \mathbb{K} \xrightarrow{1} \mathbb{K} \\
P_3^{op} &= \quad 0 \xleftarrow{0} 0 \xrightarrow{0} \mathbb{K}\varepsilon_3^* && \simeq \quad 0 \xleftarrow{0} 0 \xrightarrow{0} \mathbb{K}
\end{aligned}$$

Hence the injective indecomposable representations of Q are:

$$\begin{aligned}
I_1 &\simeq \quad \mathbb{K} \xrightarrow{0} 0 \xleftarrow{0} 0 \\
I_2 &\simeq \quad \mathbb{K} \xrightarrow{1} \mathbb{K} \xleftarrow{1} \mathbb{K} \\
I_3 &\simeq \quad 0 \xrightarrow{0} 0 \xleftarrow{0} \mathbb{K}
\end{aligned}$$

2.5 The Auslander-Reiten quiver

Finally, we want to introduce the Auslander-Reiten quiver Γ_Q associated with the quiver Q. This is a very powerful tool in representation theory, and many of the arguments and methods used in the two main parts of this document are in one or the other way related to Γ_Q.

As a preparation we first collect some more definitions and facts. Given morphisms $f \in \mathrm{Hom}_Q(X,Y)$ and $g \in \mathrm{Hom}_Q(Y,Z)$, suppose there exist morphisms $f' \in \mathrm{Hom}_Q(Y,X)$ and $g' \in \mathrm{Hom}_Q(Z,Y)$ such that $f' \circ f = \mathrm{id}_X$ and $g \circ g' = \mathrm{id}_Z$. Then f is called a section and g is called a retraction. For non-isomorphic indecomposable representations X and Z, a morphism $h \in \mathrm{Hom}_Q(X,Z)$ is called irreducible if for any factorization $h = f \circ g$ either f is a section or g is a retraction.

Note that the quiver algebra $\mathbb{K}Q$ is the direct sum

$$\mathbb{K}Q = \bigoplus_{i \in Q_0} P_i.$$

This implies that $\mathrm{Hom}_Q(\ ?\ , \mathbb{K}Q)$ is a functor from $\mathrm{rep}(Q)$ to $\mathrm{rep}(Q^{op})$ which carries the projective indecomposable representation P_i of Q to P_i^{op}. By composition with the duality functor D, we get the functor

$$F = D\,\mathrm{Hom}_Q(\ ?\ , \mathbb{K}Q) \colon\ \mathrm{rep}(Q) \longrightarrow \mathrm{rep}(Q).$$

For a non-projective indecomposable representation X, we apply F to the canonical projective resolution of X. This gives a new short exact sequence

$$0 \longrightarrow \tau X \longrightarrow \bigoplus_{\alpha \in Q_1} I_{h\alpha} \otimes X(t\alpha) \longrightarrow \bigoplus_{i \in Q_0} I_i \otimes X(i) \longrightarrow 0. \tag{4}$$

For the maps occurring in (4), see [1, §5.2]. The representation τX is called the Auslander-Reiten translate of X, and the map τ thus defined is called the Auslander-Reiten translation. Note that τ is a bijection from the isomorphism classes of non-projective indecomposable representations to those of non-injective indecomposable representations. Its inverse τ^{-1} is defined similarly, for non-injective indecomposable representations.

For a non-projective indecomposable representation X, a non-split exact sequence

$$0 \longrightarrow N \longrightarrow E \xrightarrow{f} X \longrightarrow 0$$

is called an Auslander-Reiten sequence (or an almost split sequence) if N is indecomposable and if every morphism $g \colon Y \to X$ which is not a retraction factors through f. Now there is an important result ensuring the existence of Auslander-Reiten sequences. A proof of it can be found in [1, §1].

Theorem 2.8. *For any non-projective indecomposable representation X there exists a unique Auslander-Reiten sequence*

$$0 \longrightarrow N \longrightarrow E \longrightarrow X \longrightarrow 0$$

up to isomorphism. Moreover, N is isomorphic to τX. Similarly, for any non-injective indecomposable representation Y there exists a unique Auslander-Reiten sequence

$$0 \longrightarrow Y \longrightarrow E' \longrightarrow M \longrightarrow 0$$

up to isomorphism, and M is isomorphic to $\tau^{-1}Y$.

Now we are in the position to describe the Auslander-Reiten quiver Γ_Q. As the name suggests, Γ_Q is a quiver in the sense of our definition. There is exactly one vertex in Γ_Q for each isomorphism class of indecomposable representations of Q, and we label each vertex with a representative of the corresponding isomorphism class. If X and Y are vertices of Γ_Q then there is an arrow from X to Y if and only if $\mathrm{Hom}_Q(X,Y)$ contains an irreducible morphism. According to theorem 2.4 this description implies that Γ_Q is a finite quiver if and only if Q is a Dynkin quiver.

Unless specified otherwise, from now on we will always assume that Q is a connected Dynkin quiver. The Auslander-Reiten quiver is related to Auslander-Reiten sequences in the following way: Suppose X is a vertex in Γ_Q and $f_1,\dots,f_k$ are all the arrows starting in X. Assuming that the arrow f_i ends in Y_i for $i=1,\dots,k$, there is an arrow $g_i\colon Y_i \to \tau^{-1}X$ in Γ_Q for each Y_i, and the sequence

$$\begin{tikzcd}[row sep=small] & Y_1 \arrow[dr, "g_1"] & & \\ 0 \arrow[r] & X \arrow[ur, "f_1"] \arrow[dr, "f_k"'] & \oplus \cdots \oplus & \tau^{-1}X \arrow[r] & 0 \\ & Y_k \arrow[ur, "g_k"'] & & \end{tikzcd}$$

is an Auslander-Reiten sequence. Omitting the zeros to the left and to the right, this sequence describes an elementary unit of Γ_Q called a mesh. The construction of Γ_Q can now be accomplished in the following way: Start by establishing every possible irreducible inclusion relation among the projective indecomposable representations P_i. Then every constellation

$$\begin{tikzcd}[row sep=small] & P_{i_1} \\ P_i \arrow[ur, hook] \arrow[dr, hook] & \vdots \\ & P_{i_k} \end{tikzcd}$$

such that no further irreducible inclusion for P_i is possible, is the left half of a mesh in Γ_Q. Since Auslander-Reiten sequences are exact, it is possible to compute the dimension vector of $\tau^{-1}P_i$. Note that for Dynkin quivers the isomorphism class of an indecomposable representation is uniquely determined by its dimension vector. Hence, as $\tau^{-1}P_i$ is required to be indecomposable, it can be constructed up to isomorphism from the dimension information. Thus every left half of a mesh can be completed. The process of completing meshes induces new left halves of meshes which can again be completed, etc. This "knitting" algorithm comes to an end as soon as the left hand vertices of all left halves of meshes turn out to be injective indecomposable representations. For a more formal foundation of the above description, consult [1, §6].

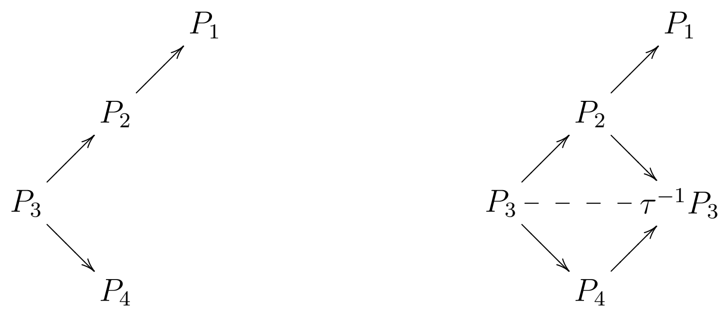
Example 2.9. Suppose Q and Q^{op} are given as follows:

$$Q\colon\quad 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \xleftarrow{\gamma} 4 \qquad\qquad Q^{op}\colon\quad 1 \xleftarrow{\alpha^*} 2 \xleftarrow{\beta^*} 3 \xrightarrow{\gamma^*} 4$$

We list the projective and injective indecomposable representations:

$$\begin{aligned}
P_1 &= \mathbb{K} \xrightarrow{1} \mathbb{K} \xrightarrow{1} \mathbb{K} \xleftarrow{0} 0 & \qquad I_1 &= \mathbb{K} \xrightarrow{0} 0 \xrightarrow{0} 0 \xleftarrow{0} 0\\
P_2 &= 0 \xrightarrow{0} \mathbb{K} \xrightarrow{1} \mathbb{K} \xleftarrow{0} 0 & I_2 &= \mathbb{K} \xrightarrow{1} \mathbb{K} \xrightarrow{0} 0 \xleftarrow{0} 0\\
P_3 &= 0 \xrightarrow{0} 0 \xrightarrow{0} \mathbb{K} \xleftarrow{0} 0 & I_3 &= \mathbb{K} \xrightarrow{1} \mathbb{K} \xrightarrow{1} \mathbb{K} \xleftarrow{1} \mathbb{K}\\
P_4 &= 0 \xrightarrow{0} 0 \xrightarrow{0} \mathbb{K} \xleftarrow{1} \mathbb{K} & I_4 &= 0 \xrightarrow{0} 0 \xrightarrow{0} 0 \xleftarrow{0} \mathbb{K}
\end{aligned}$$

Establishing the irreducible inclusions among the P_i yields the situation



(a) Irreducible inclusions (b) First mesh completed

Figure 3: First steps for the construction of Γ_Q

pictured in figure 3(a). Now P_2, P_3 and P_4 form the left half of a mesh, which is completed in figure 3(b), by adding the representation

$$\tau^{-1}P_3 \;=\; P_2 \oplus P_4/P_3 \;\simeq\; 0 \xrightarrow{0} \mathbb{K} \xrightarrow{1} \mathbb{K} \xleftarrow{1} \mathbb{K}$$

and the corresponding arrows. With the completion of the first mesh (marked by a dashed line in figure 3(b)), there arise two new left halves of meshes, the first one formed by P_4 and $\tau^{-1}P_3$, and the second one by P_1, P_2 and $\tau^{-1}P_3$. The finally resulting Auslander-Reiten quiver Γ_Q is shown in figure 4. Note that the maps within the representations have been omitted here, since the non-trivial ones can always be chosen to be 1.

$$\begin{tikzcd}[column sep=small]
 & & \mathbb{K}\,\mathbb{K}\,\mathbb{K}\,0 \arrow[rr, dashed, no head] \arrow[dr, dashed, no head] & & 0\,0\,0\,\mathbb{K} & \\
 & 0\,\mathbb{K}\,\mathbb{K}\,0 \arrow[ur] \arrow[dr] \arrow[rr, dashed, no head] & & \mathbb{K}\,\mathbb{K}\,\mathbb{K}\,\mathbb{K} \arrow[ur] \arrow[dr] & & \\
0\,0\,\mathbb{K}\,0 \arrow[ur] \arrow[dr] \arrow[rr, dashed, no head] & & 0\,\mathbb{K}\,\mathbb{K}\,\mathbb{K} \arrow[ur] \arrow[dr] \arrow[rr, dashed, no head] & & \mathbb{K}\,\mathbb{K}\,0\,0 \arrow[dr] & \\
 & 0\,0\,\mathbb{K}\,\mathbb{K} \arrow[ur] \arrow[rr, dashed, no head] & & 0\,\mathbb{K}\,0\,0 \arrow[ur] \arrow[rr, dashed, no head] & & \mathbb{K}\,0\,0\,0
\end{tikzcd}$$

Figure 4: The completed Auslander-Reiten quiver Γ_Q

3 Geometric and combinatorial methods

In this section we want to describe some geometric aspects of representation theory and also introduce some combinatorial methods based on the Auslander-Reiten quiver Γ_Q, which are used in the following two parts of this document. As before, if nothing else is specified then Q is a connected Dynkin quiver, i.e. Q is of finite representation type. However, note that most of the results presented here hold for a much bigger class than that of Dynkin quivers.

3.1 Affine spaces of representations

We will assume throughout that Q has n vertices, i.e. $Q_0 = \{1, \dots, n\}$. We call an n-tuple

$$\mathbf{d} = (d_1, \dots, d_n) \qquad (\text{all } d_i \in \mathbb{N}_0)$$

a dimension vector, and we say that a representation Z is of dimension $\mathbf{d}$ if $\dim_{\mathbb{K}} Z(i) = d_i$ for $i = 1, \dots, n$. Conversely, for a representation Z we denote its dimension vector by $\mathbf{dim}\, Z$. We define

$$\mathrm{rep}(Q, \mathbf{d}) = \{Z \in \mathrm{rep}(Q) \; ; \; \mathbf{dim}\, Z = \mathbf{d}\}$$

to be the set of all representations of Q of dimension $\mathbf{d}$. For each vertex $i \in Q_0$, we fix a basis for the vector spaces $Z(i)$ occurring in representations

$Z \in \mathrm{rep}(Q, \mathbf{d})$. Then $\mathrm{rep}(Q, \mathbf{d})$ may be identified with

$$\mathrm{rep}(Q, \mathbf{d}) = \prod_{\alpha \in Q_1} \mathrm{Mat}(d_{h\alpha} \times d_{t\alpha}, \mathbb{K}),$$

and this way it becomes obvious that $\mathrm{rep}(Q, \mathbf{d})$ is a $\mathbb{K}$-vector space of dimension

$$N = \dim_{\mathbb{K}} \mathrm{rep}(Q, \mathbf{d}) = \sum_{\alpha \in Q_1} d_{t\alpha} \cdot d_{h\alpha}.$$

Let $\mathbb{K}[\mathrm{rep}(Q, \mathbf{d})]$ be the algebra of regular (or polynomial) functions on $\mathrm{rep}(Q, \mathbf{d})$. Denoting by

$$\{X_{\alpha,i,j} \ ; \ \alpha \in Q_1, \ i \leq d_{h\alpha}, \ j \leq d_{t\alpha}\}$$

the dual basis of the standard basis

$$\{E_{\alpha,i,j} \ ; \ \alpha \in Q_1, \ i \leq d_{h\alpha}, \ j \leq d_{t\alpha}\}$$

of $\mathrm{rep}(Q, \mathbf{d})$, we get the following description for $\mathbb{K}[\mathrm{rep}(Q, \mathbf{d})]$:

$$\mathbb{K}[\mathrm{rep}(Q, \mathbf{d})] = \mathbb{K}[X_{\alpha,i,j}]_{\alpha \in Q_1, \, i \leq d_{h\alpha}, \, j \leq d_{t\alpha}}.$$

Hence $\mathbb{K}[\mathrm{rep}(Q, \mathbf{d})]$ is isomorphic to the ring of polynomials in N indeterminates over $\mathbb{K}$.

Example 3.1. Let Q be the quiver

$$Q\colon \quad 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$$

and $\mathbf{d} = (2, 2, 1)$ a dimension vector. Then the generic representation of $\mathrm{rep}(Q, \mathbf{d})$ becomes

$$X = \quad \mathbb{K}^2 \xrightarrow{\begin{pmatrix} X_{\alpha,1,1} & X_{\alpha,1,2} \\ X_{\alpha,2,1} & X_{\alpha,2,2} \end{pmatrix}} \mathbb{K}^2 \xrightarrow{(\, X_{\beta,1,1} \ X_{\beta,1,2} \,)} \mathbb{K} \, .$$

Hence $\mathbb{K}[\mathrm{rep}(Q, \mathbf{d})] \simeq \mathbb{K}[X_1, \ldots, X_6]$, with indeterminates X_i.

The ring of regular functions gives rise to defining the Zariski topology for $\mathrm{rep}(Q, \mathbf{d})$: Closed sets of $\mathrm{rep}(Q, \mathbf{d})$ are defined to be exactly the zero sets

$$\mathcal{Z}(I) = \{Z \in \mathrm{rep}(Q, \mathbf{d}) \ ; \ f(Z) = 0 \text{ for all } f \in I\},$$

of arbitrary ideals $I = (f_1, \ldots, f_k)$ of $\mathbb{K}[\mathrm{rep}(Q, \mathbf{d})]$. Note that $\mathbb{K}[\mathrm{rep}(Q, \mathbf{d})]$ is noetherian, i.e. every ideal in $\mathbb{K}[\mathrm{rep}(Q, \mathbf{d})]$ is finitely generated. Equipped with its ring of regular functions and the Zariski topology, $\mathrm{rep}(Q, \mathbf{d})$ carries the structure of an N-dimensional affine space over $\mathbb{K}$.

There is a canonical action of the group

$$\mathrm{Gl}(\mathbf{d}) = \prod_{i=1}^{n} \mathrm{GL}(d_i, \mathbb{K})$$

on $\mathrm{rep}(Q, \mathbf{d})$, given by

$$((g_1, \ldots, g_n) * Z)(\alpha) = g_{h\alpha} \cdot Z(\alpha) \cdot g_{t\alpha}^{-1}, \tag{5}$$

for any $g = (g_1, \ldots, g_n) \in \mathrm{Gl}(\mathbf{d})$, for any $Z \in \mathrm{rep}(Q, \mathbf{d})$ and for any $\alpha \in Q_1$. Equation (5) just says that the diagram

$$\begin{tikzcd} \mathbb{K}^{d_i} \arrow[r, "Z(\alpha)"] \arrow[d, "g_i"'] & \mathbb{K}^{d_j} \arrow[d, "g_j"] \\ \mathbb{K}^{d_i} \arrow[r, "g_j Z(\alpha) g_i^{-1}"] & \mathbb{K}^{d_j} \end{tikzcd}$$

commutes for any g, Z and α. Thus it becomes obvious that the $\mathrm{Gl}(\mathbf{d})$-orbits in $\mathrm{rep}(Q, \mathbf{d})$ are exactly the isomorphism classes of representations of dimension $\mathbf{d}$. Note that $\mathrm{Gl}(\mathbf{d})$ is a linear algebraic group. Indeed, it is a closed subgroup of $\mathrm{GL}(N, \mathbb{K})$.

3.2 Invariants and semi-invariants

The action of $\mathrm{Gl}(\mathbf{d})$ on $\mathrm{rep}(Q, \mathbf{d})$ given in the previous subsection induces an action of $\mathrm{Gl}(\mathbf{d})$ on the ring of regular functions $\mathbb{K}[\mathrm{rep}(Q, \mathbf{d})]$, defined by

$$(g * f)(Z) = f(g^{-1} * Z),$$

for all $g \in \mathrm{Gl}(\mathbf{d})$, all $f \in \mathbb{K}[\mathrm{rep}(Q, \mathbf{d})]$ and all $Z \in \mathrm{rep}(Q, \mathbf{d})$. This leads to the interesting question about the existence and the structure of invariant regular functions, i.e. functions $f \in \mathrm{rep}(Q, \mathbf{d})$ such that

$$g * f = f \qquad (\text{for all } g \in \mathrm{Gl}(\mathbf{d})).$$

From the definition it is clear that an invariant regular function has constant values on each isomorphism class of representations in $\mathrm{rep}(Q, \mathbf{d})$. Note that the constant functions in $\mathbb{K}[\mathrm{rep}(Q, \mathbf{d})]$ trivially are invariants.

Example 3.2. Suppose for now that Q is the oriented cycle

$$Q\colon \qquad \begin{tikzcd} 1 \arrow[r, bend left, "\alpha"] & 2 \arrow[l, bend left, "\beta"] \end{tikzcd}$$

and $\mathbf{d} = (r, s)$ a dimension vector with $r,\ s > 0$. Then the generic representation of $\mathrm{rep}(Q, \mathbf{d})$ is

$$X = \qquad \begin{tikzcd} \mathbb{K}^r \arrow[r, bend left, "(X_{\alpha,i,j})"] & \mathbb{K}^s \arrow[l, bend left, "(X_{\beta,k,l})"] \end{tikzcd} .$$

We claim that the regular function

$$f((X_{\alpha,i,j}), (X_{\beta,k,l})) = \det((X_{\beta,k,l}) \cdot (X_{\alpha,i,j}))$$

is an invariant. Indeed, for an arbitrary $g = (g_1, g_2)$ in $\mathrm{Gl}(\mathbf{d})$ we get

$$\begin{aligned}(g * f)((X_{\alpha,i,j}), (X_{\beta,k,l})) &= f(g^{-1} * ((X_{\alpha,i,j}), (X_{\beta,k,l})))\\ &= \det((g_1^{-1}(X_{\beta,k,l})g_2) \cdot (g_2^{-1}(X_{\alpha,i,j})g_1))\\ &= \det((X_{\beta,k,l}) \cdot (X_{\alpha,i,j}))\\ &= f((X_{\alpha,i,j}), (X_{\beta,k,l})).\end{aligned}$$

Unfortunately, if we return to the situation of quivers without oriented cycles then there is no hope for finding a non-constant invariant: Having no oriented cycles in Q implies that the zero representation $0_{\mathbf{d}} \in \mathrm{rep}(Q, \mathbf{d})$ belongs to the Zariski closure of the orbit $\mathrm{Gl}(\mathbf{d}) * Z$ of any representation $Z \in \mathrm{rep}(Q, \mathbf{d})$.

A somewhat less restrictive but still very promising concept is the one of rational invariants. Since $\mathbb{K}[\mathrm{rep}(Q, \mathbf{d})]$ is a domain, one can define its quotient field, denoted by $\mathbb{K}(\mathrm{rep}(Q, \mathbf{d}))$. The elements of $\mathbb{K}(\mathrm{rep}(Q, \mathbf{d}))$ are called rational functions. Such a function f can be expressed as

$$f = \frac{f_1}{f_2},$$

with $f_1,\ f_2 \in \mathbb{K}[\mathrm{rep}(Q, \mathbf{d})]$. So again, the action of $\mathrm{Gl}(\mathbf{d})$ on $\mathrm{rep}(Q, \mathbf{d})$ induces an action on $\mathbb{K}(\mathrm{rep}(Q, \mathbf{d}))$, and a function $f \in \mathbb{K}(\mathrm{rep}(Q, \mathbf{d}))$ is called a rational invariant if

$$(g * f)(Z) = \left(g * \frac{f_1}{f_2}\right)(Z) = \frac{f_1(g^{-1} * Z)}{f_2((g^{-1} * Z)} = f(Z),$$

for all $g \in \mathrm{Gl}(\mathbf{d})$ and all $Z \in \mathrm{rep}(Q, \mathbf{d})$. For instance for the class of tame quivers (also called extended Dynkin or Euclidean quivers) non-constant rational invariants exist, and they were completely classified by Ringel in [9].

For Dynkin quivers, i.e. quivers of finite representation type, again there are only constant rational invariants: We write $\mathrm{rep}(Q, \mathbf{d})$ as

$$\mathrm{rep}(Q, \mathbf{d}) = \bigcup_{Z \in \mathcal{I}} \overline{\mathrm{Gl}(\mathbf{d}) * Z},$$

where $\mathcal{I}$ is an appropriate system of representatives of the orbits of $\mathrm{rep}(Q, \mathbf{d})$. But since Q is representation finite, there are only finitely many possibilities of forming pairwise non-isomorphic representations of dimension $\mathbf{d}$, by theorem 2.3, and hence there are only finitely many orbits. So

$$\mathcal{I} = \{Z_1, \ldots, Z_k\} \subseteq \mathrm{rep}(Q, \mathbf{d})$$

is a finite set, and without loss of generality we may assume that

$$\overline{\mathrm{Gl}(\mathbf{d}) * Z_i} \nsubseteq \overline{\mathrm{Gl}(\mathbf{d}) * Z_j},$$

for all $i, j \in \mathcal{I}$ and $i \neq j$. Since $\mathrm{rep}(Q, \mathbf{d})$ is irreducible, i.e. it is not the union of two Zariski closed proper subsets, the only way to avoid a contradiction is to assume that $\mathcal{I}$ consists of only one representation Z. Hence there exists a dense and open orbit in $\mathrm{rep}(Q, \mathbf{d})$, independently of $\mathbf{d}$. And since any rational invariant is constant on each orbit, it must be constant on $\mathrm{rep}(Q, \mathbf{d})$.

Further easing the requirements leads to the notion of a semi-invariant (also called a relative invariant), which we want to describe now. As mentioned before, $\mathrm{Gl}(\mathbf{d})$ is a linear algebraic group, i.e. it carries the structure of an affine $\mathbb{K}$-variety. A group homomorphism $\chi\colon \mathrm{Gl}(\mathbf{d}) \to \mathbb{K}^*$ is called a rational character, if χ belongs to the algebra $\mathbb{K}[\mathrm{Gl}(\mathbf{d})]$ of regular functions on $\mathrm{Gl}(\mathbf{d})$. It is well known that the rational characters of $\mathrm{GL}(m, \mathbb{K})$ are exactly the integral powers $(\det_m)^z$, for $z \in \mathbb{Z}$, of the determinant function. Hence the rational characters of $\mathrm{Gl}(\mathbf{d})$ are exactly the functions of the form

$$\chi = (\det{}_{d_1})^{z_1} \cdots (\det{}_{d_n})^{z_n} \qquad (z_1, \ldots, z_n \in \mathbb{Z}).$$

Note that the set of all rational characters of an algebraic group forms an abelian group. For $\mathrm{Gl}(\mathbf{d})$, we denote this group by $C(\mathrm{Gl}(\mathbf{d}))$.

Based on the above, a non-zero regular function $f \in \mathbb{K}[\mathrm{rep}(Q, \mathbf{d})]$ is called a semi-invariant if there exists a rational character $\chi_f\colon \mathrm{Gl}(\mathbf{d}) \to \mathbb{K}^*$ such that

$$g * f = \chi_f(g) \cdot f,$$

for all $g \in \mathrm{Gl}(\mathbf{d})$. Up to a non-zero factor, the character χ_f is determined uniquely by the semi-invariant f, and

$$\chi_f = (\det{}_{d_1})^{z_1} \cdots (\det{}_{d_n})^{z_n} \qquad (z_1, \ldots, z_n \in \mathbb{Z})$$

is called the weight of f. Note that sometimes it is more convenient to refer to the vector of powers $(z_1, \ldots, z_n) \in \mathbb{Z}^n$ as the weight of f. The set of all characters occurring as weights of semi-invariants form a subgroup of $C(\mathrm{Gl}(\mathbf{d}))$, denoted by $W(\mathrm{Gl}(\mathbf{d}))$.

Example 3.3. Let Q be the quiver

$$Q\colon \quad 1 \xrightarrow{\alpha} 2$$

and $\mathbf{d} = (k, k)$ a dimension vector for Q with $k > 0$. Then the generic representation X is given by

$$X = \quad \mathbb{K}^k \xrightarrow{\begin{pmatrix} X_{\alpha,1,1} & \cdots & X_{\alpha,1,k} \\ \vdots & & \vdots \\ X_{\alpha,k,1} & \cdots & X_{\alpha,k,k} \end{pmatrix}} \mathbb{K}^k ,$$

and clearly $f = \det$ is a semi-invariant of weight

$$\chi(g) = \chi(g_1, g_2) = \det(g_2) \cdot \det{}^{-1}(g_1),$$

for all $g \in \mathrm{Gl}(\mathbf{d})$. As Q is a Dynkin quiver, it seems that with the notion of semi-invariants we have finally come upon a concept which does not fall back to its trivial case for quivers of finite representation type.

3.3 Semi-invariants of quivers

If there is an open orbit in $\mathrm{rep}(Q, \mathbf{d})$ under the action of $\mathrm{Gl}(\mathbf{d})$ then there is a nice method due to Schofield for explicitly computing all the semi-invariants of $\mathrm{rep}(Q, \mathbf{d})$. We want to give an overview of this method here.

Having an open orbit, $\mathrm{rep}(Q, \mathbf{d})$ is called a prehomogeneous vector space and $\mathbf{d}$ a prehomogeneous dimension vector. Recall that for Dynkin quivers we have already shown that any dimension vector is prehomogeneous. We will always denote by T a representative of the open orbit of $\mathrm{rep}(Q, \mathbf{d})$ with its decomposition

$$T = \bigoplus_{i=1}^{r} T_i^{\lambda_i}$$

into pairwise non-isomorphic indecomposable representations T_i, occurring with multiplicities $\lambda_i \in \mathbb{N}$. Note that by theorem 2.3 the number r does not depend on our choice for T. We will assume throughout, that $\mathbf{d}$ is sincere, i.e. $d_i \neq 0$, for $i = 1, \ldots, n$. This is no restriction as we may always reduce to the full subquiver of Q supported by $\mathbf{d}$, if $\mathbf{d}$ is not initially sincere.

Now the first step on the way to classifying the semi-invariants for quivers is due to Sato and Kimura. A proof of their result can be found in [10].

Theorem 3.4 (Sato, Kimura)**.** *Let $\mathbf{d}$ be a prehomgeneous dimension vector, and let $\mathcal{D}_1, \ldots, \mathcal{D}_s$ be the irreducible components of codimension 1 of the complement of the open orbit $\mathbb{K}[\mathrm{rep}(Q, \mathbf{d})] \setminus \mathrm{Gl}(\mathbf{d}) * T$. Denote by $f_1, \ldots, f_s$ the irreducible monic polynomials such that $\mathcal{Z}(f_i) = \mathcal{D}_i$, for all i. Then we have:*

(i) *The functions $f_1, \ldots f_s$ are algebraically independent semi-invariants.*

(ii) *Every non-constant semi-invariant $f \in \mathbb{K}[(\mathrm{rep}(Q, \mathbf{d}))]$ is of the form*

$$f = c \cdot f_1^{\mu_1} \cdots f_s^{\mu_s},$$

for $\mu_1, \ldots, \mu_s \in \mathbb{N}$, and for some constant factor $c \in \mathbb{K}^*$.

With the above result, the aim now will be to find "enough" algebraically independent irreducible semi-invariant polynomials. The next step is due to Kac and tells us what "enough" means, i.e. what the value of s is. The corresponding proof is given in [4]. We denote by

$$\mathrm{Sl}(\mathbf{d}) = \prod_{i=1}^{n} \mathrm{SL}(d_i, \mathbb{K})$$

the product of the special linear groups at all vertices of Q, and by

$$\mathbb{K}(\mathrm{rep}(Q, \mathbf{d}))^{\mathrm{Sl}(\mathbf{d})} \subseteq \mathbb{K}(\mathrm{rep}(Q, \mathbf{d}))$$

the subfield of rational $\mathrm{Sl}(\mathbf{d})$-invariant functions on $\mathrm{rep}(Q, \mathbf{d})$.

Theorem 3.5 (Kac)**.** *With the notations introduced above we have*

$$\mathrm{tr.deg}_{\mathbb{K}}(\mathbb{K}(\mathrm{rep}(Q, \mathbf{d}))^{\mathrm{Sl}(\mathbf{d})}) = n - r.$$

We claim that the field $\mathbb{K}(\mathrm{rep}(Q, \mathbf{d}))^{\mathrm{Sl}(\mathbf{d})}$ is generated from $\mathbb{K}$, by adjunction of the basic $\mathrm{Gl}(\mathbf{d})$-semi-invariants $f_1, \ldots, f_s$ postulated in theorem 3.4. From this we conclude that

$$s = n - r.$$

In order to prove the claim above, we show that the algebra $\mathbb{K}[\mathrm{rep}(Q,\mathbf{d})]^{\mathrm{Sl}(\mathbf{d})}$ is generated by $f_1,\ldots,f_s$. For this, note that there is an action of the torus

$$\mathcal{T}=\prod_{i\in Q_0}(\mathbb{K}^*)_i$$

on $\mathbb{K}[\mathrm{rep}(Q,\mathbf{d})]^{\mathrm{Sl}(\mathbf{d})}$, defined by

$$(t*f)(X)=f((t_{h\alpha}\cdot t_{t\alpha}^{-1}\cdot X(\alpha))_{\alpha\in Q_1}),$$

for all $t=(t_i)_{i\in Q_0}\in\mathcal{T}$, all $f\in\mathbb{K}[\mathrm{rep}(Q,\mathbf{d})]^{\mathrm{Sl}(\mathbf{d})}$ and all $X=(X(\alpha))_{\alpha\in Q_1}\in \mathrm{rep}(Q,\mathbf{d})$. It is well known that the rational characters of $\mathcal{T}$ are exactly the functions of the form

$$\chi(t)=\chi((t_i)_{i\in Q_0})=\prod_{i\in Q_0}t_i^{z_i}\qquad (z_i\in\mathbb{Z}).$$

We denote by $C(\mathcal{T})$ the group of rational characters of $\mathcal{T}$. For an arbitrary character $\chi\in C(\mathcal{T})$, the subset

$$\mathbb{K}[\mathrm{rep}(Q,\mathbf{d})]_\chi^{\mathrm{Sl}(\mathbf{d})}=\left\{f\in\mathbb{K}[\mathrm{rep}(Q,\mathbf{d})]^{\mathrm{Sl}(\mathbf{d})}\ ;\ (t*f)=\chi(t)\cdot f\quad(\forall t\in\mathcal{T})\right\}$$

is a submodule of $\mathbb{K}[\mathrm{rep}(Q,\mathbf{d})]^{\mathrm{Sl}(\mathbf{d})}$, considered as a $\mathcal{T}$-module with respect to the operation of $\mathcal{T}$ as described above. A character $\chi\in C(\mathcal{T})$ for which

$$\mathbb{K}[\mathrm{rep}(Q,\mathbf{d})]_\chi^{\mathrm{Sl}(\mathbf{d})}\neq 0$$

is called a weight of $\mathbb{K}[\mathrm{rep}(Q,\mathbf{d})]^{\mathrm{Sl}(\mathbf{d})}$ with respect to the action of $\mathcal{T}$. The set of all such weights form a subgroup of $C(\mathcal{T})$, denoted by $W(\mathcal{T})$. The torus $\mathcal{T}$ may be seen as a subgroup of $\mathrm{Gl}(\mathbf{d})$: We identify an arbitrary element $t=(t_i)_{i\in Q_0}\in\mathcal{T}$ with $g_t=(g_{t_i})_{i\in Q_0}\in\mathrm{Gl}(\mathbf{d})$, by setting g_{t_i} to be the diagonal matrix in $\mathrm{GL}(d_i,\mathbb{K})$, having t_i on every diagonal entry. With this, the weights $\chi\in W(\mathcal{T})$ are seen to be functions of the form

$$\chi(t)=\chi((t_i)_{i\in Q_0})=\prod_{i\in Q_0}(t_i^{d_i})^{z_i}=\prod_{i\in Q_0}(\det{}_{d_i}(g_{t_i}))^{z_i}\qquad(z_i\in\mathbb{Z}).$$

So together with the definition of weights of $\mathbb{K}[\mathrm{rep}(Q,\mathbf{d})]^{\mathrm{Sl}(\mathbf{d})}$, it becomes clear that the group $W(\mathcal{T})$ is canonically isomorphic to the group $W(\mathrm{Gl}(\mathbf{d}))$ of weights of semi-invariants.

A module over a torus can always be decomposed into a direct sum, by its weights. For details on this, consult for instance [5, §III.1.3]. So $\mathbb{K}[\mathrm{rep}(Q,\mathbf{d})]^{\mathrm{Sl}(\mathbf{d})}$ can be written as

$$\mathbb{K}[\mathrm{rep}(Q,\mathbf{d})]^{\mathrm{Sl}(\mathbf{d})}=\bigoplus_{\chi\in W(\mathcal{T})}\mathbb{K}[\mathrm{rep}(Q,\mathbf{d})]_\chi^{\mathrm{Sl}(\mathbf{d})}.$$

Since $W(\mathrm{Gl}(\mathbf{d}))$ coincides with $W(\mathcal{T})$, any $h \in \mathbb{K}[\mathrm{rep}(Q,\mathbf{d})]^{\mathrm{Sl}(\mathbf{d})}$ is of the form

$$h = h_1 + \cdots + h_k,$$

where each h_i is a $\mathrm{Gl}(\mathbf{d})$-semi-invariant, and hence is a monomial in the basic semi-invariants $f_1, \ldots, f_s$ according to theorem 3.4.

The previous steps, showing that under our assumptions for Q and $\mathbf{d}$ there are exactly $s = n - r$ irreducible algebraically independent semi-invariants, are the foundation for Schofields work, to which we want to turn now. He introduces the notion of perpendicular categories, which are certain full subcategories of $\mathrm{rep}(Q)$. Then he shows that the basic semi-invariants $f_1, \ldots, f_s$ are related to the simple objects of an appropriate such category. The reference for the material presented below is [11].

Let X be an arbitrary representation of $\mathrm{rep}(Q)$. Then the right perpendicular category $X^\perp$ is the full subcategory of $\mathrm{rep}(Q)$ having as objects

$$X^\perp = \left\{ Y \in \mathrm{rep}(Q) \; ; \; \mathrm{Hom}_Q(X,Y) = \mathrm{Ext}^1_Q(X,Y) = 0 \right\}.$$

Similarly the left perpendicular category ${}^\perp X$ is defined as the full subcategory with objects

$${}^\perp X = \left\{ Z \in \mathrm{rep}(Q) \; ; \; \mathrm{Hom}_Q(Z,X) = \mathrm{Ext}^1_Q(Z,X) = 0 \right\}.$$

The right and left perpendicular categories $X^\perp$ and ${}^\perp X$ are related to each other by the equation $X^\perp = {}^\perp\tau(X)$, where τ is the Auslander-Reiten translation for all non-projective indecomposable direct summands of X, and $\tau(P_i) = I_i$ for projective indecomposable representations.

Theorem 3.6 (Schofield)**.** *Let Q be a quiver with n vertices, $\mathbf{d}$ a prehomogeneous sincere dimension vector and $T = T_1^{\lambda_1} \oplus \cdots \oplus T_r^{\lambda_r}$ a representative of the open orbit in $\mathrm{rep}(Q,\mathbf{d})$. Then $T^\perp$ and ${}^\perp T$ are equivalent to categories $\mathrm{rep}(Q^\perp)$ and $\mathrm{rep}({}^\perp Q)$, respectively, where $Q^\perp$ and ${}^\perp Q$ are quivers with $s = n - r$ vertices.*

According the above theorem, both $T^\perp$ and ${}^\perp T$ contain exactly $s = n - r$ simple objects. We will denote them by

$$S_1^{(T^\perp)}, \ldots, S_s^{(T^\perp)} \qquad \text{and} \qquad S_1^{({}^\perp T)}, \ldots, S_s^{({}^\perp T)},$$

respectively. Note that as objects of $\mathrm{rep}(Q)$, the simple objects of $T^\perp$ and ${}^\perp T$ are indecomposable representations, but need not necessarily be simple.

Let X and Y be representations in $\mathrm{rep}(Q,\mathbf{d})$ and $\mathrm{rep}(Q,\mathbf{e})$, respectively. Constructing a non-zero morphism $f \in \mathrm{Hom}_Q(X,Y)$ amounts to solving the system of linear equations

$$Y(\alpha) \cdot f(i) - f(j) \cdot X(\alpha) = 0 \quad (\forall \alpha \colon i \to j \text{ in } Q_1), \tag{6}$$

with matrices of indeterminates $(f(i)_{kl})_{k\leq e_i, l\leq d_i}$, for all $i \in Q_0$. We denote by $M_{\mathbf{d},\mathbf{e}}(X,Y)$ the matrix of coefficients of the system of equations given in (6), with respect to some basis.

Theorem 3.7 (Schofield)**.** *With Q, $\mathbf{d}$ and T as before we get:*

(*i*) *For any $X \in \operatorname{rep}(Q,\mathbf{d})$ and any simple object $S_i^{(T^\perp)}$ the matrix*

$$M_i(X) = M_{\mathbf{d},\mathbf{dim}\, S_i^{(T^\perp)}}(X, S_i^{(T^\perp)})$$

is a square matrix and $p_i(X) = \det(M_i(X))$ defines an irreducible non-zero semi-invariant p_i.

(*ii*) *The semi-invariants $p_1,\dots,p_s$ are algebraically independent.*

(*iii*) *Each semi-invariant p_i vanishes at $X \in \operatorname{rep}(Q,\mathbf{d})$ if and only if*

$$\operatorname{Hom}_Q(X, S_i^{(T^\perp)}) \neq 0.$$

Note that up to renumbering and up to a constant factor the polynomials p_i coincide with the basic semi-invariants f_i of theorem 3.4. And theorem 3.7 not only gives an explicit algorithm for computing the basic semi-invariants, but rather as well contains a method for determining their zero sets combinatorially.

3.4 Computing semi-invariants of quivers

The theory presented in the last subsection enables us to compute all semi-invariants of $\operatorname{rep}(Q,\mathbf{d})$ under the action of $\operatorname{Gl}(\mathbf{d})$. All we require in addition is to find answers to the following questions:

- What are the indecomposable direct summands of a representative T of the open orbit?
- What are the indecomposable objects in $T^\perp$?
- What are the simple objects in $T^\perp$?

Since all of the above questions have to do with indecomposable representations and their relations among each other, it is not hard to guess that in order to find answers, we will bring in the Auslander-Reiten quiver. However before doing so, we will have to carry out some translations for the various conditions, hidden in the above questions.

For the question about finding a representative T of the open orbit for a given dimension vector $\mathbf{d}$, first note that in practice, e.g. when looking for concrete examples for testing some conjecture, things are often the other way around, meaning that one just picks appropriate indecomposable representations $T_1, \ldots, T_r$ such that the representation

$$T = \bigoplus_{i=1}^{r} T_i^{\lambda_i} \qquad \text{(with all } \lambda_i \geq 1\text{)}$$

lies in the open orbit of $\operatorname{rep}(Q, \mathbf{d})$, for $\mathbf{d} = \mathbf{dim}\, T$. The tool used for choosing appropriate direct summands T_i is the following: By the Artin-Voigt lemma a representation T belonging to the open orbit of $\operatorname{rep}(Q, \mathbf{d})$ is characterized by featuring $\operatorname{Ext}^1_Q(T, T) = 0$. For a proof of this, see [9, §2]. Now requiring $\operatorname{Ext}^1_Q(T, T) = 0$ is equivalent to requiring

$$\operatorname{Ext}^1_Q(T_i, T_j) = 0, \tag{7}$$

for all indecomposable direct summands T_i and T_j of T. Whenever as in (7), one is only interested in the dimension of extension groups, then this computation can be translated to the computation of dimensions of morphism spaces, by means of the Auslander-Reiten formula

$$\dim_{\mathbb{K}} \operatorname{Ext}^1_Q(X, \ ? \ ) = \dim_{\mathbb{K}} \operatorname{Hom}_Q(\ ? \ , \tau X),$$

for arbitrary non-projective indecomposable representations X, where τ is the Auslander-Reiten translation. Note that for any projective representation P and any injective representation I, we always have

$$\operatorname{Ext}^1_Q(P, \ ? \ ) = \operatorname{Ext}^1_Q(\ ? \ , I) = 0.$$

A proof of the Auslander-Reiten formula can be found in [1, §2]. Thus the requirement in (7) is equivalent to the condition

$$\operatorname{Hom}_Q(T_j, \tau T_i) = 0,$$

for indecomposable direct summands T_i and T_j of T, whenever T_i is non-projective.

For the second question, recall that

$$T^{\perp} = \left\{ X \in \operatorname{rep}(Q) \ ; \ \operatorname{Hom}_Q(T, X) = \operatorname{Ext}^1_Q(T, X) = 0 \right\}.$$

So when looking for indecomposable objects X of $T^{\perp}$, the condition $\operatorname{Ext}^1_Q(T, X) = 0$ can be translated to the equivalent condition

$$\operatorname{Hom}_Q(X, \tau T_i) = 0,$$

for all non-projective indecomposable direct summands T_i of T, again by applying the Auslander-Reiten formula.

Finally the third question, concerning simple objects in $T^\perp$, can as well be formulated in terms of dimensions of morphism spaces, in case Q is a Dynkin quiver. For this we need the following fact: Recall that Q has n vertices. Whenever we find a set

$$\mathcal{R} = \{R_1, \ldots, R_n\} \subseteq \mathrm{rep}(Q)$$

of n indecomposable representations, satisfying $\mathrm{Hom}_Q(R_i, R_j) = 0$ for all $i \neq j$, then

$$\mathcal{R} = \{S_1, \ldots, S_n\},$$

i.e. $\mathcal{R}$ must be the set of all simple representations of Q. Now recall that $T^\perp$ is equivalent to the category of representations of a quiver with $s = n - r$ vertices. Hence as an application of the above, if we find indecomposable objects $R_1, \ldots, R_s$ in $T^\perp$ such that $\mathrm{Hom}_Q(R_i, R_j) = 0$, for all $i \neq j$, then we have found all simple objects $S_1^{(T^\perp)}, \ldots, S_s^{(T^\perp)}$ of $T^\perp$.

With the above, the task of computing semi-invariants of quivers with Schofields method boils down to determining dimensions of morphism spaces for indecomposable representations. But these dimensions can be read from the Auslander-Reiten quiver Γ_Q as we want to show now. A more formal description of the material presented below can be found in [1].

If Q is a Dynkin quiver then Γ_Q can be seen as a full subquiver of the translation quiver $\mathbb{Z}Q$ associated with Q. In order to construct $\mathbb{Z}Q$, we start with the quiver $\mathbb{Z} \times Q$. We label its vertices with (i, j) for all $i \in \mathbb{Z}$ and all $j \in Q_0$. Moreover, for each arrow $\alpha\colon (i, k) \to (i, l)$ in $\mathbb{Z} \times Q$ we add an arrow $\alpha'\colon (i, l) \to (i + 1, k)$. This finishes the construction of $\mathbb{Z}Q$. Note that for $\mathbb{Z}Q$ the orientation of Q plays no role, i.e. for any two quivers K and L with $|K| = |L|$ we find that $\mathbb{Z}K = \mathbb{Z}L$.

We define the translation τ by setting $\tau(i, j) = (i - 1, j)$ for all vertices $(i, j) \in \mathbb{Z}Q$. Note that this translation coincides with the Auslander-Reiten translation for vertices belonging to Γ_Q when embedded in $\mathbb{Z}Q$, and which are associated to isomorphism classes of non-projective indecomposable representations. For an illustration of building a translation quiver, see example 3.10. And for the embedding of Γ_Q into $\mathbb{Z}Q$, the construction carried out for a quiver of type $\mathbb{A}_5$ in this example may be considered as "generic" for all Dynkin quivers.

We denote by $\mathbb{K}(\mathbb{Z}Q)$ the $\mathbb{K}$-category of $\mathbb{Z}Q$ and call it the mesh category associated with Q. Suppose that the vertices (i, j) and $(k, l) \in \mathbb{Z}Q$ both lie in the embedding of Γ_Q, and as such are associated to the isomorphism classes of the indecomposable representations X and Y, respectively.

Then the space of morphisms $\mathrm{Hom}_Q(X,Y)$ is isomorphic to the space of morphisms $\mathrm{Hom}((i,j),(k,l))$ in the mesh category. And for the latter we have the following combinatorial description:

$$\mathrm{Hom}((i,j),(k,l)) = \left(\bigoplus_{\substack{\sigma\colon (i,j) \rightsquigarrow (k,l),\\ \text{path in } \mathbb{Z}Q}} \mathbb{K}\sigma \right) \Big/ M((i,j),(k,l)). \tag{8}$$

The space $M((i,j),(k,l))$ divided out in (8) is generated by all the meshes in $\mathbb{Z}Q$ "lying between" (i,j) and (k,l). More precisely let $\Sigma_1,\ldots,\Sigma_p$ be the family of all meshes in $\mathbb{Z}Q$ run through by some path $\sigma\colon (i,j) \rightsquigarrow (k,l)$. Each Σ_u is of the form

$$\begin{tikzcd}
 & (c_{u,1},d_{u,1}) \arrow[dr, "\beta_{u,1}"] \arrow[dd, dotted, no head] & \\
\tau(a_u,b_u) \arrow[ur, "\alpha_{u,1}"] \arrow[dr, "\alpha_{u,v_u}"'] & & (a_u,b_u) \\
 & (c_{u,v_u},d_{u,v_u}) \arrow[ur, "\beta_{u,v_u}"'] &
\end{tikzcd}$$

Denote by $\gamma_{u,t}$ the paths of length 2 in the mesh Σ_u, given by

$$\gamma_{u,t} = \beta_{u,t} \cdot \alpha_{u,t},$$

for $t = 1,\ldots,u_v$. Then $M((i,j),(k,l))$ is the space

$$M((i,j),(k,l)) = \sum_{u=1}^{p} \sum_{\varphi_u,\psi_u} \mathbb{K}(\psi_u \cdot (\gamma_{u,1} + \cdots + \gamma_{u,u_v}) \cdot \varphi_u),$$

where φ_u and ψ_u are arbitrary paths in $\mathbb{Z}Q$ with fixed tail and head vertices as follows:

$$\begin{aligned} \varphi_u &\colon (i,j) \rightsquigarrow \tau(a_u,b_u), \\ \psi_u &\colon (a_u,b_u) \rightsquigarrow (k,l). \end{aligned}$$

So the dimensions of spaces of morphism between indecomposable representations are given by counting paths modulo mesh relations in Γ_Q.

We want to illustrate the constructions and concepts discussed so far by some "generic" examples. In one of these examples we run through the entire process of computing all the basic semi-invariants with Schofields method (see example 3.10).

Example 3.8. Suppose (i,j) is an arbitrary vertex in $\mathbb{K}Q$. As the trivial path $\varepsilon_{(i,j)}$ is the only path in $\mathbb{K}Q$, leading from (i,j) to (i,j), we conclude that

$$\mathrm{Hom}((i,j),(i,j)) \simeq \mathbb{K}\varepsilon_{(i,j)} \simeq \mathbb{K}.$$

Example 3.9. Consider the mesh

$$\begin{tikzcd} & (c_1,d_1) \arrow[dr, "\beta_1"] \arrow[dd, dotted, no head] & \\ \tau(a,b) \arrow[ur, "\alpha_1"] \arrow[dr, "\alpha_m"'] & & (a,b) \\ & (c_m,d_m) \arrow[ur, "\beta_m"'] & \end{tikzcd}$$

in $\mathbb{K}Q$ for an arbitrary vertex (a,b). Then

$$\begin{aligned} \mathrm{Hom}(\tau(a,b),(a,b)) &\simeq \left(\bigoplus_{i=1}^{m} \mathbb{K}(\beta_i \cdot \alpha_i)\right) \Big/ \mathbb{K}(\beta_1 \cdot \alpha_1 + \cdots + \beta_m \cdot \alpha_m) \\ &\simeq \mathbb{K}^{m-1}. \end{aligned}$$

Example 3.10. Suppose Q is a Dynkin quiver of type $\mathbb{A}_5$. Then the translation quiver $\mathbb{Z}Q = \mathbb{Z}\mathbb{A}_5$ is constructed as follows: Start with the quiver $\mathbb{Z} \times \overrightarrow{Q}$, where $\overrightarrow{Q}$ is the following quiver:

$$\overrightarrow{Q}: \qquad 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \xrightarrow{\gamma} 4 \xrightarrow{\delta} 5$$

Thus we get the picture shown in figure 5. We label the vertices of $\mathbb{Z} \times \overrightarrow{Q}$ with coordinates (i,j), for values $i \in \mathbb{Z}$ and $j \in Q_0$. In order to complete the translation quiver $\mathbb{Z}\mathbb{A}_5$, for every arrow $\alpha\colon (i,j) \to (i,k)$ in $\mathbb{Z} \times \overrightarrow{Q}$ we add an arrow $\alpha'\colon (i,k) \to (i+1,j)$. This way we get $\mathbb{Z}\mathbb{A}_5$, of which a part is shown in figure 6. Any dashed line segment between two vertices not "factoring" through a third vertex in the figure indicates a mesh. Reading any such line segment as an arrow from right to left, gives the translation τ of the corresponding vertex.

Now we turn to computing of dimensions of morphism spaces for this example. Starting from a fixed vertex (i,j), denoted by the symbol "$\otimes$" in figure 7, the dimension of $\mathrm{Hom}((i,j),(k,l))$ is written directly into the quiver for every vertex (k,l), where the bullet symbol stands for zero. The area of vertices (k,l) such that $\mathrm{Hom}((i,j),(k,l)) \neq 0$ is highlighted by drawing the dashed lines for meshes in this area, and omitting them otherwise. Similarly, in figure 8 the dimension of $\mathrm{Hom}((k,l),(i,j))$ is written directly into the

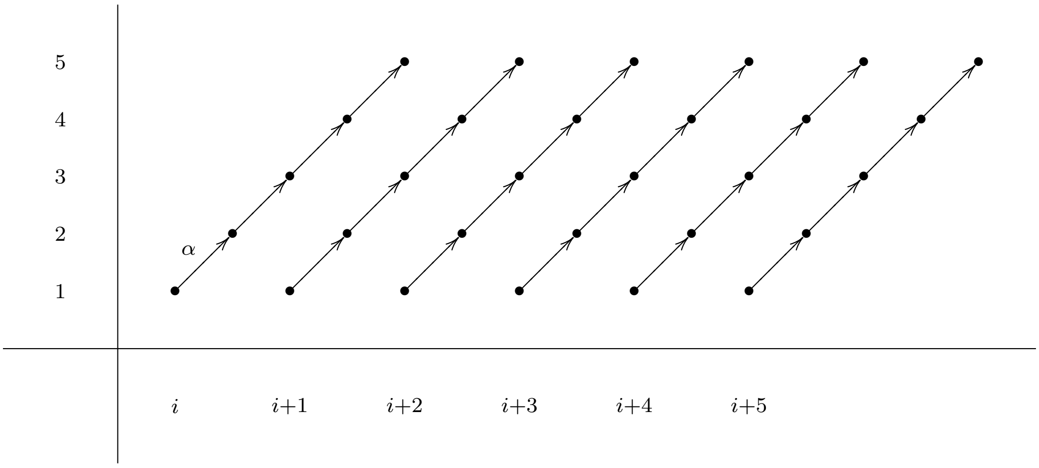


Figure 5: Construction of the translation quiver: The first step

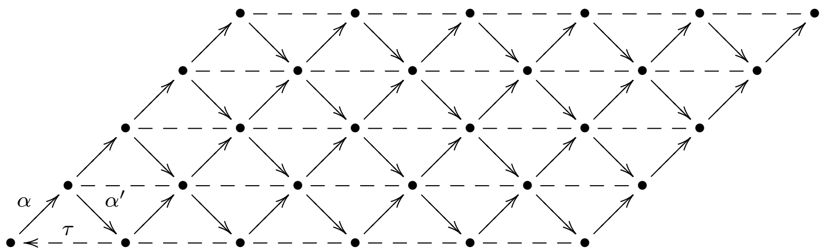


Figure 6: A part of the translation quiver

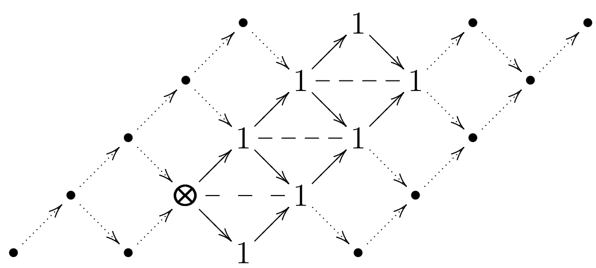


Figure 7: Dimensions of morphism spaces (“forward”)

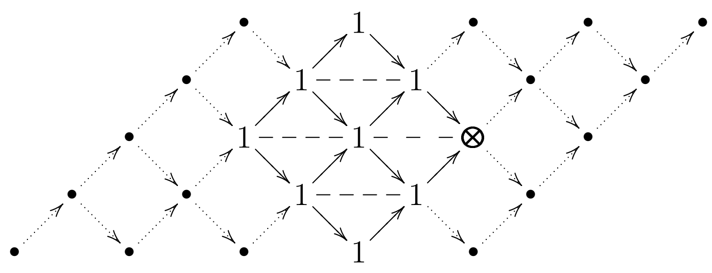


Figure 8: Dimensions of morphism spaces ("backward")

diagram for every vertex (k, l) and with respect to a fixed vertex (i, j), which is again marked by the symbol "$\otimes$".

Now suppose Q has the following orientation:

$$\begin{array}{rl} Q\colon & 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \xleftarrow{\gamma} 4 \xrightarrow{\delta} 5 \\ Q^{op}\colon & 1 \xleftarrow{\alpha^*} 2 \xleftarrow{\beta^*} 3 \xrightarrow{\gamma^*} 4 \xleftarrow{\delta^*} 5 \end{array}$$

Then Γ_Q can be seen as a full subquiver of $\mathbb{Z}\mathbb{A}_5$ in the following manner: Embed Q^{op} in $\mathbb{Z}\mathbb{A}_5$ such that every τ-orbit, i.e. every horizontal (dashed) line in figure 6, is met exactly once. Thus we get the picture shown in figure 9. The vertices of the embedded copy of Q^{op} in $\mathbb{Z}\mathbb{A}_5$ represent the projective indecomposable representations $P_1, \ldots, P_5$ of $\operatorname{rep}(Q)$. Drawing the square of

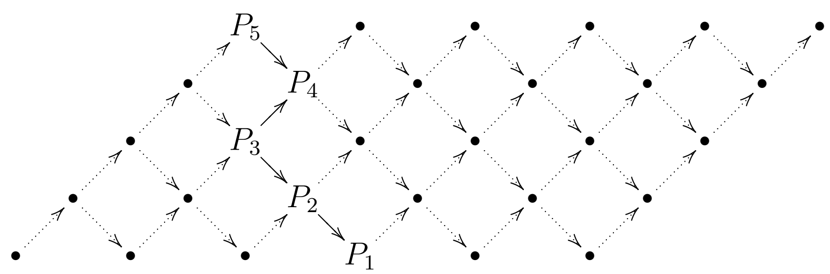


Figure 9: Embedding the opposite quiver in the translation quiver

vertices (k, l) with non-zero morphism spaces $\operatorname{Hom}(P_i, (k, l))$ for each P_i as shown before in figure 7, we get every vertex associated to an isomorphism class of indecomposable representations, by the Yoneda lemma. Thus in figure 10 we have the complete Auslander-Reiten quiver, bounded by the projective indecomposable representations $P_1, \ldots, P_5$ to the left and by the injective indecomposable representations $I_1, \ldots, I_5$ to the right. The area of vertices belonging to Γ_Q is highlighted by drawing the dashed lines indicating meshes inside Γ_Q, and omitting them otherwise.

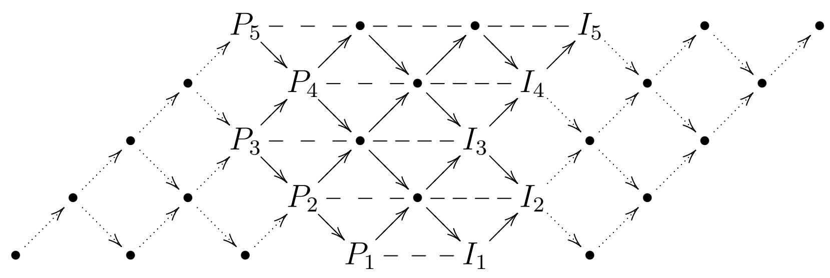


Figure 10: The Auslander-Reiten quiver embedded in the translation quiver

After all these preparations, we want to compute the semi-invariants for a dimension vector $\mathbf{d}$, specified by choosing a sincere representation T, satisfying $\mathrm{Ext}^1_Q(T,T) = 0$. One checks that T_1 and T_2 in figure 11 satisfy

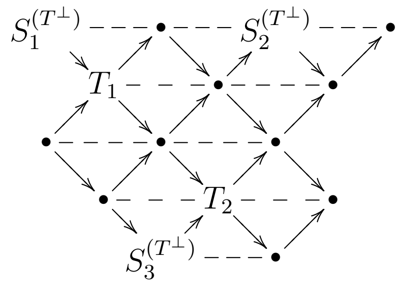


Figure 11: An open orbit and its perpendicular category

$\mathrm{Ext}^1_Q(T_i, T_j) = 0$ for $i, j = 1, 2$. Indeed, $T_1 = P_4$ is projective, and for $\mathrm{Ext}^1_Q(T_2, T_i)$, where $i = 1, 2$, we translate, by using the Auslander-Reiten formula. Moreover, setting $T = T_1 \oplus T_2$ and checking all the Hom and all the translated Ext conditions, we see that there are only three indecomposable objects up to isomorphism in $T^\perp$. Since $n = 5$ and T contains two indecomposable non-isomorphic direct summands, the number of simple objects in $T^\perp$ must be three as well. So every indecomposable object of $T^\perp$ is simple. The objects are denoted by $S_1^{(T^\perp)}$, $S_2^{(T^\perp)}$ and $S_3^{(T^\perp)}$ in figure 11. Recalling the positions of $P_1, \ldots, P_5$ in Γ_Q and applying the Yoneda lemma, we get the following list of relevant representations for this example:

$$
\begin{aligned}
T_1 &\simeq \quad 0 \xrightarrow{0} 0 \xrightarrow{0} \mathbb{K} \xleftarrow{1} \mathbb{K} \xrightarrow{1} \mathbb{K}\,, \\
T_2 &\simeq \quad \mathbb{K} \xrightarrow{1} \mathbb{K} \xrightarrow{1} \mathbb{K} \xleftarrow{1} \mathbb{K} \xrightarrow{1} \mathbb{K}\,, \\
S_1^{(T^\perp)} &\simeq \quad 0 \xrightarrow{0} 0 \xrightarrow{0} 0 \xleftarrow{0} 0 \xrightarrow{0} \mathbb{K}\,, \\
S_2^{(T^\perp)} &\simeq \quad 0 \xrightarrow{0} 0 \xrightarrow{0} 0 \xleftarrow{0} \mathbb{K} \xrightarrow{0} 0\,, \\
S_3^{(T^\perp)} &\simeq \quad \mathbb{K} \xrightarrow{1} \mathbb{K} \xrightarrow{1} \mathbb{K} \xleftarrow{0} 0 \xrightarrow{0} 0\,.
\end{aligned}
$$

We want to compute the semi-invariant denoted $p_1(X)$ in theorem 3.7 and associated to $S_1^{(T^\perp)}$. Since this object is supported only at vertex 5, the system of linear equations having to be solved turns out to be

$$S_1^{(T^\perp)}(\delta) \cdot f(4) - f(5) \cdot X(\delta) = 0.$$

But as $S_1^{(T^\perp)}(\delta) = 0$, the system further reduces to

$$f(5) \cdot X(\delta) = 0.$$

Taking generic matrices of appropriate dimensions for both, $f(5)$ and $X(\delta)$, we get

$$\begin{pmatrix} f(5)_{11} & f(5)_{12} \end{pmatrix} \cdot \begin{pmatrix} X_{\delta,1,1} & X_{\delta,1,2} \\ X_{\delta,2,1} & X_{\delta,2,2} \end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix}.$$

Executing the multiplication on the left hand side and remembering that the $f(5)_{ij}$ are the indeterminates, we get the homogeneous system of linear equations with coefficient matrix $M_1(X)$:

$$M_1(X) \cdot \begin{pmatrix} f(5)_{11} \\ f(5)_{12} \end{pmatrix} = \begin{pmatrix} X_{\delta,1,1} & X_{\delta,2,1} \\ X_{\delta,1,2} & X_{\delta,2,2} \end{pmatrix} \cdot \begin{pmatrix} f(5)_{11} \\ f(5)_{12} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

So our first semi-invariant p_1 is defined by setting

$$p_1(X) = \det(M_1(X)) = \det(X(\delta)),$$

for all $X \in \operatorname{rep}(Q, \mathbf{d})$. In a similar fashion the remaining two semi-invariants are evaluated to be

$$
\begin{aligned}
p_2(X) &= \det(X(\gamma)), \\
p_3(X) &= X_{\alpha,1,1},
\end{aligned}
$$

for all $X \in \operatorname{rep}(Q, \mathbf{d})$.

Example 3.11. With this last example we want to show that the basic semi-invariants may as well be a bit more complicated than the ones computed in example 3.10. Suppose Q is the same quiver as in example 3.10:

$$Q: \quad 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \xleftarrow{\gamma} 4 \xrightarrow{\delta} 5\ .$$

Consider the dimension vector $\mathbf{d} = (1,2,2,2,1)$. Then with the same methods as above, we get the following list of relevant representations of Q:

$$\begin{aligned}
T_1 &\simeq \quad 0 \xrightarrow{0} \mathbb{K} \xrightarrow{1} \mathbb{K} \xleftarrow{1} \mathbb{K} \xrightarrow{0} 0\ ,\\
T_2 &\simeq \quad \mathbb{K} \xrightarrow{1} \mathbb{K} \xrightarrow{1} \mathbb{K} \xleftarrow{1} \mathbb{K} \xrightarrow{1} \mathbb{K}\ ,\\
S_1^{(T^\perp)} &\simeq \quad 0 \xrightarrow{0} 0 \xrightarrow{0} \mathbb{K} \xleftarrow{1} \mathbb{K} \xrightarrow{0} 0\ ,\\
S_2^{(T^\perp)} &\simeq \quad 0 \xrightarrow{0} \mathbb{K} \xrightarrow{1} \mathbb{K} \xleftarrow{1} \mathbb{K} \xrightarrow{1} \mathbb{K}\ ,\\
S_3^{(T^\perp)} &\simeq \quad \mathbb{K} \xrightarrow{1} \mathbb{K} \xrightarrow{1} \mathbb{K} \xleftarrow{0} 0 \xrightarrow{0} 0\ .
\end{aligned}$$

With this, the corresponding semi-invariants are seen to be as follows:

$$\begin{aligned}
p_1(X) &= \det(X(\beta)),\\
p_2(X) &= \det \begin{pmatrix} X(\beta)\cdot X(\alpha) & X(\gamma) \\ 0 & X(\delta) \end{pmatrix} = \det \begin{pmatrix} (X(\beta)\cdot X(\alpha))_{11} & X(\gamma)_{11} & X(\gamma)_{12} \\ (X(\beta)\cdot X(\alpha))_{21} & X(\gamma)_{21} & X(\gamma)_{22} \\ 0 & X(\delta)_{11} & X(\delta)_{12} \end{pmatrix},\\
p_3(X) &= \det(X(\gamma)),
\end{aligned}$$

for all $X \in \operatorname{rep}(Q,\mathbf{d})$.

4 Preface to the main parts

Having presented algorithms for computing semi-invariants of quivers in the previous section, we now turn to the questions that are treated in the following two parts of this document.

4.1 The zero set of semi-invariants

The main object of investigation in these parts is the affine subvariety of common zeros of all non-constant semi-invariants of $\operatorname{rep}(Q,\mathbf{d})$, denoted and defined by

$$\mathcal{Z}_{Q,\mathbf{d}} = \{X \in \operatorname{rep}(Q,\mathbf{d})\ ;\ p_1(X) = \cdots = p_s(X) = 0\}\,,$$

respectively, where $p_1, \ldots, p_s$ are the basic semi-invariants. Note that $\mathcal{Z}_{Q,\mathbf{d}}$ can be written as

$$\mathcal{Z}_{Q,\mathbf{d}} = \mathcal{Z}(p_1) \cap \cdots \cap \mathcal{Z}(p_s),$$

i.e. as the intersection of the zero sets of the individual semi-invariants. But for the latter, Schofields theorem 3.7 gives a criterion saying at which $X \in \operatorname{rep}(Q, \mathbf{d})$ exactly they vanish: Namely $p_i(X) = 0$ if and only if

$$\operatorname{Hom}_Q(X, S_i^{(T^\perp)}) \neq 0 \qquad (i = 1, \ldots, s).$$

Thus we get a description of $\mathcal{Z}_{Q,\mathbf{d}}$ which allows us to carry out investigations, using methods similar to those discussed in the previous section:

$$\mathcal{Z}_{Q,\mathbf{d}} = \left\{ X \in \operatorname{rep}(Q, \mathbf{d})\ ;\ \operatorname{Hom}_Q(X, S_i^{(T^\perp)}) \neq 0, \text{ (for all } i = 1, \ldots, s) \right\}.$$

Note that with this description, $\mathcal{Z}_{Q,\mathbf{d}}$ is characterized purely in terms of dimensions of morphism spaces of indecomposable representations.

4.2 Complete intersections

The main goal in the two following parts is to give a description of the dimension vectors $\mathbf{d}$ for quivers Q of type $\mathbb{D}_n$, for which $\mathcal{Z}_{Q,\mathbf{d}}$ is a set theoretic complete intersection. The property of being a complete intersection for varieties can briefly be described as follows: From linear algebra we know that the set L of all solutions of a system of linear equations of rank s in N variables is always a subvariety of the N-dimensional affine space V of dimension

$$\dim L = N - s,$$

provided that L is non-empty. This of course is equivalent to saying that L, if non-empty, always has codimension

$$\operatorname{codim} L = \dim V - \dim L = s.$$

Now if we generalize to arbitrary affine subvarieties, i.e. the solution sets of systems of polynomial equations in an affine space, then the corresponding fact is no longer an equation but rather only an estimate: Suppose a non-empty affine subvariety L of the N-dimensional affine space V is described by a system of s polynomial equations $f_1, \ldots, f_s$, i.e.

$$L = \mathcal{Z}(f_1, \ldots, f_s).$$

Then the codimension of L can be estimated by

$$\operatorname{codim} L \leq s.$$

Based on this, we say that L is a set theoretic complete intersection if

$$\operatorname{codim} L = s.$$

A nice introduction to this subject, including a proof of the above estimate for $\operatorname{codim} L$, can be found in [6].

In the first of the two main parts of this document, we establish a good criterion and in the third part even a characterization of when $\mathcal{Z}_{Q,\mathbf{d}}$ is a complete intersection, for a quiver Q of type $\mathbb{D}_n$. Both criteria are in terms of the constellations of the indecomposable direct summands $T_1, \ldots, T_r$ of T in the Auslander-Reiten quiver Γ_Q, and in terms of the multiplicities λ_i of the direct summands T_i of T. Recall that T is a representative of the open orbit of $\operatorname{rep}(Q, \mathbf{d})$ with respect to the action of $\operatorname{Gl}(\mathbf{d})$.

The results presented in the following parts are refinements of and based on the work of Ch. Riedtmann and G. Zwara found in [8]. For a more general class of quivers, namely for all Dynkin and Euclidean quivers, they give a somewhat coarser criterion for $\mathcal{Z}_{Q,\mathbf{d}}$ being a complete intersection. Their criterion is based only on the multiplicities λ_i of the indecomposable direct summands T_i of a representative of the open orbit T.

4.3 Some final remarks

In this preface, we started out with the notion of a vector space over a field $\mathbb{K}$. Then we gradually moved on to the more general concepts of modules over finite dimensional $\mathbb{K}$-algebras. Now returning to the starting point, one might ask what the theory corresponding to the notions and results discussed here would look like in the context of vector spaces. The answer to this is quite simple: From the definition of quivers and their representations it becomes obvious that the category of finite dimensional vector spaces over a field $\mathbb{K}$ is equivalent to the category $\operatorname{rep}(Q)$, where Q is the quiver with only one vertex and no arrows, i.e. the quiver resembling the "trivial case" for the theory presented here. It comes as no big surprise that most of the notions are no longer too useful: For instance there is no point of defining projective or injective vector spaces, since every vector space has this property. Moreover, there is only one indecomposable vector space up to isomorphism – the simple one. So the Auslander-Reiten quiver is again just one vertex with no arrows. And in coincidence with theorem 3.5 there are no semi-invariants, etc. But still, it is interesting to note that the corresponding theory for vector spaces fits into the concepts of representation theory as a special case.

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Part II
On the zero set of semi-invariants for $\mathbb{D}_n$-quivers

Abstract

Let Q be a quiver of type $\mathbb{D}_n$, $\mathbf{d}$ a dimension vector for Q, and T a representative of the open orbit of the variety $\operatorname{rep}(Q, \mathbf{d})$ of $\mathbf{d}$-dimensional representations of Q, under the product $\mathrm{Gl}(\mathbf{d})$ of the general linear groups at all vertices of Q. Let $T = T_1^{\lambda_1} \oplus \cdots \oplus T_r^{\lambda_r}$ be a decomposition of T into pairwise non-isomorphic indecomposable representations T_i with multiplicities λ_i. We show that it depends on the multiplicity of at most one such direct summand whether or not the set of common zeros of all non-constant semi-invariants for $\operatorname{rep}(Q, \mathbf{d})$, with respect to the action of $\mathrm{Gl}(\mathbf{d})$, is a set theoretical complete intersection.

Samuel Beer

1 Introduction

Let $\mathbb{K}$ be an algebraically closed field, and let $Q = (Q_0, Q_1, t, h)$ be a finite quiver, i.e. a finite set $Q_0 = \{1, \ldots, n\}$ of vertices and a finite set Q_1 of arrows $\alpha\colon t\alpha \to h\alpha$, where $t\alpha$ and $h\alpha$ denote the tail and the head of α, respectively.

A representation of Q over $\mathbb{K}$ is a collection

$$(X(i);\ i \in Q_0)$$

of finite dimensional $\mathbb{K}$-vector spaces together with a collection

$$(X(\alpha)\colon X(t\alpha) \to X(h\alpha);\ \alpha \in Q_1)$$

of $\mathbb{K}$-linear maps. A morphism $f\colon X \to Y$ between two representations is a collection $(f(i)\colon X(i) \to Y(i))$ of $\mathbb{K}$-linear maps such that

$$f(h\alpha) \circ X(\alpha) = Y(\alpha) \circ f(t\alpha) \quad \text{for all } \alpha \in Q_1.$$

By $\sigma(X)$ we denote the number of pairwise non-isomorphic indecomposable direct summands occurring in a decomposition of X into indecomposables. According to the theorem of Krull-Schmidt, $\sigma(X)$ is well-defined. The dimension vector of a representation X of Q is the vector

$$\mathbf{dim}\, X = (\dim X(1), \ldots, \dim X(n)) \in \mathbb{N}^{Q_0}.$$

We denote the category of representations of Q by $\mathrm{rep}(Q)$, and for any vector $\mathbf{d} = (d_1, \ldots, d_n) \in \mathbb{N}^{Q_0}$

$$\mathrm{rep}(Q, \mathbf{d}) = \prod_{\alpha \in Q_1} \mathrm{Mat}(d_{h\alpha} \times d_{t\alpha}, \mathbb{K})$$

is the vector space of representations X of Q with $X(i) = \mathbb{K}^{d_i}$, $i \in Q_0$. The group

$$\mathrm{Gl}(\mathbf{d}) = \prod_{i=1}^{n} \mathrm{Gl}(d_i, \mathbb{K})$$

acts on $\mathrm{rep}(Q, \mathbf{d})$ by

$$((g_1, \ldots, g_n) \cdot X)(\alpha) = g_{h\alpha} \circ X(\alpha) \circ g_{t\alpha}^{-1}.$$

Note that the $\mathrm{Gl}(\mathbf{d})$-orbit of X consists exactly of the representations Y in $\mathrm{rep}(Q, \mathbf{d})$ which are isomorphic to X.

We call $\mathbf{d}$ a prehomogeneous dimension vector if $\mathrm{rep}(Q, \mathbf{d})$ contains an open orbit $\mathrm{Gl}(\mathbf{d}) \cdot T$. Such a representation T is characterized by $\mathrm{Ext}^1_Q(T, T) = 0$ (see [8]). If Q admits only finitely many indecomposable representations, or equivalently if the underlying graph of Q is a disjoint union of Dynkin diagrams $\mathbb{A}$, $\mathbb{D}$ or $\mathbb{E}$ (see [2]), every vector $\mathbf{d}$ is prehomogeneous. Indeed, any representation is a direct sum of indecomposables and therefore $\mathrm{rep}(Q, \mathbf{d})$ contains finitely many orbits, one of which must be open.

Let $\mathbf{d}$ be prehomogeneous, and let $f_1, \ldots, f_s \in \mathbb{K}[\mathrm{rep}(Q, \mathbf{d})]$ be the irreducible monic polynomials whose zeros $Z(f_1), \ldots, Z(f_s)$ are the irreducible components of codimension 1 of $\mathrm{rep}(Q, \mathbf{d}) \setminus \mathrm{Gl}(\mathbf{d}) \cdot T$, where $\mathrm{Gl}(\mathbf{d}) \cdot T$ is the open orbit. It is easy to see that

$$g \cdot f_i = \chi_i(g) f_i$$

for $g \in \mathrm{Gl}(\mathbf{d})$, where $\chi_i \colon \mathrm{Gl}(\mathbf{d}) \to \mathbb{K}^*$ is a character. A regular function with this property is called a semi-invariant. By [9], any semi-invariant is a scalar multiple of a monomial in $f_1, \ldots, f_s$, and the $f_1, \ldots, f_s$ are algebraically independent. We denote by

$$\mathcal{Z}_{Q,\mathbf{d}} = \{X \in \mathrm{rep}(Q, \mathbf{d});\ f_i(X) = 0,\ i = 1, \ldots, s\}$$

the closed subvariety of $\operatorname{rep}(Q, \mathbf{d})$ of the common zeros of all non-constant semi-invariants. Obviously we have $\operatorname{codim} \mathcal{Z}_{Q,\mathbf{d}} \leq s$, and equality means that $\mathcal{Z}_{Q,\mathbf{d}}$ is a set theoretic complete intersection (simply called a complete intersection in the sequel). Note that $\mathcal{Z}_{Q,\mathbf{d}}$ might be a complete intersection even if $\operatorname{codim} \mathcal{Z}_{Q,\mathbf{d}} < s$. Indeed, it is not clear whether $f_1, \dots, f_s$ always form a minimal set of generators for the ideal $(f_1, \dots, f_s)$ of $\mathbb{K}[\operatorname{rep}(Q, \mathbf{d})]$. This problem cannot be solved by the fact that non-equidimensional varieties never are complete intersections: Example 1.1 encompasses a case where $\mathcal{Z}_{Q,\mathbf{d}}$ is irreducible and $\operatorname{codim} \mathcal{Z}_{Q,\mathbf{d}} < s$.

Now suppose Q is a connected quiver of type $\mathbb{D}_n$ and let $T_1, \dots, T_r$ be pairwise non-isomorphic indecomposable representations of Q such that $\operatorname{Ext}^1_Q(T_i, T_j) = 0$, for $i, j = 1, \dots, r$. In [7], Ch. Riedtmann and G. Zwara proved that $\mathcal{Z}_{Q,\mathbf{d}}$ is a complete intersection, for any dimension vector $\mathbf{d} = \sum_{i=1}^r \lambda_i \operatorname{\mathbf{dim}} T_i$, provided that all $\lambda_i \geq 2$. They also showed that $\mathcal{Z}_{Q,\mathbf{d}}$ is irreducible if all $\lambda_i \geq 3$. For the refinements we want to give, we need some additional terminology: In §2.1 we introduce a coordinate system for the vertices of the Auslander-Reiten quiver Γ_Q of Q. Based on these coordinates, for vertices (i, j) and $(k, l) \in \Gamma_Q$, we call (i, j) high if $j \geq n-1$, and we call (i, j) higher than (k, l) if $j > l$. For an arbitrary indecomposable $U \in \operatorname{rep}(Q)$, we have $\dim U(z) \leq 2$, for all $z \in Q_0$, and we call U a 2-root if equality holds for some vertex z (see §2.2 for details). It will become clear later on that if there are any 2-roots among $T_1, \dots, T_r$ then they are totally ordered with respect to the "higher than"-relation (see lemma 3.4), and hence there is a unique highest 2-root.

Theorem. *Let Q be a connected quiver of type $\mathbb{D}_n$. Let $T_1, \dots, T_r$ be pairwise non-isomorphic indecomposable representations of Q such that $\operatorname{Ext}^1_Q(T_i, T_j) = 0$, for $i, j = 1, \dots, r$. Choose positive integers $\lambda_1, \dots, \lambda_r$ and set $\mathbf{d} = \sum_{i=1}^r \lambda_i \operatorname{\mathbf{dim}} T_i$.*

(*i*) *If there is a high indecomposable representation or if there are no 2-roots among $T_1, \dots, T_r$ then $\mathcal{Z}_{Q,\mathbf{d}}$ is a complete intersection. Moreover, $\mathcal{Z}_{Q,\mathbf{d}}$ is irreducible if all $\lambda_i \geq 2$.*

(*ii*) *If there is no high indecomposable representation and if T_l is the highest 2-root among $T_1, \dots, T_r$ and has multiplicity $\lambda_l \geq 2$ then $\mathcal{Z}_{Q,\mathbf{d}}$ is a complete intersection. Moreover, $\mathcal{Z}_{Q,\mathbf{d}}$ is irreducible if $\lambda_l \geq 3$ and all other $\lambda_i \geq 2$.*

Note that in [5], Ch. Riedtmann already showed that $\mathcal{Z}_{Q,\mathbf{d}}$ is a complete intersection if some of the T_i are high. In a forthcoming paper we will give an exact description of when $\mathcal{Z}_{Q,\mathbf{d}}$ is a complete intersection. However, the arguments used there will be much more technical than the proofs given here.

Also note that in case $\mathbb{K}$ is the field $\mathbb{C}$ of complex numbers, the fact that $\mathcal{Z}_{Q,\mathbf{d}}$ is a complete intersection implies that $\operatorname{rep}(Q,\mathbf{d})$ is cofree as a representation of the subgroup $\mathrm{Sl}(\mathbf{d})$ of $\mathrm{Gl}(\mathbf{d})$, i.e. the algebra $\mathbb{C}[\operatorname{rep}(Q,\mathbf{d})]$ is a free module over the ring $\mathbb{C}[\operatorname{rep}(Q,\mathbf{d})]^{\mathrm{Sl}(\mathbf{d})}$ of $\mathrm{Sl}(\mathbf{d})$-invariant polynomials (see [11, §17]).

Example 1.1. Consider the quiver and dimension vector

$$Q: \quad \begin{tikzcd}[row sep=small] & & 2 \arrow[dl, "\alpha_2"'] \\ 1 \arrow[r, "\alpha_1"] & 4 & \\ & & 3 \arrow[ul, "\alpha_3"] \end{tikzcd} \qquad\qquad \mathbf{e} = \quad \begin{tikzcd}[row sep=small] & & 1 \arrow[dl, dash] \\ 1 \arrow[r, dash] & 2 & \\ & & 1 \arrow[ul, dash] \end{tikzcd}$$

and set $\mathbf{d} = \lambda \cdot \mathbf{e}$, for $\lambda \in \mathbb{N}$. There is an indecomposable representation T_1 in $\operatorname{rep}(Q,\mathbf{e})$, and the complement of the open orbit of $T = T_1^\lambda$ in $\operatorname{rep}(Q,\mathbf{d})$ has three irreducible components of codimension 1, defined by

$$\det((X(\alpha_1), X(\alpha_2)) = \det((X(\alpha_1), X(\alpha_3)) = \det((X(\alpha_2), X(\alpha_3)) = 0.$$

Now X belongs to $\mathcal{Z}_{Q,\mathbf{d}}$ if and only if X either contains the simple projective P_4 or else all of the two-dimensional projective representations P_1, P_2 and P_3. It is easy to see that

$$\begin{array}{rl} \mathcal{Z}_{Q,\mathbf{e}} & \text{is irreducible and of codimension 2,} \\ \mathcal{Z}_{Q,2\cdot\mathbf{e}} & \text{has two irreducible components of codimension 3 each,} \\ \mathcal{Z}_{Q,\lambda\cdot\mathbf{e}} & \text{is irreducible and of codimension 3, for } \lambda \geq 3. \end{array}$$

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2 Preliminaries and Notations

2.1. We will assume throughout that the quiver Q is connected and of type $\mathbb{D}_n$, i.e. the underlying graph $|Q|$ is a Dynkin diagram $\mathbb{D}_n$. Following [5], we recall some notations used to describe the Auslander-Reiten quiver Γ_Q of Q. We label the vertices of $|Q|$ as follows:

$$\begin{tikzcd}[row sep=small] & & & & & n-1 \arrow[dl, dash] \\ 1 \arrow[r, dash] & 2 \arrow[r, dash] & 3 \arrow[r, dash] & \cdots \arrow[r, dash] & n-2 & \\ & & & & & n \arrow[ul, dash] \end{tikzcd}$$

By $\overrightarrow{Q}$ we denote the quiver with $\left|\overrightarrow{Q}\right| = |Q|$, for which all arrows "point to the right", i.e. if there is an edge $i - j$ in $|Q|$ and if $i < j$, then there is an arrow $\alpha\colon i \to j$ in $\overrightarrow{Q}$. The translation quiver $\mathbb{Z}\mathbb{D}_n$ is defined as follows (see [4] or [3]): Start from $\mathbb{Z} \times \overrightarrow{Q}$ and add an arrow $(i,j) \to (i+1, j-1)$ for $i \in \mathbb{Z}$ and $2 \leq j \leq n-1$, and an arrow $(i,n) \to (i+1, n-2)$ for $i \in \mathbb{Z}$. The translation is given by $\tau(i,j) = (i-1,j)$.

Note that the vertices of $\mathbb{Z}\mathbb{D}_n$ are partially ordered by defining $X \leq Y$, for $X, Y \in \mathbb{Z}\mathbb{D}_n$, if and only if there is a path from X to Y in $\mathbb{Z}\mathbb{D}_n$. For any subset $\mathcal{U}$ and any vertex A of $\mathbb{Z}\mathbb{D}_n$ we say that A lies to the left (to the right) of $\mathcal{U}$ if $A \leq X$ ($X \leq A$) for some vertex $X \in \mathcal{U}$.

We call a vertex $x \in Q_0$ low if $x \leq n-2$ and high otherwise. Similarly, for vertices of $\mathbb{Z}\mathbb{D}_n$ we call (i,j) low if $j \leq n-2$ and high otherwise. Two high vertices (i,j) and (k,l) are said to be congruent if $i + j \equiv k + l \bmod 2$. The high vertices $(i, n-1)$ and (i,n) will be called adjacent.

We will also use the following (non-reflexive) partial order relation on the set of vertices of of $\mathbb{Z}\mathbb{D}_n$: Given arbitrary vertices (i,j) and (k,l), we call (i,j) higher than (k,l) if and only if $j > l$.

The Auslander-Reiten quiver Γ_Q of Q can be viewed as a subquiver of $\mathbb{Z}\mathbb{D}_n$ in the following manner: Embed the opposite quiver Q^{op} in $\mathbb{Z}\mathbb{D}_n$ as a section, i.e. in such a way that each τ-orbit of vertices of $\mathbb{Z}\mathbb{D}_n$ is met exactly once. Define the Nakayama translate $\nu(i,j)$ of a vertex to be $(i+n-2, j)$ if (i,j) is low, and to be the high vertex with first coordinate $i+n-2$ which is congruent to (i,j) if (i,j) is high. Then the Auslander-Reiten quiver Γ_Q of Q can be identified with the full subquiver of $\mathbb{Z}\mathbb{D}_n$ whose vertices lie between Q^{op} and $\nu(Q^{op})$ (see [3]).

2.2. Recall from [3] the dimensions of the spaces of morphisms in the mesh category $\mathbb{K}(\mathbb{Z}\mathbb{D}_n)$, or equivalently in $\operatorname{rep}(Q)$ if the vertices (i,j) and (k,l) belong to Γ_Q:

Proposition 2.1.

(i) $\dim \operatorname{Hom}((i,j),(k,l)) \leq 2$.

(ii) $\dim \operatorname{Hom}((i,j),(k,l)) = 2$ *if and only if* $j, l \leq n-2$ *and* $i+1 \leq k \leq i+j-1$ *and* $i+n-1 \leq k+l \leq i+j+n-3$.

(iii) $\dim \operatorname{Hom}((i,j),(k,l)) \geq 1$ *if and only if one of the following conditions is satisfied:*

(a) $j \leq n-2$, $i \leq k \leq i+j-1$ *and* $i+j \leq k+l$,

(b) $j \leq n-2$, $l \leq n-2$, $i+n-1 \leq k+l \leq i+j+n-2$, *and* $k \leq i+n-2$,

(c) $j \in \{n-1, n\}$, $l \leq n-2$, $i+n-1 \leq k+l$ *and* $k \leq i+n-2$,

(d) $j, l \in \{n-1, n\}$, $k \leq i+n-2$ *and* (k,l) *congruent to* (i,j).

With P_x and I_x we always denote the projective and injective indecomposable representations associated with the vertex $x \in Q_0$, respectively. The coordinates of P_x in Γ_Q are those of the vertex x of Q^{op} embedded in $\mathbb{Z}\mathbb{D}_n$ when constructing Γ_Q (compare §2.1). So $P_x = (i, x)$, for some $i \in \mathbb{Z}$.

We call a vertex $x \in Q_0$ a sink if it is the head of some arrows but the tail of none. Similarly we define sources. Using the same labelling for the vertices of $|Q|$ as in §2.1, we state:

Lemma 2.2.

(i) *If* U *is an indecomposable representation of* Q *then either* $\dim U(x) \leq 1$ *for all* x *or*

$$\mathbf{dim}\, U = \begin{array}{l} \ 1 \\ 0\cdots0\ \ 1\cdots1\ \ 2\cdots2 \\ \ 1 \end{array}$$

and $\mathbf{dim}\, U$ *contains at least one* 2 *and at least three* 1.

(ii) (a) *In case* $\{n-1, n\}$ *consists of a sink and a source, an indecomposable representation* U *of* Q *is high in* Γ_Q *if and only if either* U *is the one dimensional representation supported at* $n-1$ *or* n *or else*

$$\mathbf{dim}\, U = \begin{array}{l} \ 1 \\ 0\cdots0\ \ 1\cdots1 \\ \ 1 \end{array} \qquad \textit{or} \qquad \mathbf{dim}\, U = \begin{array}{l} \ 0 \\ 0\cdots0\ \ 1\cdots1 \\ \ 0 \end{array}$$

(b) *In case* $\{n-1, n\}$ *consists of either two sinks or two sources, an indecomposable representation* U *of* Q *is high in* Γ_Q *if and only if*

$$\mathbf{dim}\, U = \begin{array}{l} \ 1 \\ 0\cdots0\ \ 1\cdots1 \\ \ 0 \end{array} \qquad \textit{or} \qquad \mathbf{dim}\, U = \begin{array}{l} \ 0 \\ 0\cdots0\ \ 1\cdots1 \\ \ 1 \end{array}$$

(c) *The pairs of dimension vectors exhibited in* (a) *and* (b) *correspond to pairs of adjacent high vertices.*

Proof. From the Yoneda lemma, we get $[P_x, V] = \dim V(x)$, for arbitrary $V \in \mathrm{rep}(Q)$ and $x \in Q_0$. Now the lemma follows from proposition 2.1, combined with the description of the coordinates of P_x in Γ_Q. $\square$

Based on the above, we call an indecomposable representation U a 2-root if there exists a vertex $x \in Q_0$ with $\dim U(x) = 2$, and we denote by $\mathcal{T}2$ the set of all 2-roots in Γ_Q. Moreover, we call U a 2_x-root if $\dim U(x) = 2$ for a vertex $x \in Q_0$ and denote by $\mathcal{T}2_x$ the set of all 2_x-roots in Γ_Q.

2.3. All varieties considered in this paper are locally closed subvarieties of some vector space, usually some $\mathrm{rep}(Q, \mathbf{d})$, with respect to the Zariski topology. Which space is always clear from the context. The term "codimension" is with reference to this ambient space.

We will assume that $T_1, \ldots, T_r$ are pairwise non-isomorphic indecomposable representations of Q with $\mathrm{Ext}^1_Q(T_i, T_j) = 0$, for $i, j = 1, \ldots, r$, and that the representation

$$T = \bigoplus_{i=1}^{r} T_i^{\lambda_i} \text{ with } \lambda_i \geq 1$$

is sincere, i.e. $T(k) \neq 0$ for all $k \in Q_0$. Note that the orbit of T is open in $\mathrm{rep}(Q, \mathbf{d})$, where $\mathbf{d} = \mathbf{dim}\, T$. The sincerity of T is no restriction as the full subquiver which supports T is a disjoint union of connected quivers $K_1, \ldots, K_m$ of types $\mathbb{A}$ and $\mathbb{D}$, implying that

$$\mathcal{Z}_{Q,\mathbf{d}} = \prod_{j=1}^{m} \mathcal{Z}_{K_j, \mathbf{d}|K_j}.$$

For quivers of type $\mathbb{A}$ there are no 2-roots and no high indecomposable representations. So only the first part of our theorem is applicable for such quivers. But the corresponding results were already shown in [7]:

Proposition 2.3. *Let K be a connected quiver of type $\mathbb{A}$. Let $D_1, \ldots, D_r$ be pairwise non-isomorphic indecomposables in* $\mathrm{rep}(K)$ *such that* $\mathrm{Ext}^1_K(D_i, D_j) = 0$*, for $i, j = 1, \ldots, r$. Choose positive integers $\mu_1, \ldots, \mu_r$ and set $\mathbf{e} = \sum_{i=1}^{r} \mu_i\, \mathbf{dim}\, D_i$. Then $\mathcal{Z}_{K,\mathbf{e}}$ is a complete intersection, independently of the multiplicities μ_i, and is irreducible if all $\mu_i \geq 2$.*

2.4. The material presented below can be found in [10]. Also compare [6]. For a representation $X \in \mathrm{rep}(Q)$, the right perpendicular category $X^{\perp}$ is the full subcategory of $\mathrm{rep}(Q)$ whose objects are

$$\left\{ A \in \mathrm{rep}(Q);\ [X, A] = {}^1[X, A] = 0 \right\},$$

where

$$[X, A] = \dim_{\mathbb{K}} \operatorname{Hom}_Q(X, A) \quad \text{and} \quad {}^1[X, A] = \dim_{\mathbb{K}} \operatorname{Ext}^1_Q(X, A).$$

Similarly, the left perpendicular category ${}^{\perp}X$ has as objects

$$\left\{A \in \operatorname{rep}(Q);\ [A, X] = {}^1[A, X] = 0\right\}.$$

Note that $X^{\perp} = {}^{\perp}(\tau X)$, where τ is the Auslander-Reiten translation for all non-projective indecomposable direct summands of X and $\tau(P_x) = I_x$ for all $x \in Q_0$. Using the same symbol for the Auslander-Reiten translation and for the translation of vertices of $\mathbb{Z}\mathbb{D}_n$ will cause no confusion. Which one is meant will always be clear from the context.

If X is sincere and ${}^1[X, X] = 0$ then the category $X^{\perp}$ is equivalent to the category of representations of a quiver with $n - \sigma(X)$ vertices. Thus $T^{\perp}$ contains $n - r$ simple objects for our representation T. If S is one of them, the set

$$\{A \in \operatorname{rep}(Q, \mathbf{d});\ [A, S] \neq 0\}$$

is an irreducible component of codimension 1 of the complement

$$\operatorname{rep}(Q, \mathbf{d}) \setminus \operatorname{Gl}(\mathbf{d}) \cdot T.$$

Non-isomorphic simple objects of $T^{\perp}$ lead to distinct irreducible components, and all irreducible components of codimension 1 are obtained in this way. Thus $\mathcal{Z}_{Q,\mathbf{d}}$ is the zero set of $n - r$ (algebraically independent) polynomials. From now on, we will denote the underlying reduced variety of $\mathcal{Z}_{Q,\mathbf{d}}$ by the same symbol. This will cause no confusion since we are only interested in the dimension and the number of irreducible components of $\mathcal{Z}_{Q,\mathbf{d}}$. We have the following descriptions:

$$\begin{aligned} \mathcal{Z}_{Q,\mathbf{d}} &= \left\{A \in \operatorname{rep}(Q, \mathbf{d});\ [A, S] \neq 0 \text{ for all simple objects } S \in T^{\perp}\right\} \\ &= \left\{A \in \operatorname{rep}(Q, \mathbf{d});\ [S', A] \neq 0 \text{ for all simple objects } S' \in {}^{\perp}T\right\}. \end{aligned}$$

2.5. We fix a sink $z \in Q_0$ which is the head of some arrows $\alpha_j \colon y_j \to z$, for $j = 1, \ldots, t$. Let E_z be the simple projective supported at z. By $\overline{Q}$ we denote the full subquiver of Q with $\overline{Q}_0 = Q_0 \setminus \{z\}$ and by $\overline{\mathbf{d}}$ the restriction of $\mathbf{d}$ to $\overline{Q}_0$. Note that if $z \in \{n-1, n\}$ then $\overline{Q}$ is of type $\mathbb{A}_{n-1}$. Otherwise $z < n - 1$ and then $\overline{Q}$ is the disjoint union

$$\overline{Q} = L \,\dot{\cup}\, H,$$

where L and H are the full subquivers of Q with vertex sets

$$L_0 = \{1, \ldots, z-1\} \quad \text{and} \quad H_0 = \{z+1, \ldots, n\}.$$

Clearly, L is always of type $\mathbb{A}_{z-1}$. If $z < n-3$ then H is of type $\mathbb{D}_{n-z}$. Otherwise H is of type $\mathbb{A}_3$ or is a disjoint union of two copies of $\mathbb{A}_1$ if $z = n-3$ or if $z = n-2$, respectively. We will also use the fact that

$$\mathcal{Z}_{\overline{Q},\overline{\mathbf{d}}} = \mathcal{Z}_{H,\mathbf{d}|H} \times \mathcal{Z}_{L,\mathbf{d}|L}.$$

By definition of E_z, we have

$$E_z^{\perp} = \{A \in \operatorname{rep}(Q);\ A(z) = 0\},$$

which we identify with $\operatorname{rep}(\overline{Q})$. Note that the orbit of the restriction

$$\overline{T} = \bigoplus_{i=1}^{r} \overline{T}_i^{\lambda_i}$$

to $\overline{Q}$ is open in $\operatorname{rep}(\overline{Q},\overline{\mathbf{d}})$. Indeed, we get ${}^1[\overline{T},\overline{T}] = 0$ by computing the Hom-Ext-sequences (T,Σ) and $(\Sigma,\overline{T})$ of the short exact sequence

$$\Sigma\colon \quad 0 \longrightarrow E_z^{d_z} \longrightarrow T \longrightarrow \overline{T} \longrightarrow 0.$$

We decompose $\overline{T}$ into indecomposable direct summands

$$\overline{T} = \bigoplus_{i=1}^{\rho} U_i^{\mu_i},$$

with pairwise non-isomorphic U_i. Then $\rho = \sigma(\overline{T})$, and it is easy to see that $\min\{\mu_i;\ i = 1,\ldots,\rho\} \geq \min\{\lambda_i;\ i = 1,\ldots,r\}$.

In order to have a unified terminology for $\mathcal{Z}_{Q,\mathbf{d}}$ as well as for $\mathcal{Z}_{\overline{Q},\overline{\mathbf{d}}}$, we fix the following: For a quiver K with some connected components of type $\mathbb{A}$ and at most one of type $\mathbb{D}$ and for a sincere dimension vector $\mathbf{e}$, suppose that

$$D = \bigoplus_{i=1}^{k} D_i^{\nu_i}.$$

is the decomposition of a representative D of the open orbit of $\operatorname{rep}(K,\mathbf{e})$ into pairwise non-isomorphic indecomposables D_i with multiplicities ν_i. We will use the following abbreviations to denote the hypotheses of our theorem: We will say that D satisfies (M_1) or (M_2), respectively, if

(M_1): All $\nu_i \geq 1$. If K contains a connected component K_1 of type $\mathbb{D}$, if none of the D_i in Γ_{K_1} are high, if there is a 2-root in Γ_{K_1}, and if D_l is the highest 2-root in Γ_{K_1}, then $\nu_l \geq 2$.

(M_2): All $\nu_i \geq 2$. If K contains a connected component K_1 of type $\mathbb{D}$, if none of the D_i in Γ_{K_1} are high, if there is a 2-root in Γ_{K_1}, and if D_l is the highest 2-root in Γ_{K_1}, then $\nu_l \geq 3$.

In order to show that $\mathcal{Z}_{K,\mathbf{e}}$ is a complete intersection if D satisfies (M_1), we will prove that under the same condition

$$\operatorname{codim} \mathcal{Z}_{K,\mathbf{e}} = \#K_0 - \sigma(D). \tag{1}$$

If equation (1) holds we will say that $\mathcal{Z}_{K,\mathbf{e}}$ has proper codimension. Note that it might be a stronger statement to say that $\mathcal{Z}_{K,\mathbf{e}}$ has proper codimension than to say that $\mathcal{Z}_{K,\mathbf{e}}$ is a complete intersection. Despite of this, for the connected components of K of type $\mathbb{A}$, we are still in the position to rely on the results of proposition 2.3. Indeed, in the proof of this proposition in [7], the approach to verify complete intersections is the same as ours, namely to show that under the hypothesis of proposition 2.3 the corresponding variety has proper codimension.

We will also frequently state that "our theorem holds for $\mathcal{Z}_{K,\mathbf{e}}$". By this we mean that if D satisfies the hypothesis (M_1) or (M_2) then $\mathcal{Z}_{K,\mathbf{e}}$ has proper codimension or is irreducible, respectively.

2.6. With the sink z as in §2.5, we define a new quiver Q' by deleting z and $\alpha_1, \ldots, \alpha_t$ and by adding a new vertex z' and arrows $\beta_j \colon z' \to y_j$, for $j = 1, \ldots, t$. Note that the simple representation $E'_{z'}$ of Q' supported at z' is injective. Let

$$\mathcal{F}_z \colon \operatorname{rep}(Q) \longrightarrow \operatorname{rep}(Q')$$

be the reflection functor associated with z (see [2] and also [7]). If E_z is not a direct summand of T, we have $d_z \leq \sum_{j=1}^{t} d_{y_j}$, and the dimension vector $\mathbf{d}' = \mathbf{dim}\,\mathcal{F}_z T$ of $\mathcal{F}_z T$ is given by

$$d'_i = \begin{cases} d_i, & i \neq z' \\ \left(\sum_{j=1}^{t} d_{y_j}\right) - d_z \geq 0, & i = z'. \end{cases}$$

In order to compare $\mathcal{Z}_{Q,\mathbf{d}}$ with $\mathcal{Z}_{Q',\mathbf{d}'}$, we decompose:

$$\mathcal{Z}_{Q,\mathbf{d}} = \mathcal{Z}'_{Q,\mathbf{d}} \,\dot{\cup}\, \mathcal{Z}''_{Q,\mathbf{d}} \quad \text{and} \quad \mathcal{Z}_{Q',\mathbf{d}'} = \mathcal{W}'_{Q',\mathbf{d}'} \,\dot{\cup}\, \mathcal{W}''_{Q',\mathbf{d}'},$$

where

$$\mathcal{Z}'_{Q,\mathbf{d}} = \{A \in \mathcal{Z}_{Q,\mathbf{d}};\ [A, E_z] = 0\}, \qquad \mathcal{Z}''_{Q,\mathbf{d}} = \{A \in \mathcal{Z}_{Q,\mathbf{d}};\ [A, E_z] > 0\},$$
$$\mathcal{W}'_{Q',\mathbf{d}'} = \{A' \in \mathcal{Z}_{Q',\mathbf{d}'};\ [E'_{z'}, A'] = 0\}, \quad \mathcal{W}''_{Q',\mathbf{d}'} = \{A' \in \mathcal{Z}_{Q',\mathbf{d}'};\ [E'_{z'}, A'] > 0\}.$$

Using these notations, we recall the following results from [7]:

Summary 2.4.

(*i*) $\mathcal{Z}''_{Q,\mathbf{d}} = \mathcal{Z}_{\overline{Q},\overline{\mathbf{d}}} \times \mathcal{N}_{\mathbf{d}}$, where $\mathcal{N}_{\mathbf{d}} = \left\{ A \in \mathrm{Mat}(d_z \times \sum_{j=1}^t d_{y_j});\ \mathrm{rank}\, A < d_z \right\}$.

(*ii*) $\mathcal{Z}''_{Q,\mathbf{d}} = \mathcal{Z}_{Q,\mathbf{d}}$, $\sigma(\overline{T}) = \sigma(T) - 1$ and $\mathrm{codim}\, \mathcal{N}_{\mathbf{d}} = 0$ if $d_z > \sum_{j=1}^t d_{y_j}$ or equivalently if E is a direct summand of T.

(*iii*) $\mathcal{Z}''_{Q,\mathbf{d}} = \mathcal{Z}_{Q,\mathbf{d}}$, $\sigma(\overline{T}) = \sigma(T)$ and $\mathrm{codim}\, \mathcal{N}_{\mathbf{d}} = 1$ if $d_z = \sum_{j=1}^t d_{y_j}$ or equivalently if $E \in T^{\perp}$.

(*iv*) $\mathrm{codim}\, \mathcal{Z}'_{Q,\mathbf{d}} = \mathrm{codim}\, \mathcal{W}'_{Q',\mathbf{d}'}$ if $d_z < \sum_{j=1}^t d_{y_j}$.

(*v*) Irreducibility of $\mathcal{W}'_{Q',\mathbf{d}'}$ implies irreducibility of $\mathcal{Z}'_{Q,\mathbf{d}}$ if $d_z < \sum_{j=1}^t d_{y_j}$.

(*vi*) $\mathrm{codim}\, \mathcal{N}_{\mathbf{d}} = d'_{z'} + 1$ if $d_z < \sum_{j=1}^t d_{y_j}$.

Note that the determinantal variety $\mathcal{N}_{\mathbf{d}}$ above is always irreducible (see [1, §1]).

3 Properties of 2-roots

For this section we fix a sink $z \in Q_0$, once for all. First we gather results which are in association with the reflection functor $\mathcal{F}_z$. We use the notations of §2.1 and §2.6. For the construction of the Auslander-Reiten quiver $\Gamma_{Q'}$, we embed $(Q')^{op}$ in $\mathbb{Z}\mathbb{D}_n$ in such a way that all vertices except for z' coincide with the vertices of the embedding of Q^{op}. From this we immediately get the following two results:

Lemma 3.1. *Let $U = (i,j) \neq E_z$ be an indecomposable representation of Q. Denote by $U' = (i',j') = \mathcal{F}_z U$ the corresponding representation of Q'. Then as elements of $\mathbb{Z}\mathbb{D}_n$ we get $(i',j') = (i,j)$.*

Lemma 3.2. *If $z \neq n-2$ then the 2-roots of Γ_Q are mapped bijectively to the 2-roots of $\Gamma_{Q'}$ by $\mathcal{F}_z$.*

Next we give a description of the sets $\mathcal{T}2_x$ of 2_x-roots. We denote by $\alpha_x \in Q_1$ the unique arrow between the vertices x and $x+1$ in Q_0, for $x \in \{1, \dots, n-3\}$. With the same arguments as in the proof of lemma 2.2 we get:

Lemma 3.3. *Set $L_{n-2} = \tau^{-1} P_{n-2}$ and $R_{n-2} = \tau I_{n-2}$. Then we have:*

$$\mathcal{T}2 = \mathcal{T}2_{n-2} = \{U \in \Gamma_Q;\ U \text{ not high and } L_{n-2} \leq U \leq R_{n-2}\}.$$

Moreover, for $x \in \{n-3, \ldots, 2\}$, we get the recursive description

$$\mathcal{T}2_x = \{U \in \mathcal{T}2_{x+1};\ L_x \leq U \leq R_x\},$$

where

$$\begin{cases} L_x = \tau^{-1} L_{x+1} \text{ and } R_x = R_{x+1} & \text{if } t(\alpha_x) = x, \\ L_x = L_{x+1} \text{ and } R_x = \tau R_{x+1} & \text{if } h(\alpha_x) = x. \end{cases}$$

Recall the Auslander-Reiten formula ${}^1[U, ?] = [?, \tau U]$, for non-projective indecomposable representations U (see [3, §2]). This formula and the requirement ${}^1[T, T] = 0$ for the representation T imply that $[T_i, \tau T_j] = 0$, for $i, j = 1, \ldots, r$. Together with the description of $\mathcal{T}2$ in the lemma above, we get:

Lemma 3.4. *The 2-roots among $T_1, \ldots, T_r$ are totally ordered, with respect to the "higher than"-relation introduced in §2.1. Moreover, if $T_l = (p, q)$ is the highest 2-root then all the other 2-roots among $T_1, \ldots, T_r$ are contained in the set*

$$\mathcal{U} = \{U \in \mathcal{T}2;\ U \textit{ to the right of } \mathcal{C} \textit{ and to the left of } \mathcal{D}\},$$

where

$$\mathcal{C} = \{(p, i) \in \Gamma_Q;\ i = 1, \ldots, q\},$$
$$\mathcal{D} = \{(p + q - j, j) \in \Gamma_Q;\ j = 1, \ldots, q\}.$$

Proposition 3.5. *Suppose $z < n - 3$. Moreover, suppose T_l is the highest 2-root among $T_1, \ldots, T_r$.*

- *(i) If T_l is neither a 2_z-root nor a 2_{z+1}-root then the restriction $T_l|H$ is indecomposable and is the highest 2-root among the indecomposable direct summands of $T|H$.*
- *(ii) If T_l is a 2_z-root or a 2_{z+1}-root then the restriction $T_l|H$ contains high indecomposable direct summands.*

Proof. The second part is a direct consequence of lemma 2.2: The restriction to H of a 2_z-root or a 2_{z+1}-root always consists of the same pair of adjacent high indecomposables.

In order to prove the first part, for an indecomposable representation U we set

$$\mathrm{b}(U) = \text{ number of 2 in } \mathbf{dim}\, U.$$

From lemmas 3.3 and 3.4 we see that $\mathrm{b}(T_l) \geq \mathrm{b}(T_k)$, for all T_k. Since T_l is neither a 2_z-root nor a 2_{z+1}-root, with lemma 2.2 we conclude that no T_k

belongs to $\mathcal{T}2_z$ or $\mathcal{T}2_{z+1}$, that the restriction $T_k|H$ of any T_k is indecomposable, and that $\mathrm{b}(T_k) = \mathrm{b}(T_k|H)$, for all T_k. From lemmas 3.3 and 3.4 we also see that if $\mathrm{b}(T_j) > \mathrm{b}(T_k)$, for a fixed j and for all $k \neq j$, then T_j is the highest 2-root of T, i.e. $j = l$. So if $\mathrm{b}(T_l) > \mathrm{b}(T_k)$, for all $k \neq l$, then $\mathrm{b}(T_l|H) > \mathrm{b}(T_k|H)$ as well, for all $k \neq l$, and hence the proposition follows in this case.

So we are left with the situation that $\mathrm{b}(T_k) = \mathrm{b}(T_l) = b$ for some $k \neq l$. Note that in order to evaluate the relation "$T_l|H$ is higher than $T_k|H$", we may consider $\mathrm{rep}(H)$ as a full subcategory of $\mathrm{rep}(Q)$ and Γ_H as embedded in Γ_Q, since the "higher than"-relation depends only on the difference of the second coordinates of $T_k|H$ and $T_l|H$. As a direct consequence of proposition 2.1 and lemma 3.3, we know that T_k and T_l as well as their restrictions to H are all contained in the "line segment"

$$\mathcal{L}_x = \mathcal{T}2_{x+1} \setminus \mathcal{T}2_x = \{U \in \Gamma_Q;\ \mathrm{b}(U) = n - 2 - x\},$$

where $x = n - 2 - b$. For arbitrary U and V in $\mathcal{L}_x$ we have

$$\begin{cases} [U,V] \neq 0 \text{ if and only if } U \text{ is higher than } V & \text{if } t(\alpha_x) = x, \\ [V,U] \neq 0 \text{ if and only if } U \text{ is higher than } V & \text{if } h(\alpha_x) = x. \end{cases}$$

So either $[T_k, T_l] \neq 0$ or $[T_l, T_k] \neq 0$, depending on the orientation of Q. And clearly $[T_k, T_l] \neq 0$ implies $[T_k|H, T_l|H] \neq 0$, and $[T_l, T_k] \neq 0$ implies $[T_l|H, T_k|H] \neq 0$. Hence $T_k|H$ cannot be higher than $T_l|H$. $\square$

Finally we want to prepare an estimate for $\operatorname{codim} \mathcal{Z}''_{Q,\mathbf{d}}$. In addition to the notations of §2.6, we set

$$x = \dim X(z) \quad \text{and} \quad x' = \Big(\sum_{j=1}^{t} \dim X(y_j)\Big) - \dim X(z),$$

for any representation X of Q. We will also need the following auxiliary result:

Lemma 3.6. *Let U, V and $W_1, \ldots, W_k$, for $k \geq 2$, be pairwise non-isomorphic indecomposable representations. Assume that $U \notin \mathcal{T}2_z$, that V is high, and that all $W_i \in \mathcal{T}2_z$. Also suppose that ${}^1[V, W_1] = {}^1[W_1, V] = 0$ and set $X = \bigoplus_{i=1}^k W_i$ and $Y = V \oplus W_1$. Then we get:*

(i) $\sigma(\overline{U}) - \sigma(U) - u' \leq 0.$

(ii) $\sigma(\overline{W_i}) - \sigma(W_i) - w_i' = 1 \quad (i = 1, \ldots, k).$

(iii) $\sigma(\overline{X}) - \sigma(X) - x' \leq 0.$

(*iv*) $\sigma(\overline{Y}) - \sigma(Y) - y' \leq 0$.

Proof. Note that $\sigma(U) = \sigma(V) = \sigma(W_i) = 1$, for all i. The first two parts are a direct consequence of the fact that

$$\sigma(\overline{A}) \leq \sum_{j=1}^{t} \dim A(y_j) = a + a',$$

for any indecomposable representation A. From the description of 2-roots and high indecomposable representations given in lemma 2.2, it is easy to see that the restriction to H of any 2_z-root always consists of the same pair of adjacent high indecomposables. Hence we get

$$\sigma(\overline{X}) \leq 2 + \sum_{i=1}^{k} w_i' = 2 + x',$$

and this proves part (*iii*). For the last part, with the Auslander-Reiten formula we translate the requirement ${}^1[V, W_1] = {}^1[W_1, V] = 0$ to $[W_1, \tau V] = [V, \tau W_1] = 0$. From this we see that if $W_1 = (p, q)$ then V belongs to the set

$$\mathcal{U} = \{Z \in \Gamma_Q;\ Z \text{ to the left of } \mathcal{C} \text{ and to the right of } \mathcal{D}\},$$

where

$$\begin{aligned}
\mathcal{C} &= \{(p, i) \in \Gamma_Q;\ i = q, \ldots, n\}, \\
\mathcal{D} &= \{(p + q - j, j) \in \Gamma_Q;\ j = q, \ldots, n - 1\} \cup \{(p + q - n + 1, n)\}.
\end{aligned}$$

Hence with the arguments of the proof of lemma 2.2, we get $\dim V(z) = 1$. And from lemma 2.2, we conclude that $V|H$ is a direct summand of $W_1|H$. Thus

$$\begin{aligned}
\sigma(\overline{Y}) &\leq \sigma(\overline{V}) + \sigma(\overline{W}_1) - 1 \\
&\leq (1 + v') + (2 + w_1') - 1 \\
&= \sigma(Y) + y'.
\end{aligned}$$

This finishes the proof. □

Proposition 3.7. *Suppose E_z is neither a direct summand of T nor an object of $T^{\perp}$. Moreover, assume that* $\operatorname{codim} \mathcal{Z}_{\overline{Q},\overline{\mathbf{d}}} = n - 1 - \sigma(\overline{T})$. *Then we have:*

(*i*) $\operatorname{codim} \mathcal{Z}''_{Q,\mathbf{d}} \geq n - \sigma(T)$ *if T satisfies* (M_1),

(*ii*) $\operatorname{codim} \mathcal{Z}''_{Q,\mathbf{d}} > n - \sigma(T)$ *if T satisfies* (M_2).

Proof. Consider the following conditions:

(a) T contains a 2_z-root and no high indecomposable direct summand.

(b) T either contains no 2_z-root or else contains a high indecomposable direct summand as well.

By means of lemma 3.6, we get the inequality

$$\sigma(\overline{T}) - \sigma(T) - \textstyle\sum_{i=1}^{r} t'_i \leq \begin{cases} 1 & \text{under condition } (a), \\ 0 & \text{under condition } (b). \end{cases}$$

Since by assumption E_z is neither a direct summand of T nor an object of $T^{\perp}$, we know that

$$\operatorname{codim} \mathcal{Z}''_{Q,\mathbf{d}} = \operatorname{codim} \mathcal{Z}_{\overline{Q},\overline{\mathbf{d}}} + \operatorname{codim} \mathcal{N}_{\mathbf{d}} = \operatorname{codim} \mathcal{Z}_{\overline{Q},\overline{\mathbf{d}}} + d'_{z'} + 1,$$

from part (vi) of summary 2.4. Note that by definition, $d'_{z'} = t' = \sum_{i=1}^{r} \lambda_i t'_i$. Now by replacing the term $\operatorname{codim} \mathcal{Z}_{\overline{Q},\overline{\mathbf{d}}}$ with $n - 1 - \sigma(\overline{T})$ and applying the inequality above, we get

$$\operatorname{codim} \mathcal{Z}''_{Q,\mathbf{d}} \geq \begin{cases} n - \sigma(T) - 1 + \sum_{i=1}^{r} (\lambda_i - 1) t'_i & \text{under condition } (a), \\ n - \sigma(T) + \sum_{i=1}^{r} (\lambda_i - 1) t'_i & \text{under condition } (b). \end{cases}$$

By lemmas 3.3 and 3.4, if T contains a 2_z-root then the highest 2-root T_l of T must be a 2_z-root as well. Since $t'_i \geq 0$ for all i, and from the fact that

$$d'_{z'} = \sum_{i=1}^{r} \lambda_i t'_i > 0$$

(see [7, §4]), we conclude that $t'_i \geq 1$, for some i. And if T_l is a 2_z-root then $t'_l \geq 1$. Using this, from the estimate for $\operatorname{codim} \mathcal{Z}''_{Q,\mathbf{d}}$ above, we conclude that under either of conditions (a) or (b) we get $\operatorname{codim} \mathcal{Z}''_{Q,\mathbf{d}} \geq n - \sigma(T)$ if T satisfies the hypothesis (M_1). And by replacing (M_1) with (M_2) in the last argument, we get the strict inequality $\operatorname{codim} \mathcal{Z}''_{Q,\mathbf{d}} > n - \sigma(T)$. □

4 Proof of the theorem

We proceed by induction on the number n of vertices of Q. For $n \leq 3$ the quiver Q is of type $\mathbb{A}$, and hence our theorem holds, by proposition 2.3. Now for $n > 3$ we may assume that the theorem holds for $\mathcal{Z}_{\overline{Q},\overline{\mathbf{d}}}$, by the inductive hypothesis. We fix a sink $z \in Q_0$ and we first treat the following cases:

(a) E_z is a direct summand of T.

(b) E_z is an object of $T^{\perp}$.

In both cases E_z is a direct summand of any $X \in \mathcal{Z}_{Q,\mathbf{d}}$, thus implying that

$$\mathcal{Z}_{Q,\mathbf{d}} = \mathcal{Z}''_{Q,\mathbf{d}} = \mathcal{Z}_{\overline{Q},\overline{\mathbf{d}}} \times \mathcal{N}_{\mathbf{d}}.$$

Indeed, E_z is a direct summand of any $X \in \operatorname{rep}(Q, \mathbf{d})$ in case (a), and E_z is a simple object in $T^{\perp}$ in case (b).

For any connected component K of $\overline{Q}$ of type $\mathbb{A}$, clearly $T|K$ satisfies (M_1) or (M_2) if T does, respectively. So suppose H is of type $\mathbb{D}_{n-z}$. If T contains a high indecomposable direct summand or does not contain any 2-root then the same is true for $T|H$, by lemma 2.2. And if T_l is the highest 2-root of T then either $T_l|H$ is the highest 2-root of $T|H$ or else contains high indecomposable direct summands, by proposition 3.5. So we conclude that $T|H$ satisfies (M_1) or (M_2) if T does, respectively. Hence by the inductive hypothesis, we know that $\mathcal{Z}_{\overline{Q},\overline{\mathbf{d}}}$ has proper codimension or is irreducible if T satisfies (M_1) or (M_2), respectively.

Now with parts (i) and (ii) of summary 2.4 in case (a), and with parts (i) and (iii) of summary 2.4 in case (b), respectively, the theorem follows for $\mathcal{Z}_{Q,\mathbf{d}}$. So we are left with the case that

(c) E_z is neither a direct summand of T nor an object of $T^{\perp}$.

The main tool for proving case (c) are reflection functors. Using the notations of §2.6, we verify the following steps:

(c_1) If T satisfies (M_1) or (M_2) then

$$\begin{aligned} &\operatorname{codim} \mathcal{Z}''_{Q,\mathbf{d}} \geq n - \sigma(T) && \text{or} \\ &\operatorname{codim} \mathcal{Z}''_{Q,\mathbf{d}} \geq n - \sigma(T) + 1, && \text{respectively.} \end{aligned}$$

(c_2) If $z \neq n-2$ and if T satisfies (M_1) or (M_2) then $\mathcal{F}_z T$ does as well, respectively.

(c_3) If $\mathcal{W}'_{Q',\mathbf{d}'}$ has proper codimension or is irreducible then $\mathcal{Z}'_{Q,\mathbf{d}}$ has these properties as well, respectively.

In order to prove (c_1), note that if T satisfies (M_2) then it automatically satisfies (M_1) too. So suppose (M_1) holds for T. Then it holds for $\overline{T}$ as well, by the same arguments as in the proof of case (a) and (b). Hence by the inductive hypothesis, $\mathcal{Z}_{\overline{Q},\overline{\mathbf{d}}}$ has proper codimension, i.e.

$$\operatorname{codim} \mathcal{Z}_{\overline{Q},\overline{\mathbf{d}}} = n - 1 - \sigma(\overline{T}).$$

Now (c_1) follows from proposition 3.7. The claim (c_2) is a direct consequence of lemmas 3.1 and 3.2. And (c_3) follows immediately from parts (iv) and (v) of summary 2.4.

By using the above claims, we reduce case (c) to either (a) or (b): Since $\mathcal{Z}''_{Q,\mathbf{d}}$ is a closed subset of $\mathcal{Z}_{Q,\mathbf{d}}$, from (c_1) we conclude that the theorem holds for $\mathcal{Z}_{Q,\mathbf{d}}$ if and only if it holds for $\mathcal{Z}'_{Q,\mathbf{d}}$, i.e. if and only if T satisfying (M_1) or (M_2) implies that $\mathcal{Z}'_{Q,\mathbf{d}}$ has proper codimension or is irreducible, respectively. Dually the theorem holds for $\mathcal{Z}_{Q',\mathbf{d}'}$ if and only if it holds for $\mathcal{W}'_{Q',\mathbf{d}'}$. Now consider the set

$$\mathcal{A} = \{X \in \Gamma_Q;\ P_{n-2} \leq X \leq I_{n-2}\}\,.$$

If there is a T_k outside of $\mathcal{A}$ then by duality, we may assume that T_k is to the left of $\mathcal{A}$. So using a finite sequence of reflection functors, none of which are associated with the vertex $n-2$, we reach the situation of (a) or (b), for $\mathcal{Z}_{Q',\mathbf{d}'}$. But then we are done, by (c_2) and (c_3).

So assume that all T_k are contained in $\mathcal{A}$. From the description of $\mathcal{T}2$ in lemma 3.3 and from the requirement ${}^1[T,T]=0$, we conclude that in $\mathcal{A}\setminus\mathcal{T}2$ it is impossible to have T_i to the left and T_j to the right of $\mathcal{T}2$ simultaneously and such that T_i and T_j are not high. So by duality, we may assume that all T_k are inside or to the left of $\mathcal{T}2$. With these constraints for T, we get the following assertion:

(c'_2) If T satisfies (M_1) or (M_2) then $\mathcal{F}_z T$ does as well, respectively, for any sink $z \in Q_0$ and so particularly for $z = n-2$.

Because of (c_2), we only have to prove (c'_2) for $z = n-2$: If T contains a high indecomposable direct summand then so does $\mathcal{F}_z T$. And assuming that T_l is the highest 2-root of T, if $\mathcal{F}_z T_l$ is a 2-root then it is the highest 2-root of $\mathcal{F}_z T$, by lemma 3.1. On the other hand, if $\mathcal{F}_z T_l$ is not a 2-root then $\mathcal{F}_z T$ contains no 2-root, by lemmas 3.3 and 3.4.

Now after a finite sequence of reflection functors at successive but otherwise arbitrary sinks, again we reach the situation of (a) or (b), for $\mathcal{Z}_{Q',\mathbf{d}'}$, and so the theorem holds for $\mathcal{Z}_{Q,\mathbf{d}}$, by (c'_2), and (c_3).

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Part III
Quivers of type $\mathbb{D}_n$, semi-invariants and complete intersections

Abstract

Let Q be a quiver of type $\mathbb{D}_n$ and $\mathbf{d}$ a dimension vector for Q. We give a necessary and sufficient condition for the set of common zeros of all non-constant semi-invariants for $\mathbf{d}$-dimensional representations of Q, under the product of the general linear groups at all vertices of Q, to be a set theoretical complete intersection.

Samuel Beer

1 Introduction

Let $Q = (Q_0, Q_1, t, h)$ be a finite quiver, i.e. a finite set $Q_0 = \{1, \ldots, n\}$ of vertices and a finite set Q_1 of arrows $\alpha\colon t\alpha \to h\alpha$, where $t\alpha$ and $h\alpha$ denote the tail and the head of α, respectively. Let $\mathbb{K}$ be an algebraically closed field of characteristic zero.

A representation of Q over $\mathbb{K}$ is a collection

$$(X(i);\ i \in Q_0)$$

of finite dimensional $\mathbb{K}$-vector spaces together with a collection

$$(X(\alpha)\colon X(t\alpha) \to X(h\alpha);\ \alpha \in Q_1)$$

of $\mathbb{K}$-linear maps. A morphism $f\colon X \to Y$ between two representations is a collection $(f(i)\colon X(i) \to Y(i))$ of $\mathbb{K}$-linear maps such that

$$f(h\alpha) \circ X(\alpha) = Y(\alpha) \circ f(t\alpha) \quad \text{for all } \alpha \in Q_1.$$

By $\sigma(X)$ we denote the number of pairwise non-isomorphic indecomposable direct summands occurring in a decomposition of X into indecomposables. According to the theorem of Krull-Schmidt, $\sigma(X)$ is well-defined. The dimension vector of a representation X of Q is the vector

$$\mathbf{dim}\, X = (\dim X(1), \ldots, \dim X(n)) \in \mathbb{N}^{Q_0}.$$

We denote the category of representations of Q by $\operatorname{rep}(Q)$, and for any vector $\mathbf{d} = (d_1, \ldots, d_n) \in \mathbb{N}^{Q_0}$

$$\operatorname{rep}(Q, \mathbf{d}) = \prod_{\alpha \in Q_1} \operatorname{Mat}(d_{h\alpha} \times d_{t\alpha}, \mathbb{K})$$

is the vector space of representations X of Q with $X(i) = \mathbb{K}^{d_i}$, $i \in Q_0$. The group

$$\operatorname{Gl}(\mathbf{d}) = \prod_{i=1}^{n} \operatorname{Gl}(d_i, \mathbb{K})$$

acts on $\operatorname{rep}(Q, \mathbf{d})$ by

$$((g_1, \ldots, g_n) \cdot X)(\alpha) = g_{h\alpha} \circ X(\alpha) \circ g_{t\alpha}^{-1}.$$

Note that the $\operatorname{Gl}(\mathbf{d})$-orbit of X consists exactly of the representations Y in $\operatorname{rep}(Q, \mathbf{d})$ which are isomorphic to X.

We call $\mathbf{d}$ a prehomogeneous dimension vector if $\operatorname{rep}(Q, \mathbf{d})$ contains an open orbit $\operatorname{Gl}(\mathbf{d}) \cdot T$. Such a representation T is characterized by $\operatorname{Ext}^1_Q(T, T) = 0$ (see [8]). If Q admits only finitely many indecomposable representations, or equivalently if the underlying graph of Q is a disjoint union of Dynkin diagrams $\mathbb{A}$, $\mathbb{D}$ or $\mathbb{E}$ (see [1]), every vector $\mathbf{d}$ is prehomogeneous. Indeed, any representation is a direct sum of indecomposables and therefore $\operatorname{rep}(Q, \mathbf{d})$ contains finitely many orbits, one of which must be open.

Let $\mathbf{d}$ be prehomogeneous, and let $f_1, \ldots, f_s \in \mathbb{K}[\operatorname{rep}(Q, \mathbf{d})]$ be the irreducible monic polynomials whose zeros $Z(f_1), \ldots, Z(f_s)$ are the irreducible components of codimension 1 of $\operatorname{rep}(Q, \mathbf{d}) \setminus \operatorname{Gl}(\mathbf{d}) \cdot T$, where $\operatorname{Gl}(\mathbf{d}) \cdot T$ is the open orbit. It is easy to see that

$$g \cdot f_i = \chi_i(g) f_i$$

for $g \in \operatorname{Gl}(\mathbf{d})$, where $\chi_i \colon \operatorname{Gl}(\mathbf{d}) \to \mathbb{K}^*$ is a rational character. A regular function with this property is called a semi-invariant. By [9], any semi-invariant is a scalar multiple of a monomial in $f_1, \ldots, f_s$, and the $f_1, \ldots, f_s$ are algebraically independent. We denote by

$$\mathcal{Z}_{Q,\mathbf{d}} = \{X \in \operatorname{rep}(Q, \mathbf{d});\ f_i(X) = 0,\ i = 1, \ldots, s\}$$

the closed subvariety of $\operatorname{rep}(Q, \mathbf{d})$ of the common zeros of all non-constant semi-invariants. Obviously we have $\operatorname{codim} \mathcal{Z}_{Q,\mathbf{d}} \leq s$, and equality means that $\mathcal{Z}_{Q,\mathbf{d}}$ is a set theoretic complete intersection (simply called a complete intersection in the sequel). Note that in general, a variety $\mathcal{Z}$ defined as the intersection of s irreducible hypersurfaces might be a complete intersection, even though $\operatorname{codim} \mathcal{Z} < s$. However, in conjunction with semi-invariants for quivers, in §2 we show that $\mathcal{Z}_{Q,\mathbf{d}}$ is a complete intersection if and only if $\operatorname{codim} \mathcal{Z}_{Q,\mathbf{d}} = s$.

Now suppose Q is a connected quiver of type $\mathbb{D}_n$ and let $T_1, \ldots, T_r$ be pairwise non-isomorphic indecomposable representations of Q such that $\operatorname{Ext}^1_Q(T_i, T_j) = 0$, for $i, j = 1, \ldots, r$. Choose positive integers $\lambda_1, \ldots, \lambda_r$ and set

$$T = \bigoplus_{i=1}^{r} T_i^{\lambda_i}$$

and $\mathbf{d} = \mathbf{dim}\, T$. In §4 we introduce the notion of folded rectangles. These are certain subsets of vertices in the Auslander-Reiten quiver Γ_Q and include a special vertex, called the bent down corner. A folded rectangle is called suitable for T if, among other rules, its bent down corner is an indecomposable direct summand of T, say T_1, with multiplicity $\lambda_1 = 1$. This gives rise to the following classification:

Theorem. *The variety $\mathcal{Z}_{Q,\mathbf{d}}$ is a complete intersection if and only if there is no folded rectangle suitable for T.*

In [7] Ch. Riedtmann and G. Zwara proved that $\mathcal{Z}_{Q,\mathbf{d}}$ is a complete intersection, for any dimension vector $\mathbf{d} = \sum_{i=1}^{r} \lambda_i \,\mathbf{dim}\, T_i$, provided that all $\lambda_i \geq 2$. By means of folded rectangles, we get the following refinement:

Corollary. *Suppose $\mathcal{Z}_{Q,\mathbf{d}}$ is not a complete intersection. Then there exists a folded rectangle suitable for T with bent down corner T_1, where T_1 is an indecomposable direct summand of T with multiplicity $\lambda_1 = 1$.*

(i) Increasing λ_1 yields a dimension vector $\mathbf{d}'$ such that $\mathcal{Z}_{Q,\mathbf{d}'}$ is a complete intersection.

(ii) Increasing any other λ_i yields a dimension vector $\mathbf{d}''$ such that $\mathcal{Z}_{Q,\mathbf{d}''}$ is not a complete intersection.

Note that in case $\mathbb{K}$ is the field $\mathbb{C}$ of complex numbers, the fact that $\mathcal{Z}_{Q,\mathbf{d}}$ is a complete intersection implies that $\operatorname{rep}(Q, \mathbf{d})$ is cofree as a representation of the subgroup $\operatorname{Sl}(\mathbf{d})$ of $\operatorname{Gl}(\mathbf{d})$, i.e. the algebra $\mathbb{C}[\operatorname{rep}(Q, \mathbf{d})]$ is a free module over the ring $\mathbb{C}[\operatorname{rep}(Q, \mathbf{d})]^{\operatorname{Sl}(\mathbf{d})}$ of $\operatorname{Sl}(\mathbf{d})$-invariant polynomials (see [11, §17]).

The paper is organized as follows: In §2 we show that for an arbitrary quiver Q and a prehomogeneous dimension vector $\mathbf{d}$, the semi-invariants $f_1, \ldots, f_s$ always form a minimal set of defining polynomials for $\mathcal{Z}_{Q,\mathbf{d}}$. The idea of the proof for this is due to G. Zwara. In §3 we fix the notations for the remaining parts and recall the relevant facts and definitions which are used later on. In §4 we introduce the notion of folded rectangles and prove a series of results which are used in §5. This last section is devoted to the proof of the theorem stated above.

Many of the proofs presented here are in terms of coordinates of vertices in some Auslander-Reiten quiver. Unfortunately the explanatory diagrams of these quivers had to be omitted due to limited space. The reader is strongly urged to redraw the pictures in order to get an easier access to the arguments.

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2 On the number of equations defining $\mathcal{Z}_{Q,\mathbf{d}}$

In this section let Q be an arbitrary finite quiver and $\mathbf{d}$ a prehomogeneous dimension vector. We set

$$\mathrm{Sl}(\mathbf{d}) = \prod_{i\in Q_0} \mathrm{Sl}(d_i, \mathbb{K}).$$

The algebraically independent $\mathrm{Gl}(\mathbf{d})$-semi-invariants $f_1, \ldots, f_s$ generate the ring of $\mathrm{Sl}(\mathbf{d})$-invariants, i.e.

$$\mathbb{K}[\mathrm{rep}(Q, \mathbf{d})]^{\mathrm{Sl}(\mathbf{d})} = \mathbb{K}[f_1, \ldots, f_s].$$

Let $\mathcal{Z}_{Q,\mathbf{d}}$ be the closed subscheme of $\mathrm{rep}(Q, \mathbf{d})$ defined by the semi-invariants $f_1, \ldots, f_s$, i.e.

$$\mathcal{Z}_{Q,\mathbf{d}} = \mathrm{Spec}\left(\mathbb{K}[\mathrm{rep}(Q, \mathbf{d})]/(f_1, \ldots, f_s)\right).$$

We want to prove the following:

Theorem 2.1. *The minimal number of generators of the ideal*

$$(f_1, \ldots, f_s) \lhd \mathbb{K}[\mathrm{rep}(Q, \mathbf{d})]$$

is s. Consequently $\mathcal{Z}_{Q,\mathbf{d}}$ is a complete intersection if and only if $\mathrm{codim}\, \mathcal{Z}_{Q,\mathbf{d}} = s$.

The quotient

$$\operatorname{rep}(Q,\mathbf{d})/\operatorname{Sl}(\mathbf{d}) = \operatorname{Spec}(\mathbb{K}[\operatorname{rep}(Q,\mathbf{d})]^{\operatorname{Sl}(\mathbf{d})}) = \operatorname{Spec}(\mathbb{K}[f_1,\dots,f_s])$$

is an s-dimensional affine space, and we consider the quotient map

$$\pi\colon \operatorname{rep}(Q,\mathbf{d}) \to \operatorname{rep}(Q,\mathbf{d})/\operatorname{Sl}(\mathbf{d})$$

induced by the inclusion

$$\mathbb{K}[\operatorname{rep}(Q,\mathbf{d})]^{\operatorname{Sl}(\mathbf{d})} \subseteq \mathbb{K}[\operatorname{rep}(Q,\mathbf{d})].$$

Of course the regular morphism π is dominant, i.e.

$$\overline{\pi(\mathbb{K}[\operatorname{rep}(Q,\mathbf{d})])} = \mathbb{K}[\operatorname{rep}(Q,\mathbf{d})]/\operatorname{Sl}(\mathbf{d}),$$

but we need more:

Proposition 2.2. *The quotient map π is surjective.*

Proof. This fact is true for base fields of characteristic zero and for reductive groups acting regularly on affine varieties (see for example [3, Theorem 4.6]). □

Lemma 2.3. *The semi-invariants $f_1,\dots,f_s$ satisfy $f_1 \notin (f_2,\dots,f_s)$.*

Proof. Suppose that $f_1 \in (f_2,\dots,f_s)$. Then

$$f_1 = g_2\cdot f_2 + \cdots + g_s\cdot f_s,$$

for some polynomials $g_i \in \mathbb{K}[\operatorname{rep}(Q,\mathbf{d})]$. We identify $\mathbb{K}[\operatorname{rep}(Q,\mathbf{d})]^{\operatorname{Sl}(\mathbf{d})}$ with the affine space $\mathbb{A}^s$ such that the point $p = (1,0,\dots,0) \in \mathbb{A}^s$ corresponds with the maximal ideal

$$\mathfrak{m} = (f_1 - 1, f_2,\dots,f_s) \lhd \mathbb{K}[\operatorname{rep}(Q,\mathbf{d})]^{\operatorname{Sl}(\mathbf{d})}.$$

By proposition 2.2, the point p belongs to the image of π, which implies that

$$\mathfrak{m} = \mathfrak{n} \cap \mathbb{K}[\operatorname{rep}(Q,\mathbf{d})]^{\operatorname{Sl}(\mathbf{d})},$$

for some maximal ideal $\mathfrak{n} \lhd \mathbb{K}[\operatorname{rep}(Q,\mathbf{d})]$. This in turn is equivalent to the fact that the ideal

$$\mathfrak{m}\cdot\mathbb{K}[\operatorname{rep}(Q,\mathbf{d})] \trianglelefteq \mathbb{K}[\operatorname{rep}(Q,\mathbf{d})]$$

is proper. But on the other hand

$$1 = -(f_1 - 1) + g_2 \cdot f_2 + \cdots + g_s \cdot f_s \in \mathfrak{m} \cdot \mathbb{K}[\mathrm{rep}(Q, \mathbf{d})],$$

which gives a contradiction. □

For $\mathbb{K}[\mathrm{rep}(Q, \mathbf{d})]$ we have the following description:

$$\mathbb{K}[\mathrm{rep}(Q, \mathbf{d})] = \mathbb{K}[X_{\alpha,i,j}]_{\alpha \in Q_1,\, i \leq d_{h\alpha},\, j \leq d_{t\alpha}},$$

where $X_{\alpha,i,j}$ corresponds to the (i, j)-entry of the generic α-matrix in $\mathrm{rep}(Q, \mathbf{d})$.

Example 2.4. Let Q be the quiver

$$\begin{tikzcd} 1 \arrow[r, "\alpha"] & 2 \arrow[r, "\beta"] \arrow[d, "\gamma"] & 3 \\ & 4 & \end{tikzcd}$$

and $\mathbf{d} = (3, 2, 1, 1)$. Then we have the variables

$$\begin{tikzcd} \mathbb{K}^3 \arrow[r, "\begin{pmatrix} X_{\alpha,1,1} & X_{\alpha,1,2} & X_{\alpha,1,3} \\ X_{\alpha,2,1} & X_{\alpha,2,2} & X_{\alpha,2,3} \end{pmatrix}"] & \mathbb{K}^2 \arrow[r, "(X_{\beta,1,1}\ X_{\beta,1,2})"] \arrow[d, "(X_{\gamma,1,1}\ X_{\gamma,1,2})"] & \mathbb{K} \\ & \mathbb{K} & \end{tikzcd}$$

For the remaining part of this section we set

$$\begin{aligned} I &= (f_1, \ldots, f_s), \\ \mathfrak{m} &= (X_{\alpha,i,j};\ \alpha \in Q_1,\ i \leq d_{h\alpha},\ j \leq d_{t\alpha}) \lhd \mathbb{K}[\mathrm{rep}(Q, \mathbf{d})]. \end{aligned}$$

Obviously $\mathfrak{m}$ is a maximal ideal and $\mathbb{K}[\mathrm{rep}(Q, \mathbf{d})]/\mathfrak{m} = \mathbb{K}$.

Lemma 2.5. *Suppose that $I = (g_1, \ldots, g_k)$. Then the residue classes of $g_1, \ldots, g_k$ generate the $\mathbb{K}$-vector space $I/I \cdot \mathfrak{m}$.*

Proof. The claim follows from the facts that the residue classes of $g_1, \ldots, g_k$ generate the $\mathbb{K}[\mathrm{rep}(Q, \mathbf{d})]$-module $I/I \cdot \mathfrak{m}$, that this module is annihilated by $\mathfrak{m}$, and that $\mathbb{K}[\mathrm{rep}(Q, \mathbf{d})]/\mathfrak{m} = \mathbb{K}$. □

As a direct consequence of the above lemma we have:

Corollary 2.6. *Suppose that $I = (g_1, \ldots, g_k)$. Then $k \geq \dim_{\mathbb{K}}(I/I \cdot \mathfrak{m})$.*

Let $(\mathbf{e}_\alpha)_{\alpha\in Q_1}$ denote the standard basis of the free commutative group $\mathbb{Z}^{Q_1}$. We have a natural grading on the $\mathbb{K}$-algebra $\mathbb{K}[\mathrm{rep}(Q,\mathbf{d})]$, by the group $\mathbb{Z}^{Q_1}$: Set the degree of the variable $X_{\alpha,i,j}$ to be $\mathbf{e}_\alpha$. If Q and $\mathbf{d}$ are as in example 2.4 then

$$\begin{aligned}\deg(X_{\alpha,2,2}) &= (1,0,0),\\ \deg(X_{\beta,1,1}^5) &= (0,5,0),\\ \deg(X_{\alpha,1,1}^2\cdot X_{\alpha,2,3}\cdot X_{\gamma,1,2}) &= (3,0,1).\end{aligned}$$

Here, the first coordinate corresponds to α, the second to β and the third to γ.

Lemma 2.7. *With respect to the above grading, the semi-invariants $f_1,\dots,f_s$ are homogeneous polynomials in $\mathbb{K}[\mathrm{rep}(Q,\mathbf{d})]$. Moreover, their degrees are pairwise different.*

Proof. Let $(\mathbf{f}_i)_{i\in Q_0}$ denote the standard basis of the free commutative group $\mathbb{Z}^{Q_0}$. Moreover, let $(\mathbf{f}_i^*)_{i\in Q_0}$ denote the dual basis in $(\mathbb{Z}^{Q_0})^*$. We identify the elements of $(\mathbb{Z}^{Q_0})^*$ with the weights of the rational characters of $\mathrm{Gl}(\mathbf{d})$. We define a $\mathbb{Z}$-linear function

$$\Psi\colon (\mathbb{Z}^{Q_1})\to(\mathbb{Z}^{Q_0})^*,\quad \Psi(\mathbf{e}_\alpha) = -\mathbf{f}_{h\alpha}^*+\mathbf{f}_{t\alpha}^*.$$

Observe that if $f\in\mathbb{K}[\mathrm{rep}(Q,\mathbf{d})]$ is a homogeneous function and $g=(g_i)_{i\in Q_0}\in\mathrm{Gl}(\mathbf{d})$, where each g_i is of the form $c_i\cdot I_{d_i}$, with $c_i\in\mathbb{K}^*$, then

$$g*f=\Psi(\deg f)(g)\cdot f=(c_1^{d_1})^{y_1}\dots(c_n^{d_n})^{y_n}\cdot f\quad(y_i\in\mathbb{Z}).\tag{1}$$

Moreover, if $f\in\mathbb{K}[\mathrm{rep}(Q,\mathbf{d})]$ is a semi-invariant and g an arbitrary element of $\mathrm{Gl}(\mathbf{d})$ then

$$g*f=\chi(g)\cdot f=(\det g_1)^{z_1}\dots(\det g_n)^{z_n}\cdot f\quad(z_i\in\mathbb{Z}).\tag{2}$$

Now suppose the semi-invariant f_i is not homogeneous. Then we may write

$$f_i=\sum_{j=1}^u f_{ij}\quad(f_{ij}\text{ homogeneous}).$$

Combining this with equations (1) and (2), for any $g\in\mathrm{Gl}(\mathbf{d})$ of the type used in equation (1), we get

$$g*f_i=\sum_{j=1}^u\chi(g)\cdot f_{ij}=\sum_{j=1}^u\Psi(\deg f_{ij})(g)\cdot f_{ij}.$$

So we conclude that

$$\Psi(\deg f_{ij}) = \Psi(\deg f_{ik}) \quad (j, k = 1, \ldots, u),$$

and hence,

$$\frac{f_{ij}}{f_{ik}} \quad (j \neq k)$$

is a non-trivial rational invariant in $\mathbb{K}(\mathrm{rep}(Q, \mathbf{d}))$. But this is a contradiction to our assumption that $\mathbf{d}$ is prehomogeneous. So the semi-invariant f_i indeed is a homogeneous function and its weight is equal to $\Psi(\deg f_i)$. Moreover, since the weights of $f_1, \ldots, f_s$ are pairwise different, the degrees of $f_1, \ldots, f_s$ must be pairwise different as well. □

We conclude from the above lemma that the ideal I is homogeneous. Obviously the maximal ideal $\mathfrak{m}$ is also homogeneous. Indeed, $\mathfrak{m}$ is described by the following property: A nonzero homogeneous polynomial $h \in \mathbb{K}[\mathrm{rep}(Q, \mathbf{d})]$ belongs to $\mathfrak{m}$ if and only if $\deg(h) \neq (0, \ldots, 0)$.

Lemma 2.8. *The residue classes of $f_1, \ldots, f_s$ in $I/I \cdot \mathfrak{m}$ are linearly independent.*

Proof. Suppose this is not the case. Then

$$a_1 \cdot f_1 + a_2 \cdot f_2 + \cdots + a_s \cdot f_s \in I \cdot \mathfrak{m},$$

for some scalars $a_i \in \mathbb{K}$, such that not all of them are zero. We may assume that $a_1 \neq 0$. Since the ideal $I \cdot \mathfrak{m}$ is homogeneous and the polynomials f_i are homogeneous of pairwise different degrees, we conclude that f_1 belongs to $I \cdot \mathfrak{m}$. So

$$f_1 = f_1 \cdot g_1 + f_2 \cdot g_2 + \cdots + f_s \cdot g_s,$$

for some polynomials $g_i \in \mathfrak{m}$. Let h_i be the component of the polynomial g_i of degree $\deg(f_1) - \deg(f_i)$. Observe that

$$f_1 = f_1 \cdot h_1 + f_2 \cdot h_2 + \cdots + f_s \cdot h_s.$$

Since h_1 has degree $(0, \ldots, 0)$ and belongs to $\mathfrak{m}$, we know that $h_1 = 0$. But this implies that $f_1 \in (f_2, \ldots, f_r)$, a contradiction with lemma 2.3. □

Now corollary 2.6 and lemma 2.8 imply that the number of generators of I is at least s. And this proves theorem 2.1.

3 Preliminaries and Notations

3.1. From now on we will assume throughout that the quiver Q is connected and of type $\mathbb{D}_n$, i.e. the underlying graph $|Q|$ is a Dynkin diagram $\mathbb{D}_n$. Following [5], we recall some notations used to describe the Auslander-Reiten quiver Γ_Q of Q. We label the vertices of $|Q|$ as follows:

$$\begin{array}{l} \hphantom{1 — 2 — 3 — \cdots — n-2 \;} n-1 \\ 1 — 2 — 3 — \cdots — n-2 \\ \hphantom{1 — 2 — 3 — \cdots — n-2 \;} n \end{array}$$

By $\vec{Q}$ we denote the quiver for which all arrows "point to the right", i.e. if there is an edge $i - j$ in $|Q|$ and if $i < j$, then there is an arrow $\alpha\colon i \to j$ in $\vec{Q}$. The translation quiver $\mathbb{Z}\mathbb{D}_n$ is defined as follows (see [4] or [2]): Start from $\mathbb{Z} \times \vec{Q}$ and add an arrow $(i, j) \to (i + 1, j - 1)$ for $i \in \mathbb{Z}$ and $2 \leq j \leq n - 1$, and an arrow $(i, n) \to (i + 1, n - 2)$ for $i \in \mathbb{Z}$. The translation is given by $\tau(i, j) = (i - 1, j)$.

Note that the vertices of $\mathbb{Z}\mathbb{D}_n$ are partially ordered by defining $X \leq Y$, for $X, Y \in \mathbb{Z}\mathbb{D}_n$, if and only if there is a path from X to Y in $\mathbb{Z}\mathbb{D}_n$. For any subset $\mathcal{U}$ and any vertex A of $\mathbb{Z}\mathbb{D}_n$ we say that A lies to the left (to the right) of $\mathcal{U}$ if $A \leq X$ ($X \leq A$) for some vertex $X \in \mathcal{U}$.

We call a vertex $x \in Q_0$ low if $x \leq n - 2$ and high otherwise. Similarly, for vertices of $\mathbb{Z}\mathbb{D}_n$ we call (i, j) low if $j \leq n - 2$ and high otherwise. Two high vertices (i, j) and (k, l) are said to be congruent if $i + j \equiv k + l \bmod 2$. The high vertices $(i, n - 1)$ and (i, n) will be called adjacent.

We will also use the following (non-reflexive) partial order relation on the set of vertices of of $\mathbb{Z}\mathbb{D}_n$: Given arbitrary vertices (i, j) and (k, l), we call (i, j) higher than (k, l) if and only if $j > l$.

The Auslander-Reiten quiver Γ_Q of Q can be viewed as a subquiver of $\mathbb{Z}\mathbb{D}_n$ in the following manner: Embed the opposite quiver Q^{op} in $\mathbb{Z}\mathbb{D}_n$ as a section, i.e. in such a way that each τ-orbit of vertices of $\mathbb{Z}\mathbb{D}_n$ is met exactly once. Define the Nakayama translate $\nu(i, j)$ of a vertex to be $(i + n - 2, j)$ if (i, j) is low, and to be the high vertex with first coordinate $i + n - 2$ and which is congruent to (i, j) if (i, j) is high. Then the Auslander-Reiten quiver Γ_Q of Q can be identified with the full subquiver of $\mathbb{Z}\mathbb{D}_n$ whose vertices lie between Q^{op} and $\nu(Q^{op})$ (see [2]).

3.2. Recall from [2] the dimensions of the spaces of morphisms in the mesh category $\mathbb{K}(\mathbb{Z}\mathbb{D}_n)$, or equivalently in $\mathrm{rep}(Q)$ if the vertices (i, j) and (k, l) belong to Γ_Q:

Proposition 3.1.

(*i*) $\dim \mathrm{Hom}((i,j),(k,l)) \leq 2$.

(*ii*) $\dim \mathrm{Hom}((i,j),(k,l)) = 2$ *if and only if* $j, l \leq n-2$ *and* $i+1 \leq k \leq i+j-1$ *and* $i+n-1 \leq k+l \leq i+j+n-3$.

(*iii*) $\dim \mathrm{Hom}((i,j),(k,l)) \geq 1$ *if and only if one of the following conditions is satisfied:*

(a) $j \leq n-2$, $i \leq k \leq i+j-1$ *and* $i+j \leq k+l$,

(b) $j \leq n-2$, $l \leq n-2$, $i+n-1 \leq k+l \leq i+j+n-2$, *and* $k \leq i+n-2$,

(c) $j \in \{n-1, n\}$, $l \leq n-2$, $i+n-1 \leq k+l$ *and* $k \leq i+n-2$,

(d) $j, l \in \{n-1, n\}$, $k \leq i+n-2$ *and* (k,l) *congruent to* (i,j).

With P_x and I_x we always denote the projective and injective indecomposable representations associated with the vertex $x \in Q_0$, respectively. The coordinates of P_x in Γ_Q are those of the vertex x of Q^{op} embedded in $\mathbb{Z}\mathbb{D}_n$ when constructing Γ_Q (compare §3.1). So $P_x = (i,x)$, for some $i \in \mathbb{Z}$, and $I_x = \nu(i,x)$.

We call a vertex $x \in Q_0$ a sink if it is the head of some arrows but the tail of none. Similarly we define sources. Using the same labelling for the vertices of $|Q|$ as in §3.1, we state:

Lemma 3.2.

(*i*) *If* U *is an indecomposable representation of* Q *then either* $\dim U(x) \leq 1$ *for all* x *or*

$$\mathbf{dim}\, U = 0 \cdots 0 \;\; 1 \cdots 1 \;\; 2 \cdots 2 \begin{matrix} 1 \\ \\ 1 \end{matrix}$$

and $\mathbf{dim}\, U$ *contains at least one* 2 *and at least three* 1.

(*ii*) (*a*) *In case* $\{n-1, n\}$ *consists of a sink and a source, an indecomposable representation* U *of* Q *is high in* Γ_Q *if and only if either* U *is the one dimensional representation supported at* $n-1$ *or* n *or else*

$$\mathbf{dim}\, U = 0 \cdots 0 \;\; 1 \cdots 1 \begin{matrix} 1 \\ \\ 1 \end{matrix} \qquad \text{or} \qquad \mathbf{dim}\, U = 0 \cdots 0 \;\; 1 \cdots 1 \begin{matrix} 0 \\ \\ 0 \end{matrix}$$

(*b*) *In case* $\{n-1, n\}$ *consists of either two sinks or two sources, an indecomposable representation* U *of* Q *is high in* Γ_Q *if and only if*

$$\mathbf{dim}\, U = 0 \cdots 0 \;\; 1 \cdots 1 \begin{matrix} 1 \\ \\ 0 \end{matrix} \qquad \text{or} \qquad \mathbf{dim}\, U = 0 \cdots 0 \;\; 1 \cdots 1 \begin{matrix} 0 \\ \\ 1 \end{matrix}$$

(*c*) *The pairs of dimension vectors exhibited in* (*a*) *and* (*b*) *correspond to pairs of adjacent high vertices.*

Proof. From the Yoneda lemma, we get $[P_x, V] = \dim V(x)$, for arbitrary $V \in \operatorname{rep}(Q)$ and $x \in Q_0$. Now the lemma follows from proposition 3.1, combined with the description of the coordinates of P_x in Γ_Q. □

Based on the above, we call an indecomposable representation U a 2-root if there exists a vertex $x \in Q_0$ with $\dim U(x) = 2$, and we denote by $\mathcal{T}2$ the set of all 2-roots in Γ_Q. Moreover, we call U a 2_x-root if $\dim U(x) = 2$ for a vertex $x \in Q_0$ and denote by $\mathcal{T}2_x$ the set of all 2_x-roots in Γ_Q.

3.3. We recall the following material from [8] and from [10]. For a quiver Q, the Euler form is the $\mathbb{Z}$-bilinear form on $\mathbb{Z}^{Q_0}$ defined by

$$\langle \mathbf{d}, \mathbf{e} \rangle = \sum_{i \in Q_0} d_i e_i - \sum_{\alpha \in Q_1} d_{t\alpha} e_{h\alpha}.$$

For $X \in \operatorname{rep}(Q, \mathbf{d})$ and $Y \in \operatorname{rep}(Q, \mathbf{e})$ it can be computed as

$$\langle \mathbf{d}, \mathbf{e} \rangle = [X, Y] - {}^1[X, Y],$$

where

$$[X, Y] = \dim_{\mathbb{K}} \operatorname{Hom}_Q(X, Y) \quad \text{and} \quad {}^1[X, Y] = \dim_{\mathbb{K}} \operatorname{Ext}^1_Q(X, Y).$$

The quadratic form

$$q(\mathbf{d}) = \langle \mathbf{d}, \mathbf{d} \rangle$$

associated with the Euler form is the Tits form of Q. It is positive definite if the underlying graph $|Q|$ is a Dynkin diagram, and so particularly if Q is of type $\mathbb{D}_n$.

3.4. The following notations and results are gathered from [12]. Given a short exact sequence of representations in $\operatorname{rep}(Q)$

$$\Sigma\colon \quad 0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0,$$

we define the additive functions

$$\delta_\Sigma(A) = [X \oplus Z, A] - [Y, A],$$
$$\delta'_\Sigma(A) = [A, X \oplus Z] - [A, Y],$$

on representations $A \in \operatorname{rep}(Q)$. For representations $X, U \in \operatorname{rep}(Q)$, where U is indecomposable, we denote by $\mu(X, U)$ the multiplicity with which U occurs as a direct summand of X. For a non-injective indecomposable representation $U \in \operatorname{rep}(Q)$ we denote by $\Sigma(U)$ an Auslander-Reiten sequence

$$\Sigma(U)\colon \quad 0 \longrightarrow U \longrightarrow E(U) \longrightarrow \tau^{-1}U \longrightarrow 0.$$

With these notations we get:

Lemma 3.3. *For $X, U \in \operatorname{rep}(Q)$, where U is non-injective and indecomposable, we have the following formulæ:*

$$\delta_{\Sigma(U)}(X) = \mu(X, U) \quad \textit{and} \quad \delta'_{\Sigma(U)}(X) = \mu(X, \tau^{-1}U).$$

3.5. All varieties considered in this paper are locally closed subvarieties of some vector space, usually some $\operatorname{rep}(Q, \mathbf{d})$, with respect to the Zariski topology. Which space is always clear from the context. The term "codimension" is with reference to this ambient space. When referring to the codimension of the Zariski closure of some orbit $\operatorname{Gl}(\mathbf{d}) \cdot X$, we usually omit the closure bar and only write $\operatorname{codim} \operatorname{Gl}(\mathbf{d}) \cdot X$. Given representations $X, Y \in \operatorname{rep}(Q, \mathbf{d})$, we call Y a degeneration of X if Y belongs to the closure of the orbit of X and denote this by $X \leq_{\deg} Y$.

We will assume that $T_1, \dots, T_r$ are pairwise non-isomorphic indecomposable representations of Q with $\operatorname{Ext}^1(T_i, T_j) = 0$, for $i, j = 1, \dots, r$, and that the representation

$$T = \bigoplus_{i=1}^{r} T_i^{\lambda_i} \text{ with } \lambda_i \geq 1$$

is sincere, i.e. $T(k) \neq 0$ for all $k \in Q_0$. Note that the orbit of T is open in $\operatorname{rep}(Q, \mathbf{d})$, where $\mathbf{d} = \mathbf{dim}\, T$. The sincerity of T is no restriction as the full subquiver which supports T is a disjoint union of connected quivers $K_1, \dots, K_m$ of types $\mathbb{A}$ and $\mathbb{D}$, implying that

$$\mathcal{Z}_{Q,\mathbf{d}} = \prod_{j=1}^{m} \mathcal{Z}_{K_j, \mathbf{d}|K_j}.$$

Note that for a quiver K of type $\mathbb{A}$ and for an arbitrary dimension vector $\mathbf{e}$ the variety $\mathcal{Z}_{K,\mathbf{e}}$ is always a complete intersection, by the results of [7].

Also recall the Auslander-Reiten formula

$$^{1}[U, ?] = [?, \tau U],$$

for non-projective indecomposable representations U (see [2, §2]). Here τ denotes the Auslander-Reiten translation. Using the same symbol as for the translation of vertices of $\mathbb{ZD}_n$ will cause no confusion. The applying translation will always be clear from the context. This formula and the requirement ${}^{1}[T, T] = 0$ for the representation T imply that $[T_i, \tau T_j] = 0$, for $i, j = 1, \ldots, r$.

3.6. The material presented below can be found in [10]. Also compare [6]. For a representation $X \in \operatorname{rep}(Q)$, the right perpendicular category $X^{\perp}$ is the full subcategory of $\operatorname{rep}(Q)$ whose objects are

$$\left\{A \in \operatorname{rep}(Q);\ [X, A] = {}^{1}[X, A] = 0\right\}.$$

Similarly, the left perpendicular category ${}^{\perp}X$ has as objects

$$\left\{A \in \operatorname{rep}(Q);\ [A, X] = {}^{1}[A, X] = 0\right\}.$$

Note that $X^{\perp} = {}^{\perp}(\tau X)$, where τ is the Auslander-Reiten translation for all non-projective indecomposable direct summands of X and $\tau(P_x) = I_x$, for all $x \in Q_0$.

If X is sincere and ${}^{1}[X, X] = 0$ then the category $X^{\perp}$ is equivalent to the category of representations of a quiver with $n - \sigma(X)$ vertices. Thus $T^{\perp}$ contains $n - r$ simple objects for our representation T. If S is one of them, the set

$$\{A \in \operatorname{rep}(Q, \mathbf{d});\ [A, S] \neq 0\}$$

is an irreducible component of codimension 1 of the complement

$$\operatorname{rep}(Q, \mathbf{d}) \setminus \operatorname{Gl}(\mathbf{d}) \cdot T.$$

Non-isomorphic simple objects of $T^{\perp}$ lead to distinct irreducible components, and all irreducible components of codimension 1 are obtained in this way. Thus $\mathcal{Z}_{Q,\mathbf{d}}$ is the zero set of $n - r$ (algebraically independent) polynomials. From now on, we will denote the underlying reduced variety of $\mathcal{Z}_{Q,\mathbf{d}}$ by the same symbol. This will cause no confusion since we are only interested in the dimension of $\mathcal{Z}_{Q,\mathbf{d}}$. We have the following descriptions:

$$\begin{aligned} \mathcal{Z}_{Q,\mathbf{d}} &= \left\{A \in \operatorname{rep}(Q, \mathbf{d});\ [A, S] \neq 0 \text{ for all simple objects } S \in T^{\perp}\right\} \\ &= \left\{A \in \operatorname{rep}(Q, \mathbf{d});\ [S', A] \neq 0 \text{ for all simple objects } S' \in {}^{\perp}T\right\}. \end{aligned}$$

3.7. We fix a sink $z \in Q_0$, i.e. a vertex which is the head of some arrows $\alpha_j \colon y_j \to z$, for $j = 1, \ldots, t$. Let E_z be the simple projective supported at z. By $\overline{Q}$ we denote the full subquiver of Q with $\overline{Q}_0 = Q_0 \setminus \{z\}$ and by $\overline{\mathbf{d}}$ the restriction of $\mathbf{d}$ to $\overline{Q}_0$. Note that if $z \in \{n-1, n\}$ then $\overline{Q}$ is of type $\mathbb{A}_{n-1}$. Otherwise $z < n-1$ and then $\overline{Q}$ is the disjoint union

$$\overline{Q} = L \,\dot{\cup}\, H,$$

where L and H are the full subquivers of Q with vertex sets

$$L_0 = \{1, \ldots, z-1\} \quad \text{and} \quad H_0 = \{z+1, \ldots, n\}\,.$$

Clearly, L is always of type $\mathbb{A}_{z-1}$. If $z < n-3$ then H is of type $\mathbb{D}_{n-z}$. Otherwise H is of type $\mathbb{A}_3$ or is a disjoint union of two copies of $\mathbb{A}_1$, respectively, if $z = n-3$ or if $z = n-2$. We will also use the fact that

$$\mathcal{Z}_{\overline{Q},\overline{\mathbf{d}}} = \mathcal{Z}_{H,\mathbf{d}|H} \times \mathcal{Z}_{L,\mathbf{d}|L}.$$

By definition of E_z, we have

$$E_z^{\perp} = \{A \in \operatorname{rep}(Q);\; A(z) = 0\}\,,$$

which we identify with $\operatorname{rep}(\overline{Q})$. Note that the orbit of the restriction

$$\overline{T} = \bigoplus_{i=1}^{r} \overline{T}_i^{\,\lambda_i}$$

to $\overline{Q}$ is open in $\operatorname{rep}(\overline{Q}, \overline{\mathbf{d}})$. Indeed, we get ${}^1[\overline{T}, \overline{T}] = 0$ by computing the Hom-Ext-sequences (T, Σ) and $(\Sigma, \overline{T})$ of the short exact sequence

$$\Sigma\colon \quad 0 \longrightarrow E_z^{\,d_z} \longrightarrow T \longrightarrow \overline{T} \longrightarrow 0.$$

Define a new quiver Q' by deleting z and $\alpha_1, \ldots, \alpha_t$ and by adding a new vertex z' and arrows $\beta_j \colon z' \to y_j$, for $j = 1, \ldots, t$. Note that the simple representation $E'_{z'}$ of Q' supported at z' is injective. Let

$$\mathcal{F}_z^- \colon \operatorname{rep}(Q) \longrightarrow \operatorname{rep}(Q')$$

be the reflection functor associated with the sink z, and dually

$$\mathcal{F}_{z'}^+ \colon \operatorname{rep}(Q') \longrightarrow \operatorname{rep}(Q)$$

the reflection functor associated with the source z' (see [1] and also [7]). If E_z is not a direct summand of T, we have $d_z \leq \sum_{j=1}^{t} d_{y_j}$, and the dimension vector $\mathbf{d}' = \mathbf{dim}\,\mathcal{F}_z^- T$ is given by

$$d_i' = \begin{cases} d_i & \text{if } i \neq z', \\ \left(\sum_{j=1}^{t} d_{y_j}\right) - d_z \geq 0 & \text{if } i = z'. \end{cases}$$

For the construction of the Auslander-Reiten quiver $\Gamma_{Q'}$, we embed $(Q')^{op}$ in $\mathbb{Z}\mathbb{D}_n$ in such a way that all vertices except for z' coincide with the vertices of the embedding of Q^{op} (compare §3.1). From this we immediately get the following result:

Lemma 3.4. *Let $U = (i,j) \neq E_z$ be an indecomposable representation of Q. Denote by $U' = (i',j') = \mathcal{F}_z^- U$ the corresponding representation of Q'. Then as elements of $\mathbb{Z}\mathbb{D}_n$ we get $(i',j') = (i,j)$.*

In order to compare $\mathcal{Z}_{Q,\mathbf{d}}$ with $\mathcal{Z}_{Q',\mathbf{d}'}$, we decompose:

$$\mathcal{Z}_{Q,\mathbf{d}} = \mathcal{Z}'_{Q,\mathbf{d}} \mathbin{\dot\cup} \mathcal{Z}''_{Q,\mathbf{d}} \quad \text{and} \quad \mathcal{Z}_{Q',\mathbf{d}'} = \mathcal{W}'_{Q',\mathbf{d}'} \mathbin{\dot\cup} \mathcal{W}''_{Q',\mathbf{d}'},$$

where

$$\begin{aligned} \mathcal{Z}'_{Q,\mathbf{d}} &= \{A \in \mathcal{Z}_{Q,\mathbf{d}};\ [A, E_z] = 0\}\,, & \mathcal{Z}''_{Q,\mathbf{d}} &= \{A \in \mathcal{Z}_{Q,\mathbf{d}};\ [A, E_z] > 0\}\,, \\ \mathcal{W}'_{Q',\mathbf{d}'} &= \{A' \in \mathcal{Z}_{Q',\mathbf{d}'};\ [E'_{z'}, A'] = 0\}\,, & \mathcal{W}''_{Q',\mathbf{d}'} &= \{A' \in \mathcal{Z}_{Q',\mathbf{d}'};\ [E'_{z'}, A'] > 0\}\,. \end{aligned}$$

For the purpose of studying $\mathcal{Z}''_{Q,\mathbf{d}}$ we set

$$\begin{aligned} \gamma''_{Q,\mathbf{d}} &= \#Q_0 - \sigma(T) - \operatorname{codim} \mathcal{Z}''_{Q,\mathbf{d}}, \\ \gamma_{\overline{Q},\overline{\mathbf{d}}} &= \#\overline{Q}_0 - \sigma(\overline{T}) - \operatorname{codim} \mathcal{Z}_{\overline{Q},\overline{\mathbf{d}}} \end{aligned}$$

and

$$\delta_{Q,\mathbf{d}} = \gamma''_{Q,\mathbf{d}} - \gamma_{\overline{Q},\overline{\mathbf{d}}}.$$

We will need to estimate the contributions to $\delta_{Q,\mathbf{d}}$ arising from the direct summands $T_1, \ldots, T_r$. For an indecomposable representation $U \neq E_z$ with dimension vector $\mathbf{u}$ we set

$$\rho(U) = \sigma(\overline{U}) - 1 - u'_{z'}.$$

With the notations introduced above we state the following results from [5]:

Summary 3.5.

(*i*) If T contains a high vertex H of Γ_Q as a direct summand then $\mathcal{Z}_{Q,\mathbf{d}}$ is a complete intersection.

(*ii*) $\delta_{Q,\mathbf{d}} = \left.\begin{cases} 0 & \text{if } d_z \geq \sum_{j=1}^{t} d_{y_j} \\ \sigma(\overline{T}) - \sigma(T) - d'_{z'} & \text{if } d_z < \sum_{j=1}^{t} d_{y_j} \end{cases}\right\} \leq 1.$

(*iii*) If $\delta_{Q,\mathbf{d}} > 0$ then $\delta_{Q,\mathbf{d}} \leq \sum\limits_{i=1}^{r} \rho(T_i)$.

We will also use the following results from [7]:

Summary 3.6.

(*i*) If $d_z \geq \sum_{j=1}^{t} d_{y_j}$ then $\mathcal{Z}_{Q,\mathbf{d}} = \mathcal{Z}''_{Q,\mathbf{d}}$.

(*ii*) If $d_z < \sum_{j=1}^{t} d_{y_j}$ then $\operatorname{codim} \mathcal{Z}'_{Q,\mathbf{d}} = \operatorname{codim} \mathcal{W}'_{Q',\mathbf{d}'}$.

3.8. For a fixed sink $z \in Q_0 \setminus \{n-1, n\}$ we describe the restriction functors

$$\mathcal{L}_z \colon \operatorname{rep}(Q) \longrightarrow \operatorname{rep}(L) \quad \text{and} \quad \mathcal{H}_z \colon \operatorname{rep}(Q) \longrightarrow \operatorname{rep}(H),$$

in terms of coordinates for indecomposable representations in Γ_Q.

Proposition 3.7. *Let $(0, z)$ be the coordinates of E_z and (i, j) an arbitrary vertex of Γ_Q. Then we have:*

(*i*) $\mathcal{L}_z(i, j) = 0$ *for the ranges:*

- (*a*) $i \leq 0$,
- (*b*) $z \leq i$ *and* $i + j \leq n - 1$,
- (*c*) $z + n - 1 \leq i + j$.

(*ii*) $\mathcal{L}_z(i, j) = (i, j)$ *for the ranges:*

- (*d*) $i + j \leq z - 1$,
- (*e*) $n - 1 \leq i$.

(*iii*) $\mathcal{L}_z(i, j) = (n - 1, i)$ *for the ranges:*

- (*f*) $1 \leq i \leq z - 1$ *and* $z \leq i + j \leq n - 1$,
- (*g*) $1 \leq i \leq z - 1$ *and* $n - 1 \leq j$.

(*iv*) $\mathcal{L}_z(i, j) = (n - 1, i + j + 1 - n)$ *for the range:*

(h) $z \leq i \leq n-2 \quad and \quad n \leq i+j \leq z+n-2$.

(v) $\mathcal{L}_z(i,j) = (n-1,i) \oplus (n-1,i+j+1-n)$ *for the range:*

(i) $i \leq z-1 \quad and \quad n \leq i+j \quad and \quad j \leq n-2$.

Proof. With the same arguments as in the proof of lemma 3.2 we conclude that $\mathcal{L}_z$, as described above, satisfies

$$\dim(\mathcal{L}_z(i,j))(k) = \begin{cases} \dim(i,j)(k) & \text{if } k < z, \\ 0 & \text{if } k \geq z, \end{cases}$$

for all $(i,j) \in \Gamma_Q$. Moreover, for representations (i,j) where $\mathcal{L}_z(i,j)$ consists of more than one indecomposable direct summand, we also have

$$^1[\mathcal{L}_z(i,j), \mathcal{L}_z(i,j)] = 0,$$

by applying the Auslander-Reiten formula (see §3.5). These two properties yield a necessary and sufficient condition for $\mathcal{L}_z$ to be the restriction functor to rep(L). □

With analogous arguments one also proves the following description of the restriction functor to rep(H):

Proposition 3.8. *Let* $(0,z)$ *be the coordinates of* E_z *and* (i,j) *an arbitrary vertex of* Γ_Q. *Then we have:*

(i) $\mathcal{H}_z(i,j) = 0$ *for the ranges:*

(a) $i+j \leq z$,

(b) $n-1 \leq i$.

(ii) $\mathcal{H}_z(i,j) = (i,j)$ *for the ranges:*

(c) $i \leq -1$,

(d) $z \leq i \quad and \quad i+j \leq n-2$,

(e) $z+n-1 \leq i+j$.

(iii) $\mathcal{H}_z(i,j) = (z, i+j-z)$ *for the range:*

(f) $0 \leq i \leq z-1 \quad and \quad z+1 \leq i+j \leq n-2$.

(iv) $\mathcal{H}_z(i,j) = (i, z+n-1-i)$ *for the range:*

(g) $z+1 \leq i \leq n-2 \quad and \quad n-1 \leq i+j \leq z+n-2$.

(*v*) $\mathcal{H}_z(i,j) = (z, n-1) \oplus (z,n)$ *for the range:*

(h) $i \leq z$ *and* $n-1 \leq i+j$ *and* $j \leq n-2$.

(*vi*) $\mathcal{H}_z(i,j) = (z,l)$ *with* $l \in \{n-1, n\}$ *such that (i,j) and (z,l) are congruent, for the range:*

(i) $0 \leq i \leq z-1$ *and* $n-1 \leq j$.

4 Folded Rectangles

Based on the coordinate system for $\mathbb{ZD}_n$ and Γ_Q introduced in §3.1 we now give the following definitions:

Definition 4.1. Let $U = (p,q)$ be a vertex in $\mathbb{ZD}_n$.

(*i*) If $q \neq n$ then by the *diagonal through* U we mean the subset of vertices in $\mathbb{ZD}_n$ denoted by

$$\mathcal{D}_U = \mathcal{D}_{(p,q)} = \{(i,j);\ i = p+q-j,\ j \in Q_0 \setminus \{n\}\} \cup \{(p+q-n+1, n)\}\,.$$

If $q = n$ we set $\mathcal{D}_U = \mathcal{D}_{(p,q-1)}$.

(*ii*) By the *codiagonal through* U we mean the subset of vertices in $\mathbb{ZD}_n$ denoted by

$$\mathcal{C}_U = \mathcal{C}_{(p,q)} = \{(p,j);\ j \in Q_0\}\,.$$

Definition 4.2. Let U be a vertex in $\mathbb{ZD}_n$. By $\mathcal{S}_{r,U}$ we denote the *sector to the right of* U *in* $\mathbb{ZD}_n$, i.e. the subset of all vertices $X \in \mathbb{ZD}_n$ which lie to the right of $\mathcal{D}_U$ and also to the right of $\mathcal{C}_U$. By $\mathcal{S}_{b,U}$ we denote the *sector below* U, i.e. the subset of all vertices $X \in \mathbb{ZD}_n$ which lie to the left of $\mathcal{D}_U$ and to the right of $\mathcal{C}_U$. In a similar way we also define the sectors *to the left of* and *above* U and denote them by $\mathcal{S}_{l,U}$ and $\mathcal{S}_{a,U}$, respectively.

Definition 4.3. Let $\mathcal{N} = (N_1, N_2)$ be a pair of low vertices with $N_1 < N_2$ in $\mathbb{ZD}_n$ and consider the following set of rules:

$$\begin{array}{rl}
(i) & \mathcal{C}_{N_1} \cap \mathcal{D}_{N_2} = \emptyset. \\
(ii) & \mathcal{D}_{N_1} \cap \mathcal{C}_{N_2} = \emptyset \quad \text{and} \quad \mathcal{D}_{N_1} \cap \mathcal{C}_{\tau N_2} \neq \emptyset. \\
(ii') & \mathcal{D}_{N_1} \cap \mathcal{C}_{N_2} \neq \emptyset.
\end{array}$$

- If $\mathcal{N}$ either satisfies ((i) and (ii)) or else ((i) and (ii')) we call the area

$$\mathcal{R}_{\mathcal{N}} = \mathcal{S}_{r,N_1} \cap \mathcal{S}_{l,N_2}$$

a *folded rectangle of type I* or of *type II* in $\mathbb{ZD}_n$, respectively.

- Given a folded rectangle $\mathcal{R}_{\mathcal{N}}$, let H_1 and H_2 be high vertices on $\mathcal{C}_{N_1}$ and on $\mathcal{D}_{N_2}$, respectively. Then we call the unique vertex $M_1 \in \mathcal{D}_{H_1} \cap \mathcal{C}_{H_2}$ the *bent down corner* of $\mathcal{R}_{\mathcal{N}}$.

- If $\mathcal{R}_{\mathcal{N}}$ is of type II we call the unique vertex $M_2 \in \mathcal{D}_{N_1} \cap \mathcal{C}_{N_2}$ the *low corner* of $\mathcal{R}_{\mathcal{N}}$.

Definition 4.4. Let $X = X_1^{\mu_1} \oplus \cdots \oplus X_r^{\mu_r}$ be a representation of Q, with pairwise non-isomorphic indecomposable direct summands X_i and positive multiplicities μ_i. Let $\mathcal{R}_{\mathcal{N}}$ be a folded rectangle contained in Γ_Q such that the following holds:

(i) The bent down corner of $\mathcal{R}_{\mathcal{N}}$ is a direct summand X_i of X with $\mu_i = 1$.

(ii) If $\mathcal{R}_{\mathcal{N}}$ is of type II the low corner of $\mathcal{R}_{\mathcal{N}}$ is a direct summand X_j of X with arbitrary μ_j.

(iii) There are no other direct summands of X except for X_i and X_j contained in $\mathcal{R}_{\mathcal{N}}$.

Then we call $\mathcal{R}_{\mathcal{N}}(X) = \mathcal{R}_{\mathcal{N}}$ a *folded rectangle suitable for* X.

Given a folded rectangle $\mathcal{R}_{\mathcal{N}}(X)$ suitable for a representation X as in definition 4.4, we set $N = N_1 \oplus N_2$. Up to renumbering, we assume X_1 to be the bent down corner of $\mathcal{R}_{\mathcal{N}}(X)$ and X_2 to be the low corner in case $\mathcal{R}_{\mathcal{N}}(X)$ is of type II. Also, we will occasionally use the notation of decomposing $X = X' \oplus X''$, where

$$X' = \begin{cases} X_1 & \text{if } \mathcal{R}_{\mathcal{N}}(X) \text{ is of type I,} \\ X_1 \oplus X_2 & \text{if } \mathcal{R}_{\mathcal{N}}(X) \text{ is of type II.} \end{cases}$$

Note that X'' may contain copies of X_2 as direct summands, even if $\mathcal{R}_{\mathcal{N}}(X)$ is of type II.

With these definitions we now gather some results which will be used in §5.1 to prove the first implication of our theorem. Recall that T is the representative of the open orbit of $\operatorname{rep}(Q, \mathbf{d})$.

Proposition 4.5. *Suppose there exists a folded rectangle $\mathcal{R}_{\mathcal{N}}(T)$ suitable for the representation $T = T' \oplus T'' \in \operatorname{rep}(Q, \mathbf{d})$. Setting $D = N \oplus T''$ we get:*

(i) $\mathbf{dim}\, D = \mathbf{d}$ *and hence* $T \leq_{\deg} D$.

(ii) $\operatorname{codim} \operatorname{Gl}(\mathbf{d}) \cdot D = 2$.

Proof. Part (i): Consider the short exact sequence

$$\Sigma\colon\quad 0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

obtained by taking the direct sum

$$\Sigma = \bigoplus_{U,\tau^{-1}U\in\mathcal{R}_{\mathcal{N}}(T)} \Sigma(U)^{\mu_U}$$

of the Auslander-Reiten sequences in $\mathcal{R}_{\mathcal{N}}(T)$ with multiplicities

$$\mu_U = \begin{cases} 1 & \text{if } U \text{ is high or if } \{U,\tau^{-1}U\} \not\subseteq \mathcal{S}_{a,T_1} \\ 2 & \text{if } U \text{ is low and if } \{U,\tau^{-1}U\} \subseteq \mathcal{S}_{a,T_1}. \end{cases}$$

Then one checks that the multiplicity of any indecomposable direct summand is the same in $X\oplus Z$ and in Y, except for N_1, N_2, and for the indecomposables of T'. The direct summands N_1 and N_2 occur exactly once in $X \oplus Z$, but never in Y. On the other hand, if the multiplicity of a direct summand T_i of T is m_i in $X \oplus Z$ then it is $m_i + 1$ in Y. This implies $\mathbf{dim}\, T' = \mathbf{dim}\, N$ and hence we get $\mathbf{dim}\, D = \mathbf{dim}\, T = \mathbf{d}$.

Part (ii): By the Artin-Voigt lemma (see [8]), stating that for arbitrary representations X we get

$$\operatorname{codim} \mathrm{Gl}(\mathbf{d}) \cdot X = {}^1[X,X],$$

and by the fact that the Tits form satisfies

$$q(\mathbf{dim}\, X) = [X,X] - {}^1[X,X],$$

we conclude

$$\begin{aligned} \operatorname{codim} \mathrm{Gl}(\mathbf{d}) \cdot D &= [D,D] - q(\mathbf{d}) \\ &= [D,D] - [T,T] + {}^1[T,T] \\ &= [D,D] - [T,T]. \end{aligned}$$

Observe that $[N,N] - [T',T'] = 2$ for both types of folded rectangles, by proposition 3.1. Hence we obtain

$$\operatorname{codim} \mathrm{Gl}(\mathbf{d}) \cdot D = 2 + [N,T''] - [T',T''] + [T'',N] - [T'',T']$$

and by the arguments of part (i) concerning the multiplicities of direct summands of the components of Σ, this is seen to be

$$\operatorname{codim} \mathrm{Gl}(\mathbf{d}) \cdot D = 2 + \delta_\Sigma(T'') + \delta'_\Sigma(T'').$$

Now since the direct summands of T'' lie outside of $\mathcal{R}_{\mathcal{N}}(T)$ or possibly at the low corner only, in case $\mathcal{R}_{\mathcal{N}}(T)$ is of type II, we get

$$\delta_{\Sigma}(T'') = \delta'_{\Sigma}(T'') = 0,$$

by applying lemma 3.3. $\square$

Lemma 4.6. *Suppose there exists a folded rectangle $\mathcal{R}_{\mathcal{N}}(T)$ suitable for the representation T and with bent down corner $T_1 = (p, q)$. Then either the adjacent high vertices H_1, H_2 with first coordinates $p + q - n + 1$ are both simple projective objects in $T^{\perp}$ or else the adjacent high vertices H_3, H_4 with first coordinates $p - 1$ are simple injective objects in $T^{\perp}$.*

Proof. The existence of $\mathcal{R}_{\mathcal{N}}(T)$ implies that the second coordinate of T_1 lies in the range $q \in \{2, \ldots, n-2\}$. For $q = n-2$ we have $\{H_1, H_2\} = \{H_3, H_4\}$ and in this case the stated property is easily seen to be true.

Now for $q \leq n-3$, the existence of $\mathcal{R}_{\mathcal{N}}(T)$ implies that the entire sector $\mathcal{S}_{a,(p-1,q+1)}$ lies in Γ_Q and hence belongs to $T_1^{\perp}$. It is clear that, as objects of $T_1^{\perp}$, the vertices H_1, H_2 are both projective and H_3, H_4 are both injective. We first show that either H_1, H_2 or else H_3, H_4 are simple in $T_1^{\perp}$: Suppose H_1, H_2 are not simple in $T_1^{\perp}$. Then the vertex A on $\mathcal{C}_{H_1}$ with second coordinate $n - q - 2$ must belong to $T_1^{\perp}$ and hence to Γ_Q. But then no vertex (k, l) on $\mathcal{D}_{H_3}$ with $l \leq n - q - 2$ belongs to Γ_Q and hence neither to $T_1^{\perp}$. This in turn implies that H_3, H_4 must be simple in $T_1^{\perp}$.

Assuming that H_1, H_2 are simple and projective in $T_1^{\perp}$, we only have to show that they belong to $T^{\perp}$, because then, of course, they are simple and projective in $T^{\perp}$ as well. Now if they do not belong to $T^{\perp}$ then this implies that T has a direct summand T_j on $\mathcal{C}_{H_1}$ being lower or equal to A. But then T_j belongs to $T_1^{\perp}$ and, in contradiction to our assumption, H_1, H_2 are not simple in $T_1^{\perp}$. A similar argument shows that if H_1, H_2 are not simple in $T_1^{\perp}$ then H_3, H_4 belong to $T^{\perp}$. $\square$

Lemma 4.7. *Suppose there exists a folded rectangle $\mathcal{R}_{\mathcal{N}}(T)$ suitable for the representation T and with bent down corner $T_1 = (p, q)$. If $\mathcal{R}_{\mathcal{N}}(T)$ is of type I we set u to be the first coordinate of N_2, and $v = 0$. If $\mathcal{R}_{\mathcal{N}}(T)$ is of type II we set $T_2 = (u, v)$. Then at least one of the following vertices belongs to $T^{\perp}$:*

$$\begin{aligned}
Y_1 &= (p, u + v - p), \\
Y_2 &= (u - 1, v + 1), \\
Y_3 &= (u - 1, p + q - u).
\end{aligned}$$

Proof. First note that Y_i belongs to $T^{\perp}$ if and only if T belongs to ${}^{\perp}Y_i$. Remembering also that T must satisfy ${}^1[T,T] = 0$, we obtain the following:

$Y_1 \notin T^{\perp}$ if and only if T contains a direct summand $T_j \in \mathcal{A}_1 \cup \mathcal{A}_2$,
$Y_2 \notin T^{\perp}$ if and only if T contains a direct summand $T_j \in \mathcal{A}_1 \cup \mathcal{A}_4$,
$Y_3 \notin T^{\perp}$ if and only if T contains a direct summand $T_j \in \mathcal{A}_3 \cup \mathcal{A}_4$.

The areas $\mathcal{A}_i$ are obtained in the following way: Considering the vertices

$$\begin{aligned}
A_1 &= (u+v-n+1, p+q-u-v-1),\\
A_2 &= (p, u-p-1),\\
A_3 &= (u+v+1, p+q-u-v-1),\\
A_4 &= (p+n, u-p-1),
\end{aligned}$$

we define

$$\begin{aligned}
\mathcal{A}_1 &= \{X \in \mathcal{C}_{A_1};\ X \text{ not higher than } A_1\},\\
\mathcal{A}_2 &= \{X \in \mathcal{C}_{A_2};\ X \text{ not higher than } A_2\},\\
\mathcal{A}_3 &= \{X \in \mathcal{D}_{A_3};\ X \text{ not higher than } A_3\},\\
\mathcal{A}_4 &= \{X \in \mathcal{D}_{A_4};\ X \text{ not higher than } A_4\}.
\end{aligned}$$

Note that there are special cases of folded rectangles, where some of the Y_i coincide. In these cases some of the A_l have non-positive second coordinates, and we consider the corresponding $\mathcal{A}_l$ to be empty. Now the result follows from the fact that $\mathcal{A}_l$ and $\mathcal{A}_k$ cannot exist in Γ_Q simultaneously if $|l-k| \geq 2$. □

Proposition 4.8. *Suppose there exists a folded rectangle $\mathcal{R}_{\mathcal{N}}(T)$ suitable for the representation T. Then there are non-trivial morphisms from N_1 to three different simple objects of $T^{\perp}$.*

Proof. The first two simple objects of $T^{\perp}$ arise from lemma 4.6, since for any folded rectangle we get $[N_1, H_i] = 1$, for $i = 1, \ldots, 4$. Moreover, also for the Y_j of lemma 4.7 we always obtain $[N_1, Y_j] = 1$, for $j = 1, \ldots, 3$. Note that the Y_j belonging to $T^{\perp}$ might not be simple in $T^{\perp}$. But by an easy filtration argument, we get a non-trivial morphism from N_1 to a simple object S of $T^{\perp}$. And S cannot be one of the H_i, since Y_j and H_i evidentially belong to different connected components of $T^{\perp}$. □

The remaining results in this section will be used in §5.2 to prove the second implication of the statement of our theorem. We fix a sink $z \in Q_0$ once for all.

Lemma 4.9. *Let U be an arbitrary vertex in Γ_Q. Any vertex $V \in \Gamma_Q$, satisfying ${}^1[U,V] = {}^1[V,U] = 0$ and not belonging to $\mathcal{S}_{a,U} \cup \mathcal{S}_{b,U}$, cannot belong to any possible folded rectangle suitable for U.*

Proof. We set $U = (p,q)$. Clearly, any vertex V satisfying the above conditions must belong to

$$\mathcal{S}_{l,(p-2,1)} \cup \mathcal{S}_{r,(p+q+1,1)}.$$

But every folded rectangle suitable for U is completely contained in

$$\mathcal{S}_{r,(p+q-n+1,n-q)} \cap \mathcal{S}_{l,(p+q-1,n-q)}$$

and therefore cannot intersect with the area of possible locations of V. $\square$

Lemma 4.10. *Suppose $\mathcal{Z}''_{Q,\mathbf{d}}$ is not a complete intersection and $\delta_{Q,\mathbf{d}} = 1$. Then we get:*

(*i*) *The sink z is not a high vertex of Q.*

(*ii*) *Exactly one direct summand of the representation T, say T_1, belongs to $\mathcal{T}_{2,z}$, and its multiplicity is $\lambda_1 = 1$.*

(*iii*) *Let $X = (u,v)$ be the lowest vertex in $\mathcal{T}_{2,z}$. Then no direct summand of T on $\mathcal{C}_{(u+1,v-1)}$ can be higher than or equal to $(u+1,v-1)$.*

(*iv*) *Let Y be the unique vertex in $\mathcal{C}_{T_1} \cap \mathcal{D}_{E_z}$. Then no other direct summand of T on $\mathcal{C}_{T_1}$, except for T_1, can be higher than or equal to Y.*

Proof. For part (*i*) suppose z is a high vertex of Q. Then E_z is a high vertex in Γ_Q and therefore cannot be a direct summand of T by part (*i*) of summary 3.5. For any other indecomposable $U \neq E_z$, we find $\rho(U) \leq 0$ by means of propositions 3.7 and 3.8. So by part (*ii*) of summary 3.5 also $\delta_{Q,\mathbf{d}} \leq 0$, which contradicts our hypothesis on $\delta_{Q,\mathbf{d}}$.

For the remaining parts we may assume $z \in Q_0 \setminus \{n-1, n\}$. Again, by means of propositions 3.7 and 3.8 we get

$$\rho(U) \begin{cases} = 1, & \text{if } U \in \mathcal{T}_{2,z} \\ \leq 0, & \text{otherwise.} \end{cases}$$

So T must contain a direct summand $T_1 \in \mathcal{T}_{2,z}$. However, for any further copy of T_1 occurring in T, or for any other direct summand of T belonging to $\mathcal{T}_{2,z}$ or to one of the ranges described in (*iii*) and (*iv*), the gain on

$$\sigma(T) + d'_{z'}$$

would strictly exceed the gain on

$$\sigma(\overline{T}).$$

But by part (*iii*) of summary 3.5, this would imply $\delta_{Q,\mathbf{d}} \leq 0$. □

Proposition 4.11. *Suppose $\mathcal{Z}''_{Q,\mathbf{d}}$ is not a complete intersection and $\delta_{Q,\mathbf{d}} = 1$. Then there exists a folded rectangle suitable for the representation T.*

Proof. Let $T_1 = (p, q)$ be the unique direct summand of T belonging to $\mathcal{T}_{2,z}$ according to part (*ii*) of lemma 4.10. The folded rectangle $\mathcal{R}_{\mathcal{N}}(T_1)$ of type I, defined by requiring $N_1 \in \mathcal{D}_{E_z}$, is clearly suitable for T_1, but not necessarily for T.

By lemma 4.9, any further direct summand of T belonging to $\mathcal{R}_{\mathcal{N}}(T_1)$ must also be in $\mathcal{S}_{b,T_1}$. By applying parts (*ii*) to (*iv*) of lemma 4.10, the range of possible direct summands of T in $\mathcal{R}_{\mathcal{N}}(T_1)$ can be further reduced as follows: Setting

$$Z = (p + 1, q - 2),$$

any further direct summand of T belonging to $\mathcal{R}_{\mathcal{N}}(T_1)$ is seen to be in

$$\mathcal{A} = \mathcal{S}_{b,Z} \cap \mathcal{R}_{\mathcal{N}}(T_1).$$

If there is no direct summand of T in $\mathcal{A}$, then $\mathcal{R}_{\mathcal{N}}(T_1)$ is also suitable for T and we are done. Otherwise let T_2 be a highest indecomposable direct summand of T in $\mathcal{A}$. Then we obtain a folded rectangle $\mathcal{R}'_{\mathcal{N}'}(T)$ of type II suitable for T as follows:

$$\mathcal{R}'_{\mathcal{N}'}(T) = \mathcal{R}_{\mathcal{N}}(T_1) \cap \mathcal{S}_{a,T_2}.$$

□

Proposition 4.12. *Suppose $\overline{Q}$ contains a connected component H of type $\mathbb{D}_{n-z}$ and there exists a folded rectangle suitable for $T|H$ in Γ_H. Then there exists a folded rectangle suitable for T.*

Proof. By proposition 3.8, the bent down corner of the folded rectangle $\mathcal{R}_{\mathcal{N}_H}(T|H)$ in Γ_H is the restriction to H of a direct summand, say T_1, of T. Assuming $\mathcal{R}_{\mathcal{N}_H}(T|H)$ to be of type I, we have to distinguish different cases, depending on the position of $T_1|H$ embedded in Γ_Q. We set $E_z = (0, z)$ and $T_1|H = (i, j)$:

(*i*) For $i = z$ and $j \leq n - z - 2$, by proposition 3.8, the possible positions for T_1 are
$$T_1 \in \{(k, z + j - k);\ k = 0, \dots, z\}\,.$$
In this situation we get a folded rectangle $\mathcal{R}_{\mathcal{N}}(T_1)$ suitable for T_1, by setting $N_1 = (N_H)_1$ and N_2 to be the unique vertex in
$$\mathcal{C}_{(N_H)_2} \cap \mathcal{D}_{(k,n-1)}.$$

(*ii*) For $z + 1 \leq i \leq n - 2$ and $i + j = z + n - 1$ the possible positions for T_1 are
$$T_1 \in \{(i, j - k);\ k = 0, \dots, z\}\,.$$
Here we obtain a folded rectangle $\mathcal{R}_{\mathcal{N}}(T_1)$ suitable for T_1, by setting $N_2 = (N_H)_2$ and N_1 to be the unique vertex in
$$\mathcal{D}_{(N_H)_1} \cap \mathcal{C}_{(i+j-k-n+1,n-1)}.$$

(*iii*) For all other possible positions of $T_1|H$ in Γ_Q, namely for

- $i \leq -1$,
- $z + 1 \leq i \quad$ and $\quad i + j \leq n - 2$,
- $z + n - 1 \leq i + j$,

we conclude that $T_1 = T_1|H$. And by setting $N_1 = (N_H)_1$ and $N_2 = (N_H)_2$, again we get a folded rectangle $\mathcal{R}_{\mathcal{N}}(T_1)$ suitable for T_1.

In all of the above cases $\mathcal{R}_{\mathcal{N}}(T_1)$ must also be suitable for T since the restrictions $U|H$ of arbitrary vertices $U \in \mathcal{R}_{\mathcal{N}}(T_1)$ lie in $\mathcal{R}_{\mathcal{N}_H}(T|H)$.

With similar arguments also folded rectangles of type II suitable for $T|H$ can be lifted to folded rectangles suitable for T in Γ_Q. $\square$

Proposition 4.13. *Suppose there is a folded rectangle suitable for $\mathcal{G}^-T$ as well as one suitable for $\mathcal{G}^+T$, where $\mathcal{G}^-$ and $\mathcal{G}^+$ are compositions of some reflection functors at successive sinks or successive sources, respectively. Then there is a folded rectangle suitable for T.*

Proof. Suppose $\mathcal{R}_{\mathcal{N}_1}(X), \dots, \mathcal{R}_{\mathcal{N}_k}(X)$ are different folded rectangles suitable for $X \in \mathrm{rep}(Q)$, with bent down corner $X_1 = (p, q)$. From the definition of folded rectangles it is clear that they are all contained in the area $\mathcal{A} \cap \Gamma_Q$, where
$$\mathcal{A} = \mathcal{S}_{r,(p+q-n+1,n-q)} \cap \mathcal{S}_{l,\nu(p+q-n+1,n-q)}.$$

Note that the vertices $(N_1)_1, \ldots, (N_k)_1$ all lie on $\mathcal{C}_{(p+q-n+1,n-q)}$.

By lemma 3.4, $\mathcal{R}_{\mathcal{F}_z^- \mathcal{N}_i}(\mathcal{F}_z^- X)$ is a folded rectangle suitable for $\mathcal{F}_z^- X$, for any $i = 1, \ldots, k$, unless $(N_i)_1 = E_z$. In the latter case, we say that $\mathcal{R}_{\mathcal{N}_i}(X)$ is destroyed by the reflection functor $\mathcal{F}_z^-$. Similarly we say that a folded rectangle suitable for $\mathcal{F}_z^- X$ is created by $\mathcal{F}_z^-$ if this rectangle is destroyed by $\mathcal{F}_{z'}^+$.

From the above facts, it is clear that if $\mathcal{R}_{\mathcal{N}_i}(X)$ is destroyed by $\mathcal{F}_z^-$ then $(N_i)_1$ is the lowest vertex among $(N_1)_1, \ldots, (N_k)_1$. Combining this with the description of $\mathcal{A}$, we conclude that if a folded rectangle $\mathcal{R}_{\mathcal{N}_i}(X)$ is destroyed by $\mathcal{F}_z^-$ then all vertices of $\mathcal{A}$ are to the left of Γ_Q. Hence no folded rectangle will ever be created under any sequence of reflection functors at successive sinks.

Now suppose there is no folded rectangle suitable for T. This implies that under the inverse reflections of $\mathcal{G}^+$ any folded rectangle suitable for $\mathcal{G}^+ T$ as well as any additional folded rectangle created at some intermediate stage is destroyed, before reaching T. But then by the above arguments, there is no folded rectangle suitable for $\mathcal{G}^- T$. And this contradicts our hypothesis. $\square$

5 Proof of the theorem

5.1. We first prove that the existence of a folded rectangle suitable for T implies that $\mathcal{Z}_{Q,\mathbf{d}}$ is not a complete intersection.

By proposition 4.5, a folded rectangle $\mathcal{R}_{\mathcal{N}}(T)$ yields a degeneration $D = N \oplus T''$ of T, with $\operatorname{codim} \mathrm{Gl}(\mathbf{d}) \cdot D = 2$. On the other hand $\mathrm{Gl}(\mathbf{d}) \cdot D$ belongs to the intersection of three different irreducible hypersurfaces of the complement of the open orbit $\mathrm{rep}(Q, \mathbf{d}) \setminus \mathrm{Gl}(\mathbf{d}) \cdot T$, by §3.6 and by proposition 4.8. Hence the variety

$$\mathcal{B} = \overline{\mathrm{Gl}(\mathbf{d}) \cdot D}$$

is not a complete intersection. Now since the zero representation of $\mathrm{rep}(Q, \mathbf{d})$ belongs to $\mathcal{B}$ as well as to any irreducible hypersurface of $\mathrm{rep}(Q, \mathbf{d}) \setminus \mathrm{Gl}(\mathbf{d}) \cdot T$, the intersection of $\mathcal{B}$ and the remaining irreducible hypersurfaces of $\mathrm{rep}(Q, \mathbf{d}) \setminus \mathrm{Gl}(\mathbf{d}) \cdot T$ must contain an irreducible component of $\mathcal{Z}_{Q,\mathbf{d}}$ which is not a complete intersection.

5.2. Now we prove that if $\mathcal{Z}_{Q,\mathbf{d}}$ is not a complete intersection then there exists a folded rectangle suitable for T.

We proceed by induction on the number n of vertices of Q. First assume $n \leq 3$. Then Q is of type $\mathbb{A}$, and hence $\mathcal{Z}_{Q,\mathbf{d}}$ is a complete intersection, by the results of [7]. As there are no folded rectangles for quivers of type $\mathbb{A}$, our claim holds for the base case. Now for $n > 3$ we fix a source y and a sink

$z \in Q_0$ and use the notations of §3.7. We have to distinguish the following cases:

(a) An irreducible component of $\mathcal{Z}_{Q,\mathbf{d}}$ with too small codimension belongs to $\mathcal{Z}''_{Q,\mathbf{d}}$ with respect to z.

(b) An irreducible component of $\mathcal{Z}_{Q,\mathbf{d}}$ with too small codimension belongs to $\mathcal{W}''_{Q,\mathbf{d}}$ with respect to y.

(c) Every irreducible component of $\mathcal{Z}_{Q,\mathbf{d}}$ with too small codimension belongs to $\mathcal{Z}'_{Q,\mathbf{d}}$ with respect to z, and belongs to $\mathcal{W}'_{Q,\mathbf{d}}$ with respect to y.

In case (a), we know that $\delta_{Q,\mathbf{d}}$ can only take the values 0 or 1, by part (ii) of summary 3.5. If $\delta_{Q,\mathbf{d}} = 1$ then there is a folded rectangle suitable for T, by proposition 4.11. So assume $\delta_{Q,\mathbf{d}} = 0$. As

$$\gamma''_{Q,\mathbf{d}} = \delta_{Q,\mathbf{d}} + \gamma_{\overline{Q},\overline{\mathbf{d}}} > 0,$$

we conclude that $\mathcal{Z}_{\overline{Q},\overline{\mathbf{d}}}$ is not a complete intersection. Hence $z < n-3$, i.e. $\overline{Q}$ contains a connected component H of type $\mathbb{D}_{n-z}$, and $\mathcal{Z}_{H,\mathbf{d}|H}$ is not a complete intersection. So by the inductive hypothesis, there exists a folded rectangle suitable for $T|H$. But then there is a folded rectangle suitable for T, by proposition 4.12.

In case (b) there is a folded rectangle suitable for T, by the dual of the arguments used in case (a).

In case (c) we have $\mathcal{Z}_{Q,\mathbf{d}} \neq \mathcal{Z}''_{Q,\mathbf{d}}$ with respect to the sink z. By summary 3.6, up to a series of reflection functors at successive sinks, we may assume that an irreducible component of $\mathcal{Z}_{Q',\mathbf{d}'}$ with too small codimension belongs to $\mathcal{Z}''_{Q',\mathbf{d}'}$ with respect to a sink z_1. So by case (a), there is a folded rectangle suitable for $\mathcal{F}_z^- T$. And by duality, we may assume that there is a folded rectangle suitable for $\mathcal{F}_y^+ T$ as well. Hence by proposition 4.13, there is a folded rectangle suitable for T.

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Part IV
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