

**On Questions of Existence and Regularity
Related to Notions of Convexity in
Carnot Groups**

Inauguraldissertation
der Philosophisch-naturwissenschaftlichen Fakultät
der Universität Bern

vorgelegt von

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Leiter der Arbeit:

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Mathematisches Institut

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Introduction

Sub-Riemannian geometry, also known as Carnot geometry or non-holonomic Riemannian geometry, is the study of sub-Riemannian manifolds, i.e. triples $(M, HM, \langle \cdot, \cdot \rangle)$ where M is a manifold, HM is a sub-bundle of the tangent bundle TM –the so-called horizontal bundle– and $\langle \cdot, \cdot \rangle$ is an inner product on the fibers of HM . An absolutely continuous curve $\gamma : [a, b] \rightarrow M$ is called horizontal if $\dot{\gamma}(t) \in H_{\gamma(t)}M$ for almost every t in $[a, b]$. One can define the Carnot–Carathéodory distance –also called sub-Riemannian distance– $\rho(p, q)$ of points $p, q \in M$ as the infimum of the sub-Riemannian lengths of absolutely continuous, horizontal curves $\gamma : [a, b] \rightarrow M$ connecting p and q , the sub-Riemannian length $L(\gamma)$ of γ being defined as

$$L(\gamma) := \int_a^b (\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle_{\gamma(t)})^{\frac{1}{2}} dt.$$

The horizontal distribution HM is called bracket-generating if for each $p \in M$ there exists an open neighbourhood U of p and a local frame $\{X_1, \dots, X_n\}$ of vector fields for HM on U (where $n = \dim(HM)$), such that among the vector fields

$$X_{j_1}, [X_{j_1}, X_{j_2}], [X_{j_1}, [X_{j_2}, X_{j_3}]], [X_{j_1}, [X_{j_2}, [X_{j_3}, \dots, X_{j_k}]] \dots], \dots$$

(where each j_i belongs to $\{1, \dots, n\}$) there exist $d = \dim(M)$ which are linearly independent.

By a classical theorem due to Carathéodory, Rashevsky and Chow ([15], [80], [18]), if M is connected and HM is bracket-generating, then any two points $p, q \in M$ may be connected by means of an absolutely continuous, horizontal curve of finite sub-Riemannian length (whence $\rho(p, q) < +\infty$), and ρ is a metric on M which generates the manifold topology.

Sub-Riemannian geometry is a natural generalization of Riemannian geometry, where sub-Riemannian metrics occur as limiting cases. It provides a unified framework for various parts of pure and applied mathematics, such as Cauchy–Riemann geometry, analysis of subelliptic differential operators, classical mechanics and control theory.

A connected, simply connected, nilpotent Lie group \mathbb{G} is called a Carnot group or a stratified group if its Lie algebra \mathfrak{g} of left invariant vector fields admits a stratification, that is a direct sum decomposition $\mathfrak{g} = \bigoplus_{i=1}^s V_i$ where V_1, \dots, V_s are non-zero subspaces, $[V_i, V_i] = V_{i+1}$ for all $1 \leq i \leq s-1$ and $[V_1, V_s] = \{0\}$. In particular, the horizontal sub-bundle $H\mathbb{G}$ spanned by the left invariant vector fields which belong to V_1 is bracket-generating. Any inner product on V_1 yields an inner product $\langle \cdot, \cdot \rangle$ on the fibers of $H\mathbb{G}$. Thus $(\mathbb{G}, H\mathbb{G}, \langle \cdot, \cdot \rangle)$ is a sub-Riemannian manifold which satisfies the hypotheses of the Carathéodory–Rashevsky–Chow theorem.

It turns out that Carnot groups are, in a suitable sense, the “tangent spaces” to sub-Riemannian manifolds. More precisely, it can be shown that the tangent cone at any regular point of a sub-Riemannian manifold is isometric to a stratified group endowed with its natural sub-Riemannian metric (cf. [69], [9], [66]). It follows from the differentiability theorem of Pansu ([79]) that the stratified group is unique, up to an isomorphism of stratified groups.

The idea of approximating sub-Riemannian manifolds by means of Carnot groups goes back to the work of Rothschild and Stein ([84]), Métivier ([68]) and Goodman ([42]), who use this technique in order to study the properties (regularity, spectra) of linear differential operators defined by means of vector fields satisfying Hörmander’s condition (cf. Hörmander’s foundational paper [50]). We refer the interested reader to the survey article [34] of Folland for an exposition of these methods.

Finally, it should be mentioned that the nilpotent subgroup $N \subseteq G$ in the Iwasawa decomposition

$$\Psi : K \times A \times N \rightarrow G$$

of a semisimple Lie group G of real rank one is a stratified group (see [59]). Moreover, the boundary at infinity $\partial X = G/MAN$ of the corresponding symmetric space $X = G/K$ has a natural sub-Riemannian manifold structure, and N is the tangent cone to this sub-Riemannian manifold at each $x \in \partial X$ (cf. [79]).

In view of the above observations, it is not surprising that substantial efforts have been made –and are being made– in order to develop analysis and geometric measure theory on stratified groups and to achieve a better understanding of their geometry. In particular, it is natural to try to extend Euclidean concepts and theorems to this sub-Riemannian setting.

Motivated by the role played by convex functions in the theory of fully nonlinear partial differential equations –the Monge–Ampère equation for instance (see e.g. [17], [44])– in Euclidean space, researchers working in the field of subelliptic partial differential equations have proposed several notions of convexity of sets and functions in the Carnot group setting, and more generally in the setting of sub-Riemannian manifolds induced by systems $\{X_1, \dots, X_n\}$ of Hörmander vector fields. The notion of horizontal convexity –h-convexity for short–, originally formulated by Caffarelli, was rediscovered by Danielli, Garofalo and Nhieu in [23]. Loosely speaking, a subset C of a Carnot group \mathbb{G} is said to be h-convex if the following condition holds: if two points on an integral curve of some left invariant, horizontal vector field on \mathbb{G} belong to C , then the whole segment of the integral curve between these points is also contained in C . A function $f : C \rightarrow \mathbb{R}$ is h-convex if it is convex along the integral curves $\gamma \subseteq C$ of the left invariant, horizontal vector fields on \mathbb{G} . The notion of horizontal convexity in the viscosity sense –v-convexity for short– was proposed and studied by Lu, Manfredi and Stroffolini in [64]. Roughly speaking, an upper semicontinuous function $f : \Omega \rightarrow \mathbb{R}$ defined on an open subset Ω of a Carnot group \mathbb{G} is v-convex if the horizontal Hessian of test functions touching f from above is positive semidefinite. It turns out that h-convexity and v-convexity are equivalent (see [62], [8], [65], [89] and the third chapter of this thesis). In [75], Monti and the author showed that the notion of geodesic convexity, which is perfectly appropriate in the Riemannian setting, is totally inadequate in the setup of the first Heisenberg group.

Convex functions $f : \Omega \rightarrow \mathbb{R}$ –where $\Omega \subseteq \mathbb{R}^d$ denotes a convex, open set– enjoy various strong regularity properties: First, a convex function is locally Lipschitz continuous, hence (by Rademacher’s theorem) almost everywhere differentiable. In addition, the supremum norm of the function on a ball is controlled by the mean of its absolute value on a concentric ball of comparable radius, and the essential supremum norm of its weak gradient on a ball of radius r is controlled by the mean of its absolute value on a concentric ball of comparable radius divided by r (L^∞ – L^1 estimates). Second order regularity results are also available for convex functions; it can be shown that their second order partial derivatives are (signed) Radon measures and, by a theorem of Aleksandrov, that they are twice differentiable almost everywhere ([1], [82], [30]). The Aleksandrov theorem has been recently generalized to the class of k -convex functions ($k > n/2$) in [16].

In the sub-Riemannian setting, the following regularity results have been obtained so far: It has been shown in [23] that continuous h-convex functions are locally Lipschitz

continuous, which, by the Rademacher type differentiability theorem of Pansu ([79]), implies that continuous h -convex functions are differentiable almost everywhere. Moreover, L^∞ - L^1 estimates for the function and its weak horizontal gradient hold ([23]). Similar results for v -convex functions have been obtained in [64]. It is proved in [65] that h -convex functions which are locally bounded above are automatically continuous. Finally, the theorem of Aleksandrov has been extended to continuous h -convex functions on the first Heisenberg group in [46], and to continuous h -convex functions on arbitrary Carnot groups of step two in [24].

We now describe the content of this thesis and the results obtained. The first chapter is a casual introduction to Carnot groups. We start by recalling the definitions and the basic properties of stratified groups; we leave the verification of most of the facts to the interested reader. We then supply some important examples; the well-known Heisenberg groups for instance appear several times in this work. Finally, we collect some definitions and elementary results related to differentiation in the horizontal directions.

The second chapter is based on the article [75]. We study the notion of geodesic convexity in the setting of the first Heisenberg group. It is modelled on the corresponding Riemannian notion of geodesic convexity. The results, which have been obtained in joint work with Roberto Monti, show that the notion of geodesic convexity is useless in this sub-Riemannian setting, since it turns out that the classes of geodesically convex sets and functions are essentially trivial.

In the third chapter, we introduce the closely related notions of h -convexity and v -convexity. We show that there is a sufficiently large supply of non-trivial convex sets and functions to make the theory interesting: For instance, in the setup of the first Heisenberg group, we can construct Weierstrass-type h -convex functions which are nowhere differentiable in the vertical direction on a dense set or on a Cantor set of vertical lines. These examples have been obtained in joint work with Zoltán Balogh (see [8]). We also prove that in any Carnot group, there exists a basis of the topology consisting of h -convex, bounded open sets with smooth boundary. This result is due to the author (cf. [83]).

In the Euclidean setting, it can be shown that h -convexity and v -convexity are equivalent, see for instance [62]. In the case of the Heisenberg groups, the equivalence of h -convexity and v -convexity is a joint result of Zoltán Balogh and of the author (cf. [8]). In this chapter, we prove that this equivalence holds in arbitrary Carnot groups. Note that this generalization has been obtained independently by Magnani ([65]), Wang ([89]), Juutinen, Lu, Manfredi and Stroffolini ([52]) and the author.

We start our investigations of the first order regularity of h -convex functions in the fourth chapter. First, we show that h -convex functions which are locally bounded above are also locally Lipschitz continuous with respect to an intrinsic metric. Second, we prove that if \mathbb{G} is an h -convex finite Carnot group, that is a Carnot group which contains a finite subset whose h -convex closure has non-empty interior, then any h -convex function defined on some h -convex, open subset of \mathbb{G} is locally bounded above. Third, we demonstrate that any stratified group of step two is finitely h -convex, but we also exhibit a stratified group of step three and topological dimension four (the Engel group) which is not finitely h -convex. This counterexample shows that finite h -convexity is not a generic property of stratified groups and that a new strategy is needed in order to prove the continuity of h -convex functions in step strictly larger than two.

The local Lipschitz continuity of locally bounded above h -convex functions –which is an easy generalization of the corresponding Euclidean assertion– has been obtained independently by Magnani ([65]) and the author. The method employed to demonstrate the local boundedness from above of h -convex functions in stratified groups of step two generalizes an idea used by Zoltán Balogh and the author in the Heisenberg groups ([8]). In

the general case however, the proofs are substantially harder and more technical. Finally, the counterexample to h-convex finiteness in step three is due to the author. The whole chapter is based on the paper [83].

Chapter five is devoted to the study of geometric/measure-theoretic properties of h-convex subsets of an arbitrary stratified group \mathbb{G} . We start by proving that there exists a constant $0 \leq c = c(\mathbb{G}, \rho) < 1$ such that the estimate

$$(0.1) \quad \frac{\mathcal{H}^Q(B(g, r) \cap C)}{\mathcal{H}^Q(B(g, r))} \leq c \quad \forall 0 < r < +\infty$$

holds whenever $C \subseteq \mathbb{G}$ is a measurable, h-convex subset and g is a point on its boundary. Here $B(g, r)$ denotes the open metric ball of radius r centered at g with respect to an intrinsic metric ρ on \mathbb{G} , Q is the homogeneous dimension of the group and \mathcal{H}^Q is the Q -dimensional Hausdorff measure induced by ρ . This estimate implies that, loosely speaking, measurable, h-convex sets do not admit inward cusps. Next, we show that measurable, h-convex sets have locally finite horizontal perimeter. Finally, we demonstrate that the measurability requirements in the previous assertions cannot be removed. Indeed, surprisingly, it turns out that even the first Heisenberg group contains non-measurable, h-convex sets.

The results of this chapter are due to the author. Estimate (0.1) is one of the main results in [83].

The fundamental estimate (0.1) has several interesting consequences, which we present in chapter six. We use (0.1) to give a concise alternative proof of the L^∞ - L^1 estimates for (continuous) h-convex functions of Danielli, Garofalo and Nhieu. This estimates are the main result in [23]. Note that L^∞ - L^1 estimates for v-convex functions were independently proved by Lu, Manfredi and Stroffolini in [64]. As a second application of (0.1), we show how this estimate can be combined with sufficient conditions proved by Danielli in [21] –see also the results of J. Björn in [10]– in order to demonstrate that boundary points of an h-convex, bounded open subset Ω of a Carnot group are regular and Hölder regular for weak solutions of the Dirichlet problem for the subelliptic p -Laplacian. Finally, we show that the existence of local upper bounds for h-convex functions satisfying a weak measurability condition (hence the local Lipschitz continuity of such functions) is a corollary of (0.1). This corollary is taken from [83].

In the last chapter, we describe the main steps which lead to the generalization of the Aleksandrov theorem to (continuous) h-convex functions on Carnot groups of step two. The results presented in this chapter are taken from [5], [23], [64] and [24]. The only contribution of the author is the proof of Theorem 7.7, which is a straightforward adaptation of the proof of the corresponding Euclidean statement (cf. Theorem 1 and its proof in section 6.4 of [30]). A similar reasoning can be found in [65] and [24].

Two interesting questions concerning the regularity of h-convex functions remain open: the first is whether h-convex functions are necessarily measurable, or whether there exists a non-measurable h-convex function on some Carnot group of step larger than two, the second is whether the Aleksandrov theorem extends to continuous, h-convex functions on general Carnot groups.

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CHAPTER 1

Carnot groups

In this chapter, we introduce the Carnot groups, also known as stratified groups. We will compare and study various notions of convexity on such groups in subsequent chapters. In the first section, we recall the definitions and we review some properties of Carnot groups, in particular the basic facts concerning the sub-Riemannian distance and the Haar measure. Examples of stratified groups are given in the second section. In the last section, we review some elementary results related to differentiation in the horizontal directions.

1. Definitions

DEFINITION 1.1. A connected, simply connected, nilpotent Lie group \mathbb{G} is called a *Carnot group* –or a *stratified group*– if its Lie algebra \mathfrak{g} of left invariant vector fields admits a *stratification*, i.e. if there exist non-zero subspaces V_1, \dots, V_s such that

- (i) $\mathfrak{g} = \bigoplus_{i=1}^s V_i$,
- (ii) $[V_1, V_i] = V_{i+1}$ ($i = 1, \dots, s-1$) and
- (iii) $[V_1, V_s] = \{0\}$.

Given a stratification $\bigoplus_{i=1}^s V_i$ of \mathfrak{g} , we let $d_i := \dim_{\mathbb{R}}(V_i)$ ($1 \leq i \leq s$). Then $d := \sum_{i=1}^s d_i$ is the topological dimension of \mathbb{G} and we define the *homogeneous dimension* of \mathbb{G} to be $Q := \sum_{i=1}^s i d_i$.

REMARK 1.1. It is easy to check that $[V_i, V_j] \subseteq V_{i+j}$ if $1 \leq i+j \leq s$ and $[V_i, V_j] = \{0\}$ if $i+j > s$. Moreover, it is clear that $\bigoplus_{i=1}^s V_i, \bigoplus_{i=2}^s V_i, \dots, \{0\}$ is precisely the descending central series of \mathfrak{g} . In particular, s is the step of \mathbb{G} and the numbers d_1, \dots, d_s as well as the homogeneous dimension Q are independent of the stratification. It is left to the reader to verify that if $\bigoplus_{i=1}^s V_i$ and $\bigoplus_{i=1}^s \tilde{V}_i$ are two stratifications of \mathfrak{g} , then there exists a Lie algebra automorphism $A : \mathfrak{g} \rightarrow \mathfrak{g}$ which maps each V_i onto \tilde{V}_i . Since \mathbb{G} is connected and simply connected, there exists a unique Lie group automorphism $a : \mathbb{G} \rightarrow \mathbb{G}$ such that $da = A$ (see for instance [90, Theorem 3.27]).

In the following, \mathbb{G} denotes a Carnot group of topological dimension d whose Lie algebra \mathfrak{g} of left invariant vector fields is endowed with a fixed stratification $\bigoplus_{i=1}^s V_i$.

The exponential mapping $\exp : \mathfrak{g} \rightarrow \mathbb{G}$ is a global diffeomorphism and

$$(1.1) \quad \exp(X) \exp(Y) = \exp(X * Y) \quad \forall X, Y \in \mathfrak{g}.$$

Here $X * Y$ is defined by the Baker–Campbell–Dynkin–Hausdorff formula

$$(1.2) \quad X * Y = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] + \frac{1}{12}[Y, [Y, X]] + \dots,$$

where the dots indicate a finite \mathbb{R} -linear combination of Lie brackets of X and Y of order at least four (see e.g. [87, Theorem 3.6.1]).

For $g_0 \in \mathbb{G}$, $l_{g_0} : \mathbb{G} \rightarrow \mathbb{G}$ and $r_{g_0} : \mathbb{G} \rightarrow \mathbb{G}$ denote respectively left and right translation by g_0 , i.e.

$$l_{g_0}(g) = g_0 g \quad \text{and} \quad r_{g_0}(g) = g g_0 \quad \forall g \in \mathbb{G}.$$

For each $\lambda > 0$, let $A_\lambda : \mathfrak{g} \rightarrow \mathfrak{g}$ be the unique Lie algebra automorphism such that $A_\lambda(X) = \lambda^i X$ if $X \in V_i$ ($i = 1, \dots, s$). By general theory (cf. e.g. [90, Theorem 3.27] and [90, Theorem 3.32]),

$$(1.3) \quad \delta_\lambda := \exp \circ A_\lambda \circ \exp^{-1}$$

is an automorphism of \mathbb{G} called *dilation with λ* . Clearly, $\{\delta_\lambda\}_{\lambda>0}$ is a 1-parameter group of automorphisms of \mathbb{G} . By convention, for all $g \in \mathbb{G}$, $\delta_0(g) := e$ and $\delta_\lambda(g) := \delta_{-\lambda}(g^{-1})$ if $\lambda < 0$.

REMARK 1.2. Notice that if $X \in V_i$ for some $1 \leq i \leq s$, then X is homogeneous of degree i with respect to dilations. Indeed, given $\lambda > 0$, an open subset $\Omega \subseteq \mathbb{G}$ and a smooth function $f : \Omega \rightarrow \mathbb{R}$, we have

$$\begin{aligned} X(f \circ \delta_\lambda)(g) &= \left. \frac{d}{dt} f(\delta_\lambda(g \exp(tX))) \right|_{t=0} \\ &= \left. \frac{d}{dt} f(\delta_\lambda(g) \exp(\lambda^i tX)) \right|_{t=0} \\ &= \lambda^i \left. \frac{d}{dt} f(\delta_\lambda(g) \exp(tX)) \right|_{t=\lambda^i 0} \\ &= \lambda^i X f(\delta_\lambda(g)). \end{aligned}$$

DEFINITION 1.2. A left invariant vector field X on \mathbb{G} is called *horizontal* if it belongs to the first layer V_1 of \mathfrak{g} . If (X_1, \dots, X_{d_1}) is a basis of V_1 , then the sub-bundle HG of $\text{T}\mathbb{G}$ spanned by $X_1(g), \dots, X_{d_1}(g)$ at each $g \in \mathbb{G}$ is called the *horizontal bundle*.

REMARK 1.3. Observe that any inner product $\langle \cdot, \cdot \rangle$ on V_1 induces a left invariant inner product –denoted again $\langle \cdot, \cdot \rangle$ – on the fibers of HG . Thus the triple $(\mathbb{G}, \text{HG}, \langle \cdot, \cdot \rangle)$ is a sub-Riemannian manifold.

We equip \mathbb{G} with a sub-Riemannian metric as follows: A curve $\gamma : [a, b] \rightarrow \mathbb{G}$ is said to be *admissible* if it is absolutely continuous and *horizontal*, i.e. if $\dot{\gamma}(t) \in \text{H}_{\gamma(t)}\mathbb{G}$ for almost every $t \in [a, b]$. If $\langle \cdot, \cdot \rangle$ is a left invariant inner product on the fibers of HG , define the *sub-Riemannian length* of γ to be

$$L(\gamma) := \int_a^b (\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle_{\gamma(t)})^{\frac{1}{2}} dt.$$

The *Carnot–Carathéodory distance* –also called *sub-Riemannian distance*– $\rho(g_1, g_2)$ of points $g_1, g_2 \in \mathbb{G}$ is defined as the infimum of sub-Riemannian lengths of admissible curves connecting g_1 with g_2 . The finiteness of $\rho(g_1, g_2)$ is guaranteed by Proposition 1.2. Standard regularization arguments (together with Proposition 1.2) show that absolutely continuous curves can be replaced by piecewise smooth curves in the above definition.

It is a simple exercise to check that ρ is left invariant and homogeneous, that is

$$\rho(l_{g_0}(g_1), l_{g_0}(g_2)) = \rho(g_1, g_2) \quad \forall g_0, g_1, g_2 \in \mathbb{G}$$

and

$$\rho(\delta_\lambda(g_1), \delta_\lambda(g_2)) = \lambda \rho(g_1, g_2) \quad \forall g_1, g_2 \in \mathbb{G} \quad \forall \lambda > 0.$$

In the following, $B(g, r)$ denotes the open ball of radius r centered at g with respect to the sub-Riemannian metric ρ , that is

$$B(g, r) := \{g' \in \mathbb{G} \mid \rho(g, g') < r\},$$

while $\overline{B}(g, r)$ denotes the closed ball of radius r centered at g with respect to the sub-Riemannian metric ρ , that is

$$\overline{B}(g, r) := \{g' \in \mathbb{G} \mid \rho(g, g') \leq r\}.$$

(This is no abuse of notation since (\mathbb{G}, ρ) is a length space).

We will now state and prove the Carnot group version of the Carathéodory–Rashevsky–Chow theorem (Proposition 1.2). Let us introduce some notation: Given $X_1, X_2, \dots \in \mathfrak{g}$ and $g_1, g_2, \dots \in \mathbb{G}$, we define

$$I^0(X_1) := X_1, \quad J^0(g_1) := g_1, \quad I^1(X_1, X_2) := [X_1, X_2], \quad J^1(g_1, g_2) := [g_1, g_2],$$

and, for $n \geq 2$,

$$I^n(X_1, \dots, X_{n+1}) := I^1(I^{n-1}(X_1, \dots, X_n), X_{n+1})$$

and

$$J^n(g_1, \dots, g_{n+1}) := J^1(J^{n-1}(g_1, \dots, g_n), g_{n+1}).$$

We will need the following corollary of the Baker–Campbell–Dynkin–Hausdorff formula in our proof of Proposition 1.2:

LEMMA 1.1. *If $X, Y \in \mathbb{G}$, then*

$$(1.4) \quad \exp(X) \exp(Y) \exp(-X) \exp(-Y) = \exp([X, Y] + R(X, Y)),$$

where $R(X, Y)$ is a finite \mathbb{R} -linear combination of brackets of X and Y of order ≥ 3 .

PROOF. Apply formulae (1.1) and (1.2) three times. \square

PROPOSITION 1.2. *Let \mathbb{G} be a stratified group of step s , $\mathfrak{g} = \bigoplus_{k=1}^s V_k$ a stratification of its Lie algebra, $\langle \cdot, \cdot \rangle$ an inner product on V_1 and ρ the sub-Riemannian distance induced by $\langle \cdot, \cdot \rangle$. Then $\rho(g_1, g_2) < +\infty$ for all $g_1, g_2 \in \mathbb{G}$ and ρ induces the manifold topology. What's more, there exist constants*

$$l = l(\mathbb{G}) \in \mathbb{N}, \quad n = n(\mathbb{G}) = \sum_{k=1}^s d_k (3 \cdot 2^{k-1} - 2) \in \mathbb{N},$$

and left invariant, horizontal vector fields $X_{i,j}^k$ with $1 \leq k \leq s$, $1 \leq i \leq d_k$, $1 \leq j \leq k$, such that any $g_1, g_2 \in \mathbb{G}$ can be connected by means of a path consisting of at most n segments of integral curves of these vector fields, each segment of length at most $l \cdot \rho(g_1, g_2)$.

PROOF. For each $k \in \{1, \dots, s\}$, let

$$\left(I^{k-1} \left(X_{1,1}^k, \dots, X_{1,k}^k \right), \dots, I^{k-1} \left(X_{d_k,1}^k, \dots, X_{d_k,k}^k \right) \right)$$

be a basis of V_k , where each $X_{i,j}^k$ belongs to V_1 (recall that V_1 generates \mathfrak{g}) and has sub-Riemannian length equal to one. Fix $\epsilon > 0$. For all $k \in \{1, \dots, s\}$, $i \in \{1, \dots, d_k\}$, define

$$f_{k,i} : (-\epsilon, \epsilon) \rightarrow \mathbb{G}, \quad f_{k,i}(t) := J^{k-1} \left(\exp \left(t^{\frac{1}{k}} X_{i,1}^k \right), \dots, \exp \left(t^{\frac{1}{k}} X_{i,k}^k \right) \right).$$

It follows from (1.4) by induction that

$$f_{k,i}(t) = \exp \left(t I^{k-1} \left(X_{i,1}^k, \dots, X_{i,k}^k \right) + O \left(t^{1+\frac{1}{k}} \right) \right).$$

In particular, $f_{k,i}$ is C^1 smooth and $d_0 f_{k,i}(\partial_t(0)) = I^{k-1} \left(X_{i,1}^k, \dots, X_{i,k}^k \right)$. Define

$$F : (-\epsilon, \epsilon)^d \rightarrow \mathbb{G}, \quad F(t_{1,1}, \dots, t_{s,d_s}) := \prod_{k=1}^s \prod_{i=1}^{d_k} f_{k,i}(t_{k,i}).$$

Then, for all $k \in \{1, \dots, s\}$, $i \in \{1, \dots, d_k\}$,

$$d_0 F(\partial_{t_{k,i}}(0)) = d_0 f_{k,i}(\partial_t(0)) = I^{k-1} \left(X_{i,1}^k, \dots, X_{i,k}^k \right).$$

Hence dF has maximal rank at 0. Consequently, $F((-\epsilon, \epsilon)^d)$ contains an open neighbourhood U of e . Each $f_{k,i}(t)$ is a product of $3 \cdot 2^{k-1} - 2$ elements of $\exp(V_1)$ of the form $\exp\left(t^{\frac{1}{k}} X_{i,j}^k\right)$, $j \in \{1, \dots, k\}$. Hence $F(t_{1,1}, \dots, t_{s,d_s})$ is a product of

$$n = n(\mathbb{G}) = \sum_{k=1}^s d_k \left(3 \cdot 2^{k-1} - 2\right)$$

elements of $\exp(V_1)$ of the form $\exp\left(t^{\frac{1}{k}} X_{i,j}^k\right)$ with $|t_{k,i}| < \epsilon$ for $k = 1, \dots, s$, $i = 1, \dots, d_k$, $j = 1, \dots, k$. Hence for ϵ sufficiently small, we have

$$(1.5) \quad U \subseteq F((-\epsilon, \epsilon)^d) \subseteq \{g \in \mathbb{G} \mid \rho(e, g) < 1\}.$$

The finiteness of the sub-Riemannian distance follows via left translations and dilations. From the definition of ρ , it is clear that the topology generated by ρ is finer than the manifold topology. The opposite direction follows from (1.5) using left translations and dilations.

Now let (X_1, \dots, X_d) be a basis of \mathfrak{g} such that (X_1, \dots, X_{d_1}) is an orthonormal basis of V_1 with respect to $\langle \cdot, \cdot \rangle$ and $(X_{n_{j-1}+1}, \dots, X_{n_j})$ is a basis of V_j for all $2 \leq j \leq s$, where $n_j := \sum_{i=1}^j d_i$. For $\delta > 0$ small enough and

$$B_\delta := \left\{ \sum_{i=1}^d x_i X_i \mid |x_i| < \delta \text{ for } i = 1, \dots, d \right\},$$

we have $\exp(\partial B_\delta) \subseteq U$. Since the sub-Riemannian distance induces the manifold topology, it follows that the set $\{\rho(e, g) \mid g \in \exp(\partial B_\delta)\}$ is bounded away from zero. Hence there exists a constant $l = l(\mathbb{G}) \in \mathbb{N}$ such that each $g \in \exp(\partial B_\delta)$ can be connected with e by means of a path consisting of at most n segments of integral curves of the vector fields $X_{i,j}^k$, each segment of length at most $l \cdot \rho(0, g)$. The full statement now follows via left translations and dilations. \square

DEFINITION 1.3. A left invariant, homogeneous metric on \mathbb{G} which induces the manifold topology is called an *intrinsic metric*.

It is not difficult to prove that left invariant, homogeneous metrics ρ_1 and ρ_2 on \mathbb{G} which both induce the manifold topology are equivalent, i.e. there exists a constant $1 \leq K < +\infty$ such that

$$\frac{1}{K} \rho_1(g_1, g_2) \leq \rho_2(g_1, g_2) \leq K \rho_1(g_1, g_2) \quad \forall g_1, g_2 \in \mathbb{G}.$$

Since ρ is intrinsic and $\exp : \mathfrak{g} \rightarrow \mathbb{G}$ is a diffeomorphism, it follows that (\mathbb{G}, ρ) is a boundedly compact –in particular complete– metric space. Recall that any rectifiable curve admits a parametrization by arc length (see for instance [12]). It can be shown that a rectifiable curve $\gamma : [a, b] \rightarrow \mathbb{G}$ parameterized by arc length is admissible (cf. e.g. [74, Proposition 1.3.3]) and that $\text{Var}(\gamma) = L(\gamma)$, that is the total variation of γ equals its sub-Riemannian length ([74, Theorem 1.3.5]). In particular, it follows that $\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle_{\gamma(t)} = 1$ for almost every $t \in [a, b]$. Finally, a standard compactness argument (see e.g. [12]) shows that any two points $g_1, g_2 \in \mathbb{G}$ can be connected by means of a (not necessarily unique) rectifiable curve parameterized by arc length which minimizes the total variation among rectifiable curves which connect these points. In view of the preceding observations, the sub-Riemannian length of such curve equals the sub-Riemannian distance of g_1 and g_2 .

DEFINITION 1.4. An admissible curve $\gamma : [a, b] \rightarrow \mathbb{G}$ whose sub-Riemannian length equals $\rho(\gamma(a), \gamma(b))$ is called a *length minimizer* or a *geodesic*.

REMARK 1.4. Golé and Karidi have shown that length minimizers in stratified groups of step two are smooth ([41]). Whether this remains true in Carnot groups of higher step is a central open question in the field. The study of geodesics and of their properties in a sub-Riemannian setting is an active area of research, see for instance [70], [41], [63], [71] and [72].

Formulae (1.1) and (1.2) and the change of variable formula for the Lebesgue measure in \mathbb{R}^d show that the image measure μ of a fixed Haar measure on the Lie algebra $(\mathfrak{g}, +)$ of \mathbb{G} under the exponential map is a bi-invariant Haar measure on the Borel σ -algebra $\mathcal{B}(\mathbb{G})$ of \mathbb{G} , that is

$$\mu(l_g(B)) = \mu(r_g(B)) = \mu(B) \quad \forall B \in \mathcal{B}(\mathbb{G}) \quad \forall g \in \mathbb{G}.$$

The bi-invariance implies that the inversion mapping, which maps the Borel set $B \in \mathcal{B}(\mathbb{G})$ to B^{-1} , preserves μ . Formula (1.3) and the change of variable formula show that μ is Q -homogeneous with respect to the dilations, i.e.

$$\mu(\delta_\lambda(B)) = \lambda^Q \mu(B) \quad \forall B \in \mathcal{B}(\mathbb{G}) \quad \forall \lambda > 0.$$

The Q -Ahlfors regularity of μ and the left invariance of ρ imply that the Q -dimensional Hausdorff measure \mathcal{H}^Q induced by ρ coincides with μ , up to multiplication with a unique positive constant (cf. [25, Lemma 1.2]).

DEFINITION 1.5. Let \mathbb{G} be a Carnot group, $\bigoplus_{i=1}^s V_i$ a stratification of its Lie algebra \mathfrak{g} of left invariant vector fields and $\langle \cdot, \cdot \rangle$ a left invariant inner product on the horizontal bundle $\mathrm{H}\mathbb{G}$. A basis (X_1, \dots, X_d) of \mathfrak{g} is said to be *adapted to the stratification* if

- (i) (X_1, \dots, X_{d_1}) is an orthonormal basis of V_1 with respect to $\langle \cdot, \cdot \rangle$ and
- (ii) $(X_{n_{j-1}+1}, \dots, X_{n_j})$ is a basis of V_j for all $2 \leq j \leq s$, where $n_j := \sum_{i=1}^j d_i$.

It will be convenient to work with a more explicit representation of a given stratified group \mathbb{G} . Such representation can be obtained as follows: By (1.1), $*$: $\mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ defines a group structure on \mathfrak{g} , and $\exp : (\mathfrak{g}, *) \rightarrow \mathbb{G}$ is a Lie group isomorphism. Given an adapted basis (X_1, \dots, X_d) of \mathfrak{g} , we use the induced \mathbb{R} -linear mapping

$$f : \mathfrak{g} \rightarrow \mathbb{R}^d, \quad f \left(\sum_{i=1}^d x_i X_i \right) = (x_1, \dots, x_d)$$

to transport the group operation $*$ on \mathfrak{g} to \mathbb{R}^d . Then $(\mathbb{R}^d, *)$ is a Lie group isomorphic with \mathbb{G} . Given $1 \leq i \leq d$, we let

$$\deg(i) := \min \left\{ 1 \leq k \leq s \mid i \leq \sum_{j=1}^k d_j \right\}.$$

Then

$$(1.6) \quad x * x' = x + x' + P \quad \forall x, x' \in \mathbb{R}^d,$$

where P_i is a polynomial in the variables x_k, x'_l with $\deg(k), \deg(l) < \deg(i)$ if $d_1 < i \leq d$, and $P_i = 0$ if $1 \leq i \leq d_1$. The unit element in $(\mathbb{R}^d, *)$ is 0, and the inverse of $x \in \mathbb{R}^d$ with respect to $*$ is $-x$. Dilation with $\lambda > 0$ is given by

$$(1.7) \quad \delta_\lambda(x_1, \dots, x_d) = \left(\lambda^{\deg(1)} x_1, \lambda^{\deg(2)} x_2, \dots, \lambda^{\deg(d)} x_d \right).$$

We can transport the horizontal sub-bundle $\mathrm{H}\mathbb{G}$ together with its left invariant inner product $\langle \cdot, \cdot \rangle$ to $\mathrm{T}\mathbb{R}^d$ by means of $d(f \circ \exp^{-1})$. The sub-bundle $\mathrm{H}\mathbb{R}^d$ of $\mathrm{T}\mathbb{R}^d$ obtained in

this way is spanned by the left invariant vector fields X on $(\mathbb{R}^d, *)$ uniquely determined by the condition

$$X(0) = \sum_{i=1}^{d_1} x_i \partial_i(0),$$

where $x_1, \dots, x_{d_1} \in \mathbb{R}$ are arbitrary, and the left invariant vector fields X_1, \dots, X_{d_1} uniquely determined by the condition $X_i(0) = \partial_i(0)$ form an orthonormal basis of the horizontal subspace $H_x \mathbb{R}^d$ at each $x \in \mathbb{R}^d$ with respect to the push-forward of $\langle \cdot, \cdot \rangle$ under $f \circ \exp^{-1}$. The integral curve $\gamma : \mathbb{R} \rightarrow \mathbb{R}^d$ of the left invariant vector field X determined by $X(0) = \sum_{i=1}^{d_1} x_i \partial_i(0)$ which satisfies the initial condition $\gamma(0) = p \in \mathbb{R}^d$ is given by

$$(1.8) \quad \gamma(t) = p * \delta_t(x_1, \dots, x_{d_1}, 0, \dots, 0) \quad \forall t \in \mathbb{R}.$$

It isn't difficult to see that the restriction of γ to any compact interval is a length minimizer for the sub-Riemannian distance on \mathbb{R}^d induced by the push-forward of $\langle \cdot, \cdot \rangle$ under $d(f \circ \exp^{-1})$, and that γ is a parametrization proportional to intrinsic arc length of $\gamma(\mathbb{R})$.

Finally, it follows from (1.6) and (1.7) that d -dimensional Hausdorff measure \mathcal{H}_E^d built with respect to the Euclidean distance on \mathbb{R}^d is a Q -homogeneous (with respect to the dilations (1.7)) Haar measure on the Borel σ -algebra $\mathcal{B}(\mathbb{R}^d)$. In particular, there exists a unique constant $0 < \alpha < +\infty$ such that $\mathcal{H}^Q = \alpha \mathcal{H}_E^d$.

Throughout this work, the notation $\mathbb{G} \equiv (\mathbb{R}^d, *)$ indicates that an adapted basis of the Lie algebra of \mathbb{G} has been chosen and that the group operation of \mathbb{G} has been transported to \mathbb{R}^d by means of the procedure described above. If $\mathbb{G} \equiv (\mathbb{R}^d, *)$, then (\cdot, \cdot) denotes the standard inner product on \mathbb{R}^d , $\|\cdot\|$ the induced Euclidean norm and $B_E(x, r)$ the open ball of radius r centered at $x \in \mathbb{R}^d$ with respect to the Euclidean metric ρ_E .

The following useful proposition shows that the flow generated by a horizontal vector field of sub-Riemannian length one is measure-preserving. Moreover, it gives a Fubini-type decomposition formula for stratified groups.

PROPOSITION 1.3. *Let $\mathbb{G} \equiv (\mathbb{R}^d, *)$ be a Carnot group and let $X \neq 0$ be the left invariant, horizontal vector field on $(\mathbb{R}^d, *)$ uniquely determined by the condition*

$$X(0) = \sum_{i=1}^{d_1} x_i \partial_i(0).$$

Then there exists a diffeomorphism $\Phi_X : \mathbb{R}^{d-1} \times \mathbb{R} \rightarrow \mathbb{R}^d$ such that

$$\Phi_X(y, t) = \Phi_X(y, 0) * \exp(tX) \quad \forall (y, t) \in \mathbb{R}^{d-1} \times \mathbb{R}.$$

Φ_X is measure-preserving if the sub-Riemannian length of X is one.

PROOF. Let Π denote the orthogonal complement of $v_1 := (x_1, \dots, x_{d_1}, 0, \dots, 0)$. Let (v_2, \dots, v_d) be an orthonormal basis of Π such that (v_1, \dots, v_{d_1}) is an orthogonal basis of $\mathbb{R}^{d_1} \times \{0\} \subseteq \mathbb{R}^d$ and $v_i = e_i$ if $d_1 + 1 \leq i \leq d$. Here e_i denotes the i -th vector in the standard basis of \mathbb{R}^d . Now define

$$\Phi_X : \mathbb{R}^{d-1} \times \mathbb{R} \rightarrow \mathbb{R}^d, \quad \Phi_X(y, t) := \left(\sum_{i=1}^{d-1} y_i v_{i+1} \right) * t v_1. \quad \forall (y, t) \in \mathbb{R}^{d-1} \times \mathbb{R}.$$

Using (1.6), it is a simple matter to verify the claims. \square

2. Example of Carnot groups

According to Bellaïche ([9]), there are many algebraically non-isomorphic stratified groups having the same topological dimension d , uncountably many if $d \geq 6$. In this section, we review some important examples.

2.1. The Heisenberg groups. $\mathbb{H}_n = (\mathbb{R}^{2n+1}, *)$ with the group law

$$(x, y, t) * (x', y', t') = \left(x + x', y + y', t + t' + 2 \sum_{i=1}^n x'_i y_i - x_i y'_i \right)$$

for all $(x, y, t), (x', y', t') \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ is the n -th Heisenberg group. Its Lie algebra is

$$\mathfrak{h}_n = \text{span}_{\mathbb{R}}\{X_1, \dots, X_n, Y_1, \dots, Y_n\} \oplus \text{span}_{\mathbb{R}}\{T\},$$

where

$$X_i = \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial t}, \quad Y_i = \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial t}, \quad T = \frac{\partial}{\partial t}.$$

We have

$$[X_i, Y_i] = -[Y_i, X_i] = -4T \quad \text{for } 1 \leq i \leq n,$$

and all other commutators vanish. For each $\lambda > 0$, the dilation $\delta_\lambda : \mathbb{H}_n \rightarrow \mathbb{H}_n$ induced by the stratification is given by

$$\delta_\lambda(x, y, t) = (\lambda x, \lambda y, \lambda^2 t) \quad \forall (x, y, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}.$$

Observe that \mathfrak{h}_n is isomorphic with the Lie algebra

$$\text{span}_{\mathbb{R}}\{X_1, \dots, X_n, P_1, \dots, P_n\} \oplus \text{span}_{\mathbb{R}}\{Id\},$$

generated by the operators $X_1, \dots, X_n, P_1, \dots, P_n$ and Id from quantum mechanics, which act on complex valued functions depending on the real variables x_1, \dots, x_n . More specifically, X_i is multiplication with x_i , $P_i = \frac{\hbar}{i} \partial_{x_i}$ and Id is the identity.

The Heisenberg group \mathbb{H}_n arises as the nilpotent part N in the Iwasawa decomposition

$$\Psi : K \times A \times N \rightarrow SU(1, n + 1)$$

of the simple group of real rank one $SU(1, n + 1)$. The Heisenberg group can also be identified with the boundary of the domain

$$D = \left\{ z \in \mathbb{C}^{n+1} \mid \Im(z_{n+1}) - \sum_{k=1}^n |z_k|^2 > 0 \right\},$$

which is biholomorphically equivalent to the open unit ball B^{n+1} in \mathbb{C}^{n+1} . Hence the structure of the unit sphere S^{2n+1} in \mathbb{C}^{n+1} and of the Heisenberg group \mathbb{H}_n are locally the same. In particular, both are equipped with a Cauchy–Riemann structure—hence a contact structure—, since they arise as boundaries of strictly pseudoconvex domains.

2.2. H -type groups.

DEFINITION 1.6. Let \mathfrak{n} be a nilpotent Lie algebra of step two equipped with an inner product $\langle \cdot, \cdot \rangle$. Let \mathfrak{z} denote the center of \mathfrak{n} and let $\mathfrak{v} = \mathfrak{z}^\perp$. We say that \mathfrak{n} is H -type if $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ is a stratification of \mathfrak{n} and the restriction of ad_X to $(\ker(\text{ad}_X))^\perp$ is an isometry onto \mathfrak{z} for all $X \in \mathfrak{v}$ such that $\langle X, X \rangle = 1$. A Carnot group \mathbb{G} is said to be H -type if its Lie algebra \mathfrak{g} is H -type.

H -type groups were introduced by Kaplan in [53]. He showed that the sublaplacian on an H -type group \mathbb{G} admits an explicit fundamental solution $\Phi : \mathbb{G} \setminus \{e\} \rightarrow \mathbb{R}$ of an elementary form, thereby generalizing a result of Folland on the Heisenberg group ([33]). This fundamental solution is expressed as

$$\Phi(g) = c N(g)^{-m},$$

where $m = \dim_{\mathbb{R}}(\mathfrak{g}) + \dim_{\mathbb{R}}(\mathfrak{z}) - 2$, $0 < c < +\infty$ is a constant and $N : \mathbb{G} \rightarrow \mathbb{R}$ is the homogeneous gauge given by

$$N(\exp(X + Z)) = (|X|^4 + 16|Z|^2)^{\frac{1}{4}} \quad \forall X \in \mathfrak{v}, \forall Z \in \mathfrak{z}.$$

It was shown by Cygan ([20]) that the gauge N is also subadditive, i.e.

$$N(g_1 g_2) \leq N(g_1) + N(g_2) \quad \forall g_1, g_2 \in \mathbb{G},$$

which implies that

$$\rho_N(g_1, g_2) := N(g_1^{-1} g_2) \quad \forall g_1, g_2 \in \mathbb{G}$$

defines an intrinsic metric on \mathbb{G} .

It can be shown that the nilpotent subgroup $N \subseteq G$ in the Iwasawa decomposition

$$\Psi : K \times A \times N \rightarrow G$$

of a semisimple Lie group G of real rank one is of type H (see for instance [59, Proposition 1.1]). The classical groups $SO_0(1, n)$, $SU(1, n)$, $Sp(1, n)$ and the exceptional 15-dimensional group $F_{4(-20)}$ are simple Lie groups of real rank one. The globally symmetric Riemannian spaces of real rank one and of negative curvature arise precisely as the quotients $SO_0(1, n)/SO(n)$ (real hyperbolic spaces), $SU(1, n)/U(n)$ (complex hyperbolic spaces), $Sp(1, n)/Sp(n)$ (quaternionic hyperbolic spaces) and $F_{4(-20)}/Spin(9)$ (Cayley hyperbolic plane). The boundary at infinity $\partial X = G/MAN$ of the corresponding symmetric space $X = G/K$ has a natural sub-Riemannian manifold structure, and N is the tangent cone to this sub-Riemannian manifold at each $x \in \partial X$ (cf. [79]). For real hyperbolic spaces N is Euclidean, and for complex hyperbolic spaces N is a Heisenberg group. Pansu proved a rigidity result for quasiconformal mappings on the nilpotent subgroups in the Iwasawa decomposition of $Sp(1, n)$ ($n \geq 2$) and $F_{4(-20)}$ (cf. [79]). He then used this result in order to simplify Mostow's original proof ([77]) of the rigidity of symmetric spaces in the quaternionic and Cayley number case. For recent results related to the rigidity of H -type groups, we refer the interested reader to [81] and [14].

2.3. The Engel group. Consider the stratified group

$$\mathbb{E} = (\mathbb{R}^4, *) = (\{(x_1, x_2, y, z) \mid x_1, x_2, y, z \in \mathbb{R}\}, *)$$

with the group law

$$(x_1, x_2, y, z) * (x'_1, x'_2, y', z') := (x_1 + x'_1, x_2 + x'_2, y + y', z + z') + P$$

for all $(x_1, x_2, y, z), (x'_1, x'_2, y', z') \in \mathbb{R}^4$, where

$$P = \left(0, 0, \frac{(x_1 x'_2 - x_2 x'_1)}{2}, \frac{(x_1 y' - y x'_1)}{2} + \frac{(x_1 - x'_1)(x_1 x'_2 - x_2 x'_1)}{12} \right).$$

If X_1, X_2, Y, Z denote the left invariant vector fields uniquely determined by the conditions

$$X_1(0) = \partial_{x_1}(0), \quad X_2(0) = \partial_{x_2}(0), \quad Y(0) = \partial_y(0), \quad Z(0) = \partial_z(0),$$

then the commutation relations

$$[X_1, X_2] = Y, \quad [X_1, Y] = Z, \quad [X_2, Y] = [X_1, Z] = [X_2, Z] = [Y, Z] = 0$$

hold, and

$$\text{span}_{\mathbb{R}}\{X_1, X_2\} \oplus \text{span}_{\mathbb{R}}\{Y\} \oplus \text{span}_{\mathbb{R}}\{Z\}$$

is a stratification of the Lie algebra \mathfrak{e} of left invariant vector fields on $(\mathbb{R}^4, *)$. Hence \mathbb{E} is a stratified group of step 3 and of homogeneous dimension $Q = 7$. \mathbb{E} is called the *Engel group* and \mathfrak{e} the *Engel algebra*.

For each $\lambda > 0$, the dilation $\delta_\lambda : \mathbb{E} \rightarrow \mathbb{E}$ induced by the stratification is given by the formula

$$\delta_\lambda(x_1, x_2, y, z) = (\lambda x_1, \lambda x_2, \lambda^2 y, \lambda^3 z) \quad \forall (x_1, x_2, y, z) \in \mathbb{R}^4.$$

In the fourth chapter, we will see that the Engel group provides an example of a stratified group of step three which is not finitely h-convex (compare Theorem 4.5 and Theorem 4.10).

3. Differentiation in the horizontal directions

In the following \mathbb{G} is a Carnot group, $\Omega \subseteq \mathbb{G}$ is an open subset, $\oplus_{i=1}^s V_i$ is a stratification of the Lie algebra \mathfrak{g} of left invariant vector fields on \mathbb{G} , $\{\delta_\lambda\}_{\lambda>0}$ is the family of dilations induced by the stratification, $\langle \cdot, \cdot \rangle$ is an inner product on V_1 , (X_1, \dots, X_{d_1}) is an orthonormal basis of V_1 with respect to $\langle \cdot, \cdot \rangle$, ρ is the sub-Riemannian distance induced by $\langle \cdot, \cdot \rangle$ and \mathcal{H}^Q is the Q -dimensional Hausdorff measure induced by ρ .

We start with an integration by parts formula for the derivative of a function in a horizontal direction:

LEMMA 1.4. *Let $\varphi \in C^1(\Omega)$, $\psi \in C_c^\infty(\Omega)$ and $X \in V_1$. Then*

$$\int_{\Omega} X\varphi(g)\psi(g) d\mathcal{H}^Q(g) = - \int_{\Omega} \varphi(g)X\psi(g) d\mathcal{H}^Q(g).$$

PROOF. Let $\Omega' \Subset \Omega'' \Subset \Omega$ such that the support of ψ is contained in Ω' . Using dominated convergence and the right invariance of \mathcal{H}^Q , we obtain

$$\begin{aligned} \int_{\Omega} X\varphi(g)\psi(g) d\mathcal{H}^Q(g) &= \int_{\Omega'} X\varphi(g)\psi(g) d\mathcal{H}^Q(g) \\ &= \int_{\Omega'} \left(\lim_{t \downarrow 0} \frac{\varphi(g \exp(tX)) - \varphi(g)}{t} \right) \psi(g) d\mathcal{H}^Q(g) \\ &= \lim_{t \downarrow 0} \int_{\Omega'} \left(\frac{\varphi(g \exp(tX)) - \varphi(g)}{t} \right) \psi(g) d\mathcal{H}^Q(g) \\ &= \lim_{t \downarrow 0} \int_{r_{\exp(tX)}(\Omega')} \varphi(g) \left(\frac{\psi(g(\exp(tX))^{-1}) - \psi(g)}{t} \right) d\mathcal{H}^Q(g) \\ &= \int_{\Omega''} \varphi(g) \left(\lim_{t \downarrow 0} \frac{\psi(g \exp(t(-X))) - \psi(g)}{t} \right) d\mathcal{H}^Q(g) \\ &= - \int_{\Omega} \varphi(g)X\psi(g) d\mathcal{H}^Q(g) \end{aligned}$$

□

We use a generalized version of the integration by parts formula to define the distributional derivative of a function in a horizontal direction.

DEFINITION 1.7. Let $X \in V_1$. If $f, h \in L_{loc}^1(\Omega)$ satisfy

$$\int_{\Omega} h(g)\psi(g) d\mathcal{H}^Q(g) = - \int_{\Omega} f(g)X\psi(g) d\mathcal{H}^Q(g) \quad \forall \psi \in C_c^\infty(\Omega),$$

we say that h is the *weak derivative of f in the direction X* —notice that there exists at most one h with this property up to almost everywhere equality—and we denote $Xf := h$. If the weak derivatives of $f \in L_{loc}^1(\Omega)$ in the directions X_1, \dots, X_{d_1} exist, we define

$$\nabla_{\mathbb{H}} f := \sum_{i=1}^{d_1} X_i f X_i \quad \text{and} \quad |\nabla_{\mathbb{H}} f| := (\langle \nabla_{\mathbb{H}} f, \nabla_{\mathbb{H}} f \rangle)^{\frac{1}{2}},$$

and we say that $\nabla_{\mathbb{H}} f$ is the *weak horizontal gradient of f* . Note that all orthonormal bases of V_1 yield the same weak horizontal gradient (up to almost everywhere equality).

The following lemma shows that the weak horizontal derivatives inherit the left invariance and homogeneity properties of the ordinary horizontal derivatives of smooth functions:

LEMMA 1.5. *Let $f \in L^1_{loc}(\Omega)$ and suppose that the weak horizontal derivative Xf of f in the direction $X \in V_1$ exists. Given $g_0 \in \mathbb{G}$, $\lambda > 0$, consider $f \circ \delta_\lambda \in L^1_{loc}(\delta_{1/\lambda}(\Omega))$ and $f \circ l_{g_0} \in L^1_{loc}(l_{g_0^{-1}}(\Omega))$. Then the weak horizontal derivatives $X(f \circ l_{g_0})$ of $f \circ l_{g_0}$ and $X(f \circ \delta_\lambda)$ of $f \circ \delta_\lambda$ in the direction X exist and equal $Xf \circ l_{g_0}$ and $\lambda(Xf \circ \delta_\lambda)$ respectively.*

PROOF. Let $\varphi \in C_c^\infty(\delta_{1/\lambda}(\Omega))$ and $\psi \in C_c^\infty(l_{g_0^{-1}}(\Omega))$. By left invariance of X and \mathcal{H}^Q , we have

$$\begin{aligned} \int_{l_{g_0^{-1}}(\Omega)} f \circ l_{g_0}(g) X\psi(g) d\mathcal{H}^Q(g) &= \int_{\Omega} f(g) X\psi(l_{g_0^{-1}}(g)) d\mathcal{H}^Q(g) \\ &= \int_{\Omega} f(g) X(\psi \circ l_{g_0^{-1}})(g) d\mathcal{H}^Q(g) \\ &= - \int_{\Omega} Xf(g) (\psi \circ l_{g_0^{-1}})(g) d\mathcal{H}^Q(g) \\ &= - \int_{l_{g_0^{-1}}(\Omega)} Xf \circ l_{g_0}(g) \psi(g) d\mathcal{H}^Q(g). \end{aligned}$$

Next, using Remark 1.2 and the Q -homogeneity of \mathcal{H}^Q , we compute

$$\begin{aligned} \int_{\delta_{1/\lambda}(\Omega)} f \circ \delta_\lambda(g) X\varphi(g) d\mathcal{H}^Q(g) &= \int_{\Omega} f(g) \frac{1}{\lambda} X\varphi(\delta_{1/\lambda}(g)) \frac{1}{\lambda^{Q-1}} d\mathcal{H}^Q(g) \\ &= \int_{\Omega} f(g) X(\varphi \circ \delta_{1/\lambda})(g) \frac{1}{\lambda^{Q-1}} d\mathcal{H}^Q(g) \\ &= - \int_{\Omega} Xf(g) (\varphi \circ \delta_{1/\lambda})(g) \frac{1}{\lambda^{Q-1}} d\mathcal{H}^Q(g) \\ &= - \int_{\delta_{1/\lambda}(\Omega)} \lambda(Xf \circ \delta_\lambda)(g) \varphi(g) d\mathcal{H}^Q(g). \end{aligned}$$

□

As in the Euclidean setting, it is often convenient to work with smooth functions. It is therefore useful to have a procedure which allows to regularize a function: Let $\eta \in C_c^\infty(\mathbb{G})$ such that $\eta \geq 0$ in \mathbb{G} , $\int_{\mathbb{G}} \eta(g) d\mathcal{H}^Q(g) = 1$ and the support of η is contained in $B(e, 1)$. Given $\epsilon > 0$, let

$$\Omega_\epsilon := \{g \in \Omega \mid \text{dist}(g, \partial\Omega) > \epsilon\}$$

and let $\eta_\epsilon : \mathbb{G} \rightarrow \mathbb{R}$ be given by

$$\eta_\epsilon(g) := \frac{1}{\epsilon^Q} \eta(\delta_{1/\epsilon}(g)) \quad \forall g \in \mathbb{G}.$$

Then $\int_{\mathbb{G}} \eta_\epsilon(g) d\mathcal{H}^Q(g) = 1$, and the support of η is contained in $B(e, \epsilon)$. Given $f \in L^1_{loc}(\Omega)$, we define $f_\epsilon : \Omega_\epsilon \rightarrow \mathbb{R}$ by the formula

$$f_\epsilon(g) := \eta_\epsilon \star f(g) := \int_{B(e, \epsilon)} \eta_\epsilon(h) f(h^{-1}g) d\mathcal{H}^Q(h) = \int_{r_g(B(e, \epsilon))} \eta_\epsilon(gh^{-1}) f(h) d\mathcal{H}^Q(h).$$

Some important properties of the mollification are collected in the following

LEMMA 1.6. *Let $f \in L^1_{loc}(\Omega)$. Then*

- (i) $f_\epsilon \in C^\infty(\Omega_\epsilon)$,
- (ii) $f_\epsilon \rightarrow f$ locally uniformly in Ω as $\epsilon \downarrow 0$ provided f is continuous and
- (iii) $Xf_\epsilon = (Xf)_\epsilon$ in Ω_ϵ whenever the weak derivative of f in the direction $X \in V_1$ exists.

PROOF. We leave the proof of (i) and (ii) to the reader. Let us prove (iii). Let $\psi \in C_c^\infty(\Omega_\epsilon)$ arbitrary. By the theorem of Fubini and the left invariance of X and \mathcal{H}^Q , we obtain

$$\begin{aligned}
\int_{\Omega_\epsilon} \psi(g)(Xf)_\epsilon(g) d\mathcal{H}^Q(g) &= \int_{\Omega_\epsilon} \psi(g) \int_{B(e,\epsilon)} \eta_\epsilon(h)(Xf)(h^{-1}g) d\mathcal{H}^Q(h) d\mathcal{H}^Q(g) \\
&= \int_{B(e,\epsilon)} \eta_\epsilon(h) \int_{\Omega_\epsilon} \psi(g)(Xf)(h^{-1}g) d\mathcal{H}^Q(g) d\mathcal{H}^Q(h) \\
&= \int_{B(e,\epsilon)} \eta_\epsilon(h) \int_{l_{h^{-1}}(\Omega_\epsilon)} \psi(hg)(Xf)(g) d\mathcal{H}^Q(g) d\mathcal{H}^Q(h) \\
&= - \int_{B(e,\epsilon)} \eta_\epsilon(h) \int_{l_{h^{-1}}(\Omega_\epsilon)} X\psi(hg)f(g) d\mathcal{H}^Q(g) d\mathcal{H}^Q(h) \\
&= - \int_{B(e,\epsilon)} \eta_\epsilon(h) \int_{\Omega_\epsilon} X\psi(g)f(h^{-1}g) d\mathcal{H}^Q(g) d\mathcal{H}^Q(h) \\
&= - \int_{\Omega_\epsilon} X\psi(g) \int_{B(e,\epsilon)} \eta_\epsilon(h)f(h^{-1}g) d\mathcal{H}^Q(h) d\mathcal{H}^Q(g) \\
&= - \int_{\Omega_\epsilon} X\psi(g)f_\epsilon(g) d\mathcal{H}^Q(g) \\
&= \int_{\Omega_\epsilon} \psi(g)Xf_\epsilon(g) d\mathcal{H}^Q(g).
\end{aligned}$$

The claim follows. \square

Lemma 1.7 shows that the pointwise derivative of a locally Lipschitz function in a horizontal direction exists almost everywhere and yields the weak horizontal derivative of the function in that direction.

LEMMA 1.7. *Let $f : \Omega \rightarrow \mathbb{R}$ be a locally Lipschitz function and let $X \in V_1$. Then*

$$Xf(g) = \lim_{t \downarrow 0} \frac{f(g \exp(tX)) - f(g)}{t}$$

exists for almost every $g \in \Omega$ and $Xf \in L_{loc}^\infty(\Omega)$ is the weak derivative of f in the direction X .

PROOF. We identify \mathbb{G} with $(\mathbb{R}^d, *)$ in the usual manner. It is not restrictive to assume that the sub-Riemannian length of X is equal to one. Let $\Phi_X : \mathbb{R}^{d-1} \times \mathbb{R} \rightarrow \mathbb{R}^d$ be a measure-preserving diffeomorphism, as in Proposition 1.3. We can cover Ω with countably many sets of the form $\Phi_X((y_0, t_0) + (-r, r)^{d-1} \times (-r, r)) \Subset \Omega$, where $(y_0, t_0) \in \mathbb{R}^{d-1} \times \mathbb{R}$ and $r > 0$. Fix $y \in (-r, r)^{d-1}$. Then the function $\gamma : (-r, r) \rightarrow \mathbb{R}$ given by

$$\gamma(t) = f(\Phi_X(y_0 + y, t_0 + t)) = f(\Phi_X(y_0 + y, t_0) \exp(tX))$$

is Lipschitz continuous, hence differentiable almost everywhere by Rademacher's theorem. It follows that

$$Xf(g) = \lim_{t \downarrow 0} \frac{f(g \exp(tX)) - f(g)}{t}$$

exists for almost every $g \in \Phi_X((y_0, t_0) + (-r, r)^{d-1} \times (-r, r))$.

It is clear that Xf belongs to $L_{loc}^\infty(\Omega)$. Hence all that is left to show is that Xf is the weak derivative of f in the direction X . Let $\psi \in C_c^\infty(\Omega)$ and $\Omega' \Subset \Omega'' \Subset \Omega$ such that the support of ψ is contained in Ω' . Using dominated convergence and the right invariance of

\mathcal{H}^Q , we obtain

$$\begin{aligned}
-\int_{\Omega} f(g)X\psi(g) d\mathcal{H}^Q(g) &= \int_{\Omega'} f(g) \left(\lim_{t \downarrow 0} \frac{\psi(g \exp(t(-X))) - \psi(g)}{t} \right) d\mathcal{H}^Q(g) \\
&= \lim_{t \downarrow 0} \int_{\Omega'} f(g) \left(\frac{\psi(g \exp(t(-X))) - \psi(g)}{t} \right) d\mathcal{H}^Q(g) \\
&= \lim_{t \downarrow 0} \int_{r_{\exp(t(-X))}(\Omega')} \left(\frac{f(g \exp(tX)) - f(g)}{t} \right) \psi(g) d\mathcal{H}^Q(g) \\
&= \int_{\Omega''} \left(\lim_{t \downarrow 0} \frac{f(g \exp(tX)) - f(g)}{t} \right) \psi(g) d\mathcal{H}^Q(g) \\
&= \int_{\Omega} Xf(g)\psi(g) d\mathcal{H}^Q(g).
\end{aligned}$$

□

REMARK 1.5. Let $f : \Omega \rightarrow \mathbb{R}$ be a locally Lipschitz function. By the Rademacher type differentiability theorem of Pansu (cf. [79]), for almost every $g_0 \in \Omega$, there exists a homogeneous group homomorphism $D_{g_0}f : \mathbb{G} \rightarrow \mathbb{R}$ such that the quotient

$$\frac{f(g_0\delta_\lambda(g)) - f(g_0)}{\lambda}$$

converges locally uniformly in g to $D_{g_0}f(g)$ as $\lambda \downarrow 0$. If $D_{g_0}f$ exists at some $g_0 \in \Omega$, then the limits

$$X_i f(g_0) = \lim_{t \downarrow 0} \frac{f(g_0 \exp(tX_i)) - f(g_0)}{t}$$

exist for $1 \leq i \leq d_1$. Moreover, it is not difficult to see that

$$D_{g_0}f(\exp(X)) = \begin{cases} 0 & X \in \oplus_{i=2}^s V_i \\ \sum_{i=1}^{d_1} \alpha_i X_i f(g_0) & X = \sum_{i=1}^{d_1} \alpha_i X_i \in V_1. \end{cases}$$

Hence there exists a polynomial $P_{g_0} : \mathbb{G} \rightarrow \mathbb{R}$ of homogeneous degree at most one (compare the first section of the last chapter for a definition) such that

$$\lim_{g \rightarrow g_0} \frac{f(g) - P_{g_0}(g)}{\rho(g_0, g)} = 0.$$

The Euclidean notion of divergence of a vector field admits the following natural generalization to the Carnot group setting:

DEFINITION 1.8. If $\varphi : \Omega \rightarrow \mathbb{H}\mathbb{G}$ is any C^1 smooth section of the horizontal bundle, the *horizontal divergence* of φ is

$$\operatorname{div}_{\mathbb{H}}(\varphi) := \sum_{i=1}^{d_1} X_i \langle \varphi, X_i \rangle.$$

Clearly, all orthonormal bases of V_1 yield the same horizontal divergence.

The left invariance and homogeneity of the horizontal divergence is given by the following

LEMMA 1.8. *Given a C^1 smooth section $\varphi : \Omega \rightarrow \mathbb{H}\mathbb{G}$ of the horizontal bundle, write $\varphi_i := \langle \varphi, X_i \rangle$ for $1 \leq i \leq d_1$. Given $g_0 \in \mathbb{G}$ and $\lambda > 0$, define*

$$\varphi \circ l_{g_0} := \sum_{i=1}^{d_1} (\varphi_i \circ l_{g_0}) X_i \quad \text{and} \quad \varphi \circ \delta_\lambda := \sum_{i=1}^{d_1} (\varphi_i \circ \delta_\lambda) X_i.$$

Then $\varphi \circ l_{g_0} : l_{g_0}^{-1}(\Omega) \rightarrow \mathbb{H}\mathbb{G}$ and $\varphi \circ \delta_\lambda : \delta_{1/\lambda}(\Omega) \rightarrow \mathbb{H}\mathbb{G}$ are C^1 smooth sections of the horizontal bundle,

$$\operatorname{div}_{\mathbb{H}}(\varphi \circ l_{g_0}) = \operatorname{div}_{\mathbb{H}}(\varphi) \circ l_{g_0}$$

and

$$\operatorname{div}_{\mathbb{H}}(\varphi \circ \delta_\lambda) = \lambda(\operatorname{div}_{\mathbb{H}}(\varphi) \circ \delta_\lambda).$$

PROOF. We have

$$\operatorname{div}_{\mathbb{H}}(\varphi \circ l_{g_0}) = \sum_{i=1}^{d_1} X_i \langle \varphi \circ l_{g_0}, X_i \rangle = \sum_{i=1}^{d_1} X_i(\varphi_i \circ l_{g_0}) = \sum_{i=1}^{d_1} X_i \varphi_i \circ l_{g_0}$$

by left invariance of X_1, \dots, X_{d_1} and

$$\operatorname{div}_{\mathbb{H}}(\varphi \circ \delta_\lambda) = \sum_{i=1}^{d_1} X_i \langle \varphi \circ \delta_\lambda, X_i \rangle = \sum_{i=1}^{d_1} X_i(\varphi_i \circ \delta_\lambda) = \sum_{i=1}^{d_1} \lambda(X_i \varphi_i \circ \delta_\lambda)$$

by homogeneity of X_1, \dots, X_{d_1} (cf. Remark 1.2). □

CHAPTER 2

Geodetic convexity

In a Riemannian manifold (M, g) , one can consider the following generalization of the Euclidean notion of convexity for sets and functions:

DEFINITION 2.1. A set $C \subseteq M$ is *convex* if (the image of) any geodesic connecting arbitrary points $p_1, p_2 \in C$ is contained in C and *strongly convex* if the relative interior of (the image of) any geodesic connecting arbitrary points $p_1, p_2 \in \overline{C}$ is contained in C . A function $u : C \rightarrow \mathbb{R}$ defined on some convex subset $C \subseteq M$ is said to be *convex* if $u \circ \gamma : [a, b] \rightarrow \mathbb{R}$ is convex (in the Euclidean sense) whenever $\gamma : [a, b] \rightarrow C$ is a geodesic parameterized proportionally to arc length.

Here geodesic is understood in the sense of global length minimizer. This definition of convexity is appropriate in the Riemannian setting since there exists a sufficiently large supply of non-trivial convex sets and functions. Indeed, it is well-known that any $p \in M$ admits a strongly convex open neighbourhood $U = U(p)$ (see for instance [27]). Moreover, if (M, g) is complete and simply connected and if the sectional curvature is non-positive in U , then the square of the distance function from p is a convex function in U (cf. e.g. [51]).

In this chapter, we show that a notion of geodetic convexity modelled on the Riemannian definition is inadequate in the sub-Riemannian setting of Carnot groups. In the first section, we introduce the necessary terminology and describe the geodesics of the first Heisenberg group. In the second section, we show that the notion of geodetic convexity gives rise to a trivial class of sets and functions in the first Heisenberg group.

1. Geometry of the first Heisenberg group

In the following,

$$\mathbb{H} = \mathbb{H}_1 = (\mathbb{R}^3, *) = (\{(x, y, t) \mid x, y, t \in \mathbb{R}\}, *)$$

denotes the first Heisenberg group (see also the second section of the first chapter) with the group law

$$(x, y, t) * (x', y', t') = (x + x', y + y', t + t' + 2(x'y - xy')) \quad \forall (x, y, t), (x', y', t') \in \mathbb{R}^3.$$

One can check that the unit element is $0 \in \mathbb{R}^3$ and that the inverse of $p = (x, y, t)$ is $p^{-1} = (-x, -y, -t)$. The t -axis $Z = \{(0, 0, t) \mid t \in \mathbb{R}\}$ is the center of the group.

The differential structure on \mathbb{H} is determined by the left invariant vector fields

$$X = \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial t}, \quad Y = \frac{\partial}{\partial y} - 2x \frac{\partial}{\partial t} \quad \text{and} \quad T = \frac{\partial}{\partial t}.$$

Let $V_1 := \text{span}_{\mathbb{R}}\{X, Y\}$ and $V_2 := \text{span}_{\mathbb{R}}\{T\}$. Then $\mathfrak{h} = V_1 \oplus V_2$ is a stratification of the Lie algebra \mathfrak{h} of left invariant vector fields on \mathbb{H} . The sub-Riemannian distance on \mathbb{H} induced by the inner product on V_1 for which $\{X, Y\}$ is an orthonormal basis of V_1 is denoted ρ . Observe that if $\gamma : [a, b] \rightarrow \mathbb{H}$ is an absolutely continuous curve, then its sub-Riemannian length $L(\gamma)$ is

$$L(\gamma) = \int_a^b (\dot{\gamma}_1^2(t) + \dot{\gamma}_2^2(t))^{\frac{1}{2}} dt.$$

For each $\lambda > 0$, the dilation $\delta_\lambda : \mathbb{H} \rightarrow \mathbb{H}$ induced by the stratification is given by the formula

$$\delta_\lambda(x, y, t) = (\lambda x, \lambda y, \lambda^2 t) \quad \forall (x, y, t) \in \mathbb{R}^3.$$

Geodesics in the metric space (\mathbb{H}, ρ) can be computed explicitly and we refer, for instance, to [40], [59], [9], [73] or [7] for a discussion of the problem. Precisely, geodesics starting from the origin $0 \in \mathbb{H}$ are smooth curves $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ with

$$(2.1) \quad \begin{cases} \gamma_1(t) = \frac{\alpha \sin(\varphi t) + \beta(1 - \cos(\varphi t))}{\varphi} \\ \gamma_2(t) = \frac{\beta \sin(\varphi t) - \alpha(1 - \cos(\varphi t))}{\varphi} \\ \gamma_3(t) = 2 \frac{\varphi t - \sin(\varphi t)}{\varphi^2}. \end{cases}$$

The real parameters α, β, φ specify the geodesic. The condition ensuring arc length parametrization is $\alpha^2 + \beta^2 = 1$ and the curve must be consequently defined on an interval $[0, L]$, where $L = L(\gamma)$ is the sub-Riemannian length of γ . In the case $\varphi = 0$, the formulae (2.1) must be understood in the limit sense. If $\varphi \neq 0$ the curve γ in (2.1) is length minimizing if and only if $L \leq 2\pi/|\varphi|$. For $t > 2\pi/|\varphi|$ the curve γ is not a geodesic anymore.

Geodesics starting from an arbitrary point can be recovered from (2.1) by left translations. Note that isometries of (\mathbb{H}, ρ) and dilations transform geodesics into geodesics.

In the following proposition, we list some known properties of geodesics that can be derived from (2.1) and will be used in the sequel.

PROPOSITION 2.1.

- (i) For any $p \in \mathbb{H} \setminus Z$ there exists a unique geodesic connecting 0 and p .
- (ii) For any $p \in Z$, $p \neq 0$, and for any pair $(\alpha, \beta) \in \mathbb{R}^2$ with $\alpha^2 + \beta^2 = 1$, there exists a unique geodesic γ connecting 0 and p such that $\dot{\gamma}(0) = \alpha \partial_x(0) + \beta \partial_y(0)$. Moreover, the union of the images of the geodesics connecting 0 and p is the boundary of a convex open set which is invariant with respect to the rotations of \mathbb{R}^3 that fix Z .
- (iii) The image of the geodesic connecting the points $p = (x, y, t)$ and $p^* = (-x, -y, t)$ is the line segment $[p, p^*]$.
- (iv) For $\varphi \neq 0$ the projection onto the (x, y) -plane of the geodesic γ in (2.1) is an arc of circle with radius $1/|\varphi|$.
- (v) A geodesic $\gamma : [0, L] \rightarrow \mathbb{H}$ with parameter $\varphi \in \mathbb{R}$ and $0 < L < 2\pi/|\varphi|$ can be uniquely extended to $[0, 2\pi/|\varphi|]$ ($\varphi \neq 0$), respectively $[0, \tilde{L}]$ for any $L \leq \tilde{L} < +\infty$ ($\varphi = 0$).
- (vi) The mapping

$$\Phi : \{(\alpha, \beta, \varphi, t) \mid \alpha^2 + \beta^2 = 1, \varphi \in \mathbb{R}, t \in (0, 2\pi/|\varphi|)\} \rightarrow \mathbb{H} \setminus Z$$

where

$$\Phi(\alpha, \beta, \varphi, t) = (\Phi_1(\alpha, \beta, \varphi, t), \Phi_2(\alpha, \beta, \varphi, t), \Phi_3(\alpha, \beta, \varphi, t))$$

with

$$\begin{cases} \Phi_1(\alpha, \beta, \varphi, t) = \frac{\alpha \sin(\varphi t) + \beta(1 - \cos(\varphi t))}{\varphi} \\ \Phi_2(\alpha, \beta, \varphi, t) = \frac{\beta \sin(\varphi t) - \alpha(1 - \cos(\varphi t))}{\varphi} \\ \Phi_3(\alpha, \beta, \varphi, t) = 2 \frac{\varphi t - \sin(\varphi t)}{\varphi^2} \end{cases}$$

is a homeomorphism.

Let us now state some definitions and preliminary results that will be needed in the proofs of Theorem 2.5 and Theorem 2.7.

DEFINITION 2.2. We say that a set $C \subseteq \mathbb{H}$ is *geodetically convex* if for all $p_0, p_1 \in C$ and all geodesics $\gamma : [0, L] \rightarrow \mathbb{H}$ with $L = \rho(p_0, p_1)$, $\gamma(0) = p_0$ and $\gamma(L) = p_1$ we have $\gamma([0, L]) \subseteq C$. The *geodesic convex envelope* $\mathcal{C}(A)$ of $A \subseteq \mathbb{H}$ is the smallest geodetically convex subset of \mathbb{H} containing A .

A function $u : \mathbb{H} \rightarrow \mathbb{R}$ is said to be *geodetically convex* if for any $p_0, p_1 \in \mathbb{H}$ and any geodesic $\gamma : [0, \rho(p_0, p_1)] \rightarrow \mathbb{H} \equiv (\mathbb{R}^3, *)$ parameterized by arc length connecting p_0 and p_1 , the function $t \mapsto u(\gamma(t))$ is convex in the usual sense.

DEFINITION 2.3. For $p_0, p_1 \in \mathbb{H}$, $\Gamma(p_0, p_1)$ denotes the set of images of geodesics connecting p_0 and p_1 . Given $A \subseteq \mathbb{H}$ we define $\mathcal{G}(A) := \bigcup_{p_0, p_1 \in A} \Gamma(p_0, p_1)$, $\mathcal{G}^0(A) := A$ and $\mathcal{G}^{n+1}(A) := \mathcal{G}(\mathcal{G}^n(A))$ for all $n \in \mathbb{N}_0$.

LEMMA 2.2. For $A \subseteq \mathbb{H}$ we have $\mathcal{C}(A) = \bigcup_{n \in \mathbb{N}_0} \mathcal{G}^n(A)$.

PROOF. Using the fact that $\mathcal{G}^n(A) \subseteq \mathcal{G}^{n+1}(A)$ for all $n \in \mathbb{N}_0$, one easily checks that $\bigcup_{n \in \mathbb{N}_0} \mathcal{G}^n(A)$ is geodetically convex and contains A . This gives $\mathcal{C}(A) \subseteq \bigcup_{n \in \mathbb{N}_0} \mathcal{G}^n(A)$. On the other hand $A \subseteq \mathcal{C}(A)$, and if $\mathcal{G}^n(A) \subseteq \mathcal{C}(A)$ for some $n \in \mathbb{N}_0$, then $\mathcal{G}^{n+1}(A) \subseteq \mathcal{C}(A)$. \square

In the following, \mathcal{R} denotes the set of rotations of \mathbb{R}^3 that fix the center Z . Precisely,

$$(2.2) \quad \mathcal{R} = \left\{ R_\theta = \begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid \theta \in [0, 2\pi) \right\}.$$

The rotations R_θ , as well as the mapping

$$(2.3) \quad G : \mathbb{H} \rightarrow \mathbb{H}, \quad G(x, y, t) = (-x, y, -t) \quad \forall (x, y, t) \in \mathbb{H},$$

are isometries of (\mathbb{H}, ρ) .

LEMMA 2.3. Let $A \subseteq \mathbb{H}$, $p \in \mathbb{H}$, $\lambda > 0$ and $R \in \mathcal{R}$. Then $\mathcal{C}(l_p(A)) = l_p(\mathcal{C}(A))$, $\mathcal{C}(\delta_\lambda(A)) = \delta_\lambda(\mathcal{C}(A))$, $\mathcal{C}(R(A)) = R(\mathcal{C}(A))$ and $\mathcal{C}(G(A)) = G(\mathcal{C}(A))$.

PROOF. We prove the statement for R . Since isometries of (\mathbb{H}, ρ) map geodesics to geodesics, it follows easily by induction that

$$R(\mathcal{G}^n(A)) = \mathcal{G}^n(R(A)) \quad \forall n \in \mathbb{N}_0.$$

Moreover, by Lemma 2.2

$$R(\mathcal{C}(A)) = R\left(\bigcup_{n \in \mathbb{N}_0} \mathcal{G}^n(A)\right) = \bigcup_{n \in \mathbb{N}_0} R(\mathcal{G}^n(A)) = \bigcup_{n \in \mathbb{N}_0} \mathcal{G}^n(R(A)) = \mathcal{C}(R(A)).$$

\square

LEMMA 2.4. We have $\{(0, 0, t) \mid -1 \leq t \leq 1\} \subseteq \mathcal{G}^2(\{(0, 0, -1), (0, 0, 1)\})$.

PROOF. For $-1 < \tau < 1$, consider the plane

$$\Pi(\tau) := \{(x, y, t) \in \mathbb{H} \mid t = \tau\}.$$

By Proposition 2.1 (ii), the intersection of $\mathcal{G}^1(\{(0, 0, -1), (0, 0, 1)\})$ with $\Pi(\tau)$ is a circle of radius $r(\tau) > 0$ centered at $(0, 0, \tau)$. Pick any point p on the circle and denote by p^* the reflection of p with respect to $(0, 0, \tau)$ in $\Pi(\tau)$. From Proposition 2.1 (iii), it follows that $(0, 0, \tau) \in \mathcal{G}^2(\{(0, 0, -1), (0, 0, 1)\})$. \square

2. Non-existence of non-trivial geodetically convex sets and functions

THEOREM 2.5. *Let $A = \{(x, y, t_1), (x, y, t_2)\} \subseteq \mathbb{H}$, $t_1 \neq t_2$. Then $\mathcal{C}(A) = \mathbb{H}$.*

PROOF. By Lemma 2.3, it is enough to show that

$$\mathcal{C}(A) = \mathbb{H} \quad \text{for} \quad A = \{(0, 0, -1), (0, 0, 1)\}.$$

The proof is divided in three steps. First, we show that rotations that fix Z and reflections with respect to the (x, y) -plane map $\mathcal{C}(A)$ onto itself. Second, we prove that the function $h : [0, +\infty) \rightarrow [0, +\infty]$ defined by

$$h(r) := \sup\{t \geq 0 \mid (r, 0, t) \in \mathcal{C}(A)\}$$

is non-increasing and that

$$(2.4) \quad \{(x, y, t) \in \mathbb{R}^3 \mid 0 \leq |t| < h(r)\} \subseteq \mathcal{C}(A)$$

if $r \in [0, +\infty)$ and $h(r) > 0$. The last step consists in showing that h is nowhere finite.

1. Let $\theta \in [0, 2\pi)$ and denote by $R_\theta \in \mathcal{R}$ the rotation around Z with angle θ , as in (2.2). By Lemma 2.3, we have

$$R_\theta(\mathcal{C}(A)) = \mathcal{C}(R_\theta(A)) = \mathcal{C}(A).$$

We denote by $S : \mathbb{H} \rightarrow \mathbb{H}$, $S(x, y, t) = (x, y, -t)$, the reflection with respect to the (x, y) -plane. We claim that $S(\mathcal{C}(A)) = \mathcal{C}(A)$. The rotational symmetry of $\mathcal{C}(A)$ implies that $S(\mathcal{C}(A)) = G(\mathcal{C}(A))$, where $G(x, y, t) = (-x, y, -t)$. By Lemma 2.2,

$$S(\mathcal{C}(A)) = G(\mathcal{C}(A)) = \mathcal{C}(G(A)) = \mathcal{C}(A).$$

2. We prove (2.4) first. By definition of h , it suffices to show that for any pair of points $(x, y, -t), (x, y, t) \in \mathcal{C}(A)$ we have

$$\{(x, y, t') \mid 0 \leq |t'| \leq |t|\} \subseteq \mathcal{C}(A).$$

Notice that

$$\mathcal{C}(\{(x, y, -t), (x, y, t)\}) \subseteq \mathcal{C}(A)$$

and

$$\mathcal{C}(\{(x, y, -t), (x, y, t)\}) = \mathcal{C}(l_p \circ \delta_r(\{(0, 0, -1), (0, 0, 1)\}))$$

with $p = (x, y, 0)$ and $r = \sqrt{|t|}$. By Lemma 2.3 and Lemma 2.4, we obtain

$$\begin{aligned} \{(x, y, t') \mid 0 \leq |t'| \leq |t|\} &= l_p \circ \delta_r(\{(0, 0, t') \mid 0 \leq |t'| \leq 1\}) \\ &\subseteq l_p \circ \delta_r(\mathcal{C}(\{(0, 0, -1), (0, 0, 1)\})) \\ &= \mathcal{C}(l_p \circ \delta_r(\{(0, 0, -1), (0, 0, 1)\})) \\ &= \mathcal{C}(\{(x, y, -t), (x, y, t)\}). \end{aligned}$$

Hence (2.4) follows.

Now we prove that h is non-increasing. Otherwise we can find $0 \leq r_1 < r_2 < +\infty$ such that $0 \leq h(r_1) < h(r_2)$. Pick $h(r_1) < t < h(r_2)$. Then, by Proposition 2.1 (iii), the curve

$$\gamma : [0, 2r_2] \rightarrow \mathbb{H}, \quad \gamma(s) = (-r_2, 0, t) + s(1, 0, 0)$$

is a geodesic such that $\gamma(0) = (-r_2, 0, t) \in \mathcal{C}(A)$ and $\gamma(2r_2) = (r_2, 0, t) \in \mathcal{C}(A)$. This contradicts $h(r_1) < t$.

3. Our goal now is to show that h is nowhere finite, which concludes the proof of Theorem 2.5. Assume by contradiction that there exists $0 < r < +\infty$ with $0 \leq h(r) < +\infty$. Without loss of generality we can also assume $h(r) > 0$. Indeed, if h was nowhere positive and finite, then

$$0 < r_0 := \inf\{r \geq 0 \mid h(r) = 0\} < +\infty,$$

whence $Z \subseteq \mathcal{C}(A)$. Since $\mathcal{G}^2(\{(0, 0, -1), (0, 0, 1)\})$ contains an open neighbourhood of 0 and since

$$\delta_{\sqrt{t}}(\mathcal{G}^2(\{(0, 0, -1), (0, 0, 1)\})) = \mathcal{G}^2(\{(0, 0, -t), (0, 0, t)\}) \subseteq \mathcal{C}(A) \quad \forall t > 0,$$

$h(r) = 0$ for $r > r_0$ is impossible, a contradiction.

Let $p^- = (r, 0, -h(r))$, $p^+ = (r, 0, h(r))$. By Proposition 2.1 (ii) (modulo left translation), there exists a geodesic $\gamma : [0, \rho(p^-, p^+)] \rightarrow \mathbb{H}$ connecting p^- and p^+ with

$$\dot{\gamma}(0) = \frac{1}{\sqrt{2}}(X(p^-) + Y(p^-)) = \frac{1}{\sqrt{2}}(\partial_x(0) + (\partial_y(0) - 2r\partial_t(0))) = \frac{1}{\sqrt{2}}(1, 1, -2r).$$

Hence, for some $s > 0$ we have $\gamma(s) < -h(r)$ and $\gamma_1^2(s) + \gamma_2^2(s) > r^2$. But then the same is true for the geodesic $\tilde{\gamma} : [0, \rho((r, 0, -t), (r, 0, t))] \rightarrow \mathbb{H}$ defined by

$$\tilde{\gamma} = l_p \circ \delta_\lambda \circ l_{p^{-1}} \circ \gamma, \quad \lambda = \sqrt{t/h(r)}, \quad p = (r, 0, 0)$$

which connects $(r, 0, -t)$ and $(r, 0, t)$, provided that $t \in (0, h(r))$ is sufficiently close to $h(r)$. This contradicts the fact that h is non-increasing. \square

We need the following lemma in our proof of Theorem 2.7 below:

LEMMA 2.6. *Let $A \subseteq \mathbb{H}$. Suppose there exist $q \in \mathbb{R}^2$, a neighbourhood U of q in \mathbb{R}^2 and a continuous function $f : U \rightarrow \mathbb{R}$ such that*

$$\{(x, y, t) \in A \mid (x, y) \in U\} = \{(x, y, f(x, y)) \mid (x, y) \in U\}.$$

Then A is not geodetically convex.

PROOF. Without loss of generality we can assume that $q = (0, 0)$. Suppose by contradiction that A is geodetically convex. Note that, by Proposition 2.1 (i) (modulo left translation), for a given pair of points in A there is a unique geodesic connecting them.

1. Choose $r > 0$ such that $B := \{(x, y) \mid x^2 + y^2 < r^2\} \Subset U$ and define $g : \partial B \rightarrow \mathbb{R}$ by

$$g(x, y) := f(x, y) - f(-x, -y).$$

Since $g(x, y) = -g(-x, -y)$, the continuity of g implies $g(x, y) = 0$ for some $(x, y) \in \partial B$, i.e. $f(x, y) = f(-x, -y)$ for a pair of points $(x, y), (-x, -y)$ in ∂B . Since A is geodetically convex, $\{(sx, sy, f(x, y)) \mid s \in [-1, 1]\} \subseteq A$. Hence $f(sx, sy) = f(x, y)$ for all $s \in [-1, 1]$. By Lemma 2.3, we may assume that $x > 0$, $y = 0$ and $f(0) = 0$.

2. Let $v \in S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\} \subseteq \mathbb{R}^2$. Then the map $s \mapsto f(sv)$ from $[-r, r]$ to \mathbb{R} is monotonic. Otherwise, by continuity of f , we could find $s_1 < s_2$ in $[-r, r]$ such that $f(s_1v) = f(s_2v)$, but either $f(s_1v) > \min\{f(sv) \mid s \in [s_1, s_2]\}$ or $f(s_1v) < \max\{f(sv) \mid s \in [s_1, s_2]\}$, $i = 1, 2$, which is not possible because the image of the geodesic connecting $(s_1v, f(s_1v))$ and $(s_2v, f(s_2v))$ is a line segment contained in A .

3. The continuity of f on U , the continuity of ρ with respect to the Euclidean distance on $\mathbb{H} \cong \mathbb{R}^3$ and the fact that the length of a geodesic is actually the Euclidean length of its projection onto the (x, y) -plane imply that the projection of the image of a geodesic connecting $(x_1, y_1, f(x_1, y_1))$ and $(x_2, y_2, f(x_2, y_2))$ is contained in an arbitrary small neighbourhood of $\{(x_1, y_1), (x_2, y_2)\}$ provided (x_1, y_1) and (x_2, y_2) are chosen sufficiently close to each other.

4. For $0 < \epsilon < \pi/2$, let

$$\begin{aligned} p_0 &:= (r/2(\cos(-\epsilon), \sin(-\epsilon)), f(r/2(\cos(-\epsilon), \sin(-\epsilon))))), \\ p_1 &:= (r/2(\cos(\epsilon), \sin(\epsilon)), f(r/2(\cos(\epsilon), \sin(\epsilon))))), \\ q_0 &:= (r/2(\cos(\pi - \epsilon), \sin(\pi - \epsilon)), f(r/2(\cos(\pi - \epsilon), \sin(\pi - \epsilon))))), \\ q_1 &:= (r/2(\cos(\pi + \epsilon), \sin(\pi + \epsilon)), f(r/2(\cos(\pi + \epsilon), \sin(\pi + \epsilon))))). \end{aligned}$$

Let $\gamma^p : [0, \rho(p_0, p_1)] \rightarrow \mathbb{H}$ and $\gamma^q : [0, \rho(q_0, q_1)] \rightarrow \mathbb{H}$ be the geodesics satisfying $\gamma^p(0) = p_0$, $\gamma^p(\rho(p_0, p_1)) = p_1$, $\gamma^q(0) = q_0$ and $\gamma^q(\rho(q_0, q_1)) = q_1$. If $\epsilon > 0$ is chosen small enough, then (cf. **3.**)

$$\begin{aligned} P \circ \gamma^p([0, \rho(p_0, p_1)]) &\subseteq (\{(x, y) \in B \mid 0 < x < r\}), \\ P \circ \gamma^q([0, \rho(q_0, q_1)]) &\subseteq (\{(x, y) \in B \mid -r < x < 0\}), \end{aligned}$$

where P denotes the orthogonal projection

$$P : \mathbb{R}^3 \rightarrow \{(x, y, t) \in \mathbb{R}^3 \mid t = 0\}.$$

5. The vectors ∂_x and $\partial_y - 2x\partial_t$ span the horizontal plane at $(x, 0, 0)$. Now notice that there exists $s_p \in (0, \rho(p_0, p_1))$ with the properties

$$\gamma_2^p(s_p) = \gamma_3^p(s_p) = 0, \quad \dot{\gamma}_1^p(s_p) > 0 \quad \text{and} \quad \dot{\gamma}_2^p(s_p) > 0.$$

Indeed, γ^p must cross the (x, t) -plane, $\gamma^p([0, \rho(p_0, p_1)]) \subseteq A$, $f \equiv 0$ on $\{(x, 0) \mid |x| \leq r\}$ (cf. **1.**) and $P \circ \gamma^p([0, \rho(p_0, p_1)])$ is a line segment or an arc of circle by Proposition 2.1 (iii) and (iv). It follows that

$$\dot{\gamma}_3^p(s_p) = 2\dot{\gamma}_1^p(s_p)\dot{\gamma}_2^p(s_p) - 2\dot{\gamma}_2^p(s_p)\dot{\gamma}_1^p(s_p) = -2\dot{\gamma}_2^p(s_p)\dot{\gamma}_1^p(s_p) < 0.$$

Similarly, there exists $s_q \in (0, \rho(q_0, q_1))$ with the properties

$$\gamma_2^q(s_q) = \gamma_3^q(s_q) = 0, \quad \dot{\gamma}_1^q(s_q) < 0, \quad \dot{\gamma}_2^q(s_q) < 0 \quad \text{and} \quad \dot{\gamma}_3^q(s_q) < 0.$$

In particular, since $\gamma^p([0, \rho(p_0, p_1)])$, $\gamma^q([0, \rho(q_0, q_1)]) \subseteq A$ and f is continuous, we can find $0 < \delta < \epsilon$ and $0 < \tilde{r}_1, \tilde{r}_2 < r$ such that

$$f(\tilde{r}_1(\cos(\delta), \sin(\delta))) < 0 \quad \text{and} \quad f(\tilde{r}_2(\cos(\pi + \delta), \sin(\pi + \delta))) < 0.$$

Now, since $f(0) = 0$ and f is continuous on $\{\tilde{r}(\cos(\delta), \sin(\delta)) \mid -r \leq \tilde{r} \leq r\}$, there exist $0 < r_1, r_2 < r$ such that

$$t = f(r_1(\cos(\delta), \sin(\delta))) = f(r_2(\cos(\pi + \delta), \sin(\pi + \delta))) < 0.$$

Since the image of the geodesic connecting

$$(r_1(\cos(\delta), \sin(\delta)), t) \quad \text{and} \quad (r_2(\cos(\pi + \delta), \sin(\pi + \delta)), t)$$

is a line segment contained in A , we get $(0, 0, t) \in A$, a contradiction. \square

THEOREM 2.7. *Let $p_1, p_2, p_3 \in \mathbb{H}$ be three points not lying on the same geodesic. Then $\mathcal{C}(\{p_1, p_2, p_3\}) = \mathbb{H}$.*

PROOF. Without loss of generality, we can assume $p_1 = 0$.

1. We claim that there exist two points $q_1, q_2 \in \mathcal{C}(\{p_1, p_2, p_3\})$ such that $q_1 \neq q_2$ and $P(q_1) = P(q_2)$. Assume by contradiction that no such pair of points exists. Then there is always a unique geodesic connecting any given two points in $\mathcal{C}(\{p_1, p_2, p_3\})$.

2. Consider the geodesic $\kappa : [0, \rho(p_2, p_3)] \rightarrow \mathbb{H}$ with $\kappa(0) = p_2$ and $\kappa(\rho(p_2, p_3)) = p_3$. For $\sigma \in [0, \rho(p_2, p_3)]$, let $\gamma_\sigma : [0, \rho(p_1, \kappa(\sigma))] \rightarrow \mathbb{H}$ be the unique geodesic such that $\gamma_\sigma(0) = p_1$ and $\gamma_\sigma(\rho(p_1, \kappa(\sigma))) = \kappa(\sigma)$. We show that if $\sigma < \tau$, then $\gamma_\sigma \cap \gamma_\tau = \{p_1\}$. If the intersection is larger, let $t_1 := \max\{t \in [0, \rho(p_1, \kappa(\sigma))] \mid \gamma_\sigma(t) \in \gamma_\tau\}$ and let t_2 be the unique element in $[0, \rho(p_1, \kappa(\tau))]$ with $\gamma_\tau(t_2) = \gamma_\sigma(t_1)$. By uniqueness of geodesics, $t_1 = t_2$ and $\gamma_\sigma|_{[0, t_1]} = \gamma_\tau|_{[0, t_2]}$. It then follows from Proposition 2.1 (v) that either $t_1 = \rho(p_1, \kappa(\sigma))$ or $t_2 = \rho(p_1, \kappa(\tau))$ and hence $\gamma_\sigma \subseteq \gamma_\tau$ or $\gamma_\tau \subseteq \gamma_\sigma$. Consider for instance the case $\gamma_\sigma \subseteq \gamma_\tau$. Clearly, $\kappa([0, \tau]) \subseteq \gamma_\tau$. By Proposition 2.1 (v), it follows easily that $\kappa(0) = p_2 \in \gamma_\tau$ and so $\gamma_0 \cup \kappa|_{[0, \kappa(\tau)]} \subseteq \gamma_\tau$. The maximal extension $\tilde{\gamma}_\tau$ of γ_τ must contain $\gamma_0 \cup \kappa$. Consequently, $p_1, p_2, p_3 \in \tilde{\gamma}_\tau$, contradicting our assumption.

3. Consider the open set

$$U = \{(\sigma, s) \in \mathbb{R}^2 \mid \sigma \in (0, \rho(p_2, p_3)), s \in (0, \rho(p_1, \kappa(\sigma)))\},$$

and the mapping $F : U \rightarrow \mathbb{R}^2$ given by

$$F(\sigma, s) := P(\gamma_\sigma(s)).$$

By **2.**, F is injective. Moreover, by Proposition 2.1 (vi), $(\sigma, s) \rightarrow \gamma_\sigma(s)$ is continuous, because the endpoint $\kappa(\sigma)$ varies continuously. By the theorem on the invariance of domains –see for instance Proposition 7.4 in the fourth chapter of [28]– the mapping F is open. In particular the set $V := F(U)$ is open and the inverse mapping $F^{-1} : V \rightarrow U$ is continuous. But then so is the function $f : V \rightarrow \mathbb{R}$ defined by

$$f(x, y) := g(F^{-1}(x, y)),$$

where $g : U \rightarrow \mathbb{R}$ is the third component of $(\sigma, s) \mapsto \gamma_\sigma(s)$. We have

$$\{(x, y, t) \in \mathcal{C}(\{p_1, p_2, p_3\}) \mid (x, y) \in V\} = \{(x, y, f(x, y)) \mid (x, y) \in V\},$$

and by Lemma 2.6 the set $\mathcal{C}(\{p_1, p_2, p_3\})$ cannot be geodetically convex. This contradiction concludes the proof. \square

Thus, the only geodetically convex subsets of \mathbb{H} are the empty set, points, segments of geodesics and the whole group. The lack of geodetically convex sets has its counterpart in the lack of geodetically convex functions on \mathbb{H} :

COROLLARY 2.8. *If $u : \mathbb{H} \rightarrow \mathbb{R}$ is geodetically convex, then u is constant.*

PROOF. First we show that u must be constant on the vertical axis Z . Assume by contradiction this is not true.

Case 1: There exist three distinct points $(0, 0, t_1), (0, 0, t_2), (0, 0, t_3) \in Z$ such that $u(0, 0, t_1) \leq u(0, 0, t_2) < u(0, 0, t_3)$. The set $C := \{p \in \mathbb{H} \mid u(p) < u(0, 0, t_3)\}$ is geodetically convex, because u is convex on geodesics. Moreover $(0, 0, t_1), (0, 0, t_2) \in C$ and by Theorem 2.5, it follows that $C = \mathbb{H}$, contradicting $(0, 0, t_3) \notin C$.

Case 2: u assumes exactly two values on the vertical axis (say 0 and 1), $u(0, 0, t) = 0$ for some $t \in \mathbb{R}$ and $u(p) = 1$ for any $p \neq (0, 0, t)$ on the vertical axis (otherwise we are in Case 1). Consider two distinct geodesics γ and κ connecting $(0, 0, t)$ and $(0, 0, -t)$ (we can assume $t \neq 0$). We have $\gamma \cap \kappa = \{(0, 0, -t), (0, 0, t)\}$. By convexity of u on γ and κ , we can find $p \in \gamma \setminus (\gamma \cap \kappa)$ and $q \in \kappa \setminus (\gamma \cap \kappa)$ with $u(p), u(q) < 1$. The set $C := \{p' \in \mathbb{H} \mid u(p') < 1\}$ is geodetically convex and contains $(0, 0, t), p$ and q . Since these points do not lie on the same geodesic, Theorem 2.7 gives $C = \mathbb{H}$ which contradicts $(0, 0, t') \notin C$ when $t' \neq t$.

By left translation, the previous argument shows that u must be constant on any vertical line. Suppose now we could find two vertical lines v_1 and v_2 and $c_1 < c_2$, such that $c_i, i = 1, 2$, is the value of u restricted to v_i . But then, if we choose two points on v_1 sufficiently far apart, the union of images of geodesics connecting these two points will intersect v_2 , which is impossible since $u \leq c_1$ on this union by geodesic convexity. \square

CHAPTER 3

Horizontal convexity and horizontal convexity in the viscosity sense

In view of the results of the previous chapter, more appropriate notions of convexity in Carnot groups must be found. In the first section of this chapter, we introduce two closely related notions of convexity, horizontal convexity –h-convexity– and horizontal convexity in the viscosity sense –v-convexity–, and we record some basic properties of convex sets and functions. We provide examples in the second section. Finally, in the third section, we show that h-convexity and v-convexity define roughly the same classes of functions.

1. Definitions

The following definition of convexity, due to Caffarelli, was rediscovered by Danielli, Garofalo and Nhieu in [23].

DEFINITION 3.1. Let \mathbb{G} be a stratified group. A subset $C \subseteq \mathbb{G}$ is said to be *h-convex* if $\gamma([a, b]) \subseteq C$ whenever $\gamma : [a, b] \rightarrow \mathbb{G}$ is an integral curve of some left invariant, horizontal vector field and $\gamma(a), \gamma(b) \in C$. If $A \subseteq \mathbb{G}$ is any subset, the *h-convex closure* $\mathcal{C}(A)$ of A is the smallest h-convex set which contains A . A function $u : C \rightarrow \mathbb{R}$ defined on some h-convex subset $C \subseteq \mathbb{G}$ is said to be *h-convex* if $u \circ \gamma : [a, b] \rightarrow \mathbb{R}$ is convex (in the Euclidean sense) whenever $\gamma : [a, b] \rightarrow C$ is a segment of an integral curve of some left invariant, horizontal vector field.

REMARK 3.1. Notice that the above definitions depend on the stratification of the Lie algebra \mathfrak{g} of \mathbb{G} . However, if $\oplus_{i=1}^s V_i$ and $\oplus_{i=1}^s \tilde{V}_i$ are two stratifications of \mathfrak{g} , then we can find a Lie algebra automorphism $A : \mathfrak{g} \rightarrow \mathfrak{g}$ which maps each V_i onto \tilde{V}_i and a unique Lie group automorphism $a : \mathbb{G} \rightarrow \mathbb{G}$ such that $da = A$ (cf. Remark 1.1). Note that $C \subseteq \mathbb{G}$ is h-convex with respect to $\oplus_{i=1}^s V_i$ if and only if $a(C)$ is h-convex with respect to $\oplus_{i=1}^s \tilde{V}_i$, and $u : C \rightarrow \mathbb{R}$ is h-convex with respect to $\oplus_{i=1}^s V_i$ if and only if $u \circ a^{-1} : a(C) \rightarrow \mathbb{R}$ is h-convex with respect to $\oplus_{i=1}^s \tilde{V}_i$.

We list elementary facts about h-convex sets and h-convex functions in Lemma 3.1 below:

LEMMA 3.1. *Let \mathbb{G} be a Carnot group, $C \subseteq \mathbb{G}$ an h-convex subset, $u, v : C \rightarrow \mathbb{R}$ h-convex functions, $g \in \mathbb{G}$, $\lambda > 0$ and $c \in [0, +\infty)$. Then*

- (i) $l_{g^{-1}}(C)$ is h-convex and $u \circ l_g : l_{g^{-1}}(C) \rightarrow \mathbb{R}$ is h-convex,
- (ii) $\delta_{1/\lambda}(C)$ is h-convex and $u \circ \delta_\lambda : \delta_{1/\lambda}(C) \rightarrow \mathbb{R}$ is h-convex,
- (iii) $cu : C \rightarrow \mathbb{R}$ is h-convex and
- (iv) $u + v : C \rightarrow \mathbb{R}$ is h-convex.

Moreover,

- (v) the pointwise limit of a sequence of h-convex functions is h-convex,
- (vi) the supremum of a sequence of h-convex functions which admits pointwise upper bounds is h-convex,
- (vii) the intersection of any collection of h-convex subsets of \mathbb{G} is h-convex,
- (viii) $l_g(\mathcal{C}(A)) = \mathcal{C}(l_g(A))$ for all $A \subseteq \mathbb{G}$ and $g \in \mathbb{G}$, and

(ix) $\delta_\lambda(\mathcal{C}(A)) = \mathcal{C}(\delta_\lambda(A))$ for all $A \subseteq \mathbb{G}$ and $\lambda > 0$.

PROOF. The verifications are left to the reader. \square

Lemma 3.2 says that smoothing of a function (see §3 in the first chapter) preserves h-convexity.

LEMMA 3.2. *Let $\Omega \subseteq \mathbb{G}$ be an h-convex, open subset, $u \in L^1_{loc}(\Omega)$ h-convex and $\Omega' \Subset \Omega$. Given $\epsilon > 0$ such that $\Omega' \subseteq \Omega_\epsilon$, let $u_\epsilon : \Omega' \rightarrow \mathbb{R}$ denote the regularization of u (restricted to Ω'). Then $u_\epsilon \circ \gamma : (a, b) \rightarrow \mathbb{R}$ is convex whenever $\gamma : (a, b) \rightarrow \Omega'$ is a segment of an integral curve of some left invariant, horizontal vector field. In particular, $u_\epsilon : \Omega' \rightarrow \mathbb{R}$ is h-convex if Ω' is.*

PROOF. Let $t_1, t_2 \in (a, b)$ and $t = \alpha_1 t_1 + \alpha_2 t_2$, where $\alpha_1, \alpha_2 \geq 0$ with $\alpha_1 + \alpha_2 = 1$. Then

$$\begin{aligned} u_\epsilon(\gamma(\alpha_1 t_1 + \alpha_2 t_2)) &= \int_{B(e, \epsilon)} \eta_\epsilon(h) u(h^{-1} \gamma(\alpha_1 t_1 + \alpha_2 t_2)) d\mathcal{H}^Q(h) \\ &\leq \int_{B(e, \epsilon)} \eta_\epsilon(h) (\alpha_1 u(h^{-1} \gamma(t_1)) + \alpha_2 u(h^{-1} \gamma(t_2))) d\mathcal{H}^Q(h) \\ &= \alpha_1 u_\epsilon(\gamma(t_1)) + \alpha_2 u_\epsilon(\gamma(t_2)). \end{aligned}$$

\square

Given an open set $\Omega \subseteq \mathbb{G}$ and a basis (X_1, \dots, X_{d_1}) of V_1 , $C^2_{\mathbb{H}}(\Omega)$ denotes the set of continuous functions $u : \Omega \rightarrow \mathbb{R}$ whose weak horizontal derivatives $X_i u$ ($1 \leq i \leq d_1$), and $X_i X_j u$ ($1 \leq i, j \leq d_1$) exist and have continuous representatives. In the following, given $u \in C^2_{\mathbb{H}}(\Omega)$, $X_i u$ and $X_i X_j u$ always denote the precise representatives.

We now define the symmetrized horizontal Hessian, which plays the same role for h-convex functions in general stratified groups as does the usual Hessian for convex functions in Euclidean spaces.

DEFINITION 3.2. Let (X_1, \dots, X_{d_1}) be an orthonormal basis of V_1 . Let $\Omega \subseteq \mathbb{G}$ be an h-convex, open subset, $u \in C^2_{\mathbb{H}}(\Omega)$ and $g \in \Omega$. The symmetric \mathbb{R} -bilinear form

$$D^2_{\mathbb{H}} u(g) : \mathbb{H}_g \mathbb{G} \times \mathbb{H}_g \mathbb{G} \rightarrow \mathbb{R}$$

uniquely determined by the requirements

$$D^2_{\mathbb{H}} u(g)(X_i(g), X_j(g)) = \frac{X_i X_j u(g) + X_j X_i u(g)}{2} \quad \forall 1 \leq i, j \leq d_1$$

is called the *symmetrized horizontal Hessian* of u at g . (It is easy to check that $D^2_{\mathbb{H}} u(g)$ is independent of the choice of the basis).

PROPOSITION 3.3. *Let Ω be an h-convex, open subset of \mathbb{G} and $u \in C^2_{\mathbb{H}}(\Omega)$. Then u is h-convex in Ω if and only if $D^2_{\mathbb{H}} u$ is positive semidefinite in Ω .*

PROOF. Let $\Omega' \Subset \Omega$ and assume $u \in C^\infty(\Omega')$.

Suppose first that $u \circ \gamma$ is convex whenever $\gamma : (a, b) \rightarrow \Omega'$ is an integral curve of some $X \in V_1$. Let $g \in \Omega'$ and $X \in V_1$. Direct computation gives

$$\left. \frac{d^2}{dt^2} u(g \exp(tX)) \right|_{t=0} = D^2_{\mathbb{H}} u(g)(X(g), X(g)),$$

whence $D^2_{\mathbb{H}} u(g)(X(g), X(g)) \geq 0$. It follows that $D^2_{\mathbb{H}} u$ is positive semidefinite in Ω' .

Suppose conversely that $D^2_{\mathbb{H}} u$ is positive semidefinite in Ω' . Let $g \in \Omega'$, $X \in V_1$ and $(a, b) \subseteq \mathbb{R}$ such that $g \exp(tX) \in \Omega'$ whenever $t_0 \in (a, b)$. Then

$$\left. \frac{d^2}{dt^2} u(g \exp(tX)) \right|_{t=t_0} = D^2_{\mathbb{H}} u(g \exp(t_0 X))(X(g \exp(t_0 X)), X(g \exp(t_0 X))) \geq 0$$

for all $t_0 \in (a, b)$ and $t \mapsto u(g \exp(tX))$ is a convex function on (a, b) .

The full statement follows via regularization of u , using Lemma 1.6 and Lemma 3.2. \square

Recall that a function $u : \Omega \rightarrow \mathbb{R}$ defined on some open subset $\Omega \subseteq \mathbb{G}$ is *upper semicontinuous at $g \in \Omega$* if $\limsup_{h \rightarrow g} u(h) \leq u(g)$. A function φ defined on an open neighbourhood $U \subseteq \Omega$ of g *touches u from above at g* if $\varphi(g) = u(g)$ and $\varphi \geq u$ in U .

The following notion of convexity in Carnot groups, which is motivated by Proposition 3.3, was introduced by Lu, Manfredi and Stroffolini in [64].

DEFINITION 3.3. Let $\Omega \subseteq \mathbb{G}$ be an open subset and $u : \Omega \rightarrow \mathbb{R}$ be upper semicontinuous. u is said to be *horizontally convex in the viscosity sense* –*v-convex* for short– if $D_{\mathbb{H}}^2 \varphi(g)$ is positive semidefinite whenever $g \in \Omega$ and $\varphi \in C_{\mathbb{H}}^2(U)$ touches u from above at g .

2. Examples of h-convex sets and h-convex functions

Basic examples of h-affine –in particular h-convex– functions on a stratified group are given by the following:

LEMMA 3.4. Let \mathbb{G} be a Carnot group, $\oplus_{i=1}^s V_i$ a stratification of its Lie algebra \mathfrak{g} of left invariant vector fields and $\langle \cdot, \cdot \rangle$ an inner product on V_1 . Given $Y \in \mathfrak{g}$, we let $(Y)_1$ denote the V_1 -component of Y . We have:

- (i) Any constant function on \mathbb{G} is h-convex.
- (ii) The function $g \mapsto \langle X, (\exp^{-1}(g))_1 \rangle$ is h-convex whenever $X \in V_1$.

DEFINITION 3.4. Given a Carnot group $\mathbb{G} \equiv (\mathbb{R}^d, *)$, we say that a subset $C \subseteq \mathbb{R}^d$ is *E-convex* if it is an h-convex subset of the abelian Carnot group $(\mathbb{R}^d, +)$. Similarly, a function $f : C \rightarrow \mathbb{R}$ defined on an E-convex subset of \mathbb{R}^d is *E-convex* if it is h-convex in $(\mathbb{R}^d, +)$.

LEMMA 3.5. In a stratified group $\mathbb{G} \equiv (\mathbb{R}^d, *)$ of step at most two, any E-convex set $C \subseteq \mathbb{R}^d$ is h-convex and any E-convex function $u : C \rightarrow \mathbb{R}$ is h-convex.

PROOF. The claim is an immediate consequence of the following fact, which is a straightforward consequence of formula (1.2), formula (1.8) and the condition on the step of the group: If $\gamma : [a, b] \rightarrow (\mathbb{R}^d, *)$ is a segment of an integral curve of some left invariant, horizontal vector field, then γ is a segment of an integral curve of a left invariant vector field on $(\mathbb{R}^d, +)$. \square

The condition on the step of the group in Lemma 3.5 cannot be relaxed. In the Engel group $\mathbb{E} \equiv (\mathbb{R}^4, *)$ for instance (compare the second section of the first chapter for a definition), the function $f : \mathbb{E} \rightarrow \mathbb{R}$ given by $f(x_1, x_2, y, z) := z$ is E-convex but not h-convex, and its sublevel sets are E-convex but not h-convex. Let us however point out that the function $g : \mathbb{E} \rightarrow \mathbb{R}$ given by $g(x_1, x_2, y, z) := (x_1 y)/6 + z$ is h-convex but not E-convex. It is left to the reader to verify these assertions.

Notice that if $\oplus_{i=1}^s V_i$ is a stratification of the Lie algebra \mathfrak{g} of left invariant vector fields on \mathbb{G} and if $W_1 \subseteq V_1$ is a subspace of dimension $d_1 - 1$, then the “hyperplane” $\exp(W_1 \oplus (\oplus_{i=2}^s V_i))$ is a normal subgroup in \mathbb{G} which separates two h-convex, unbounded open sets. On the other hand, the Carnot–Carathéodory ball in the first Heisenberg group $\mathbb{H} \equiv \mathbb{H}_1$ (compare §2 of the first chapter or §1 of the second for a definition) is not h-convex, and since E-convex subsets of a stratified group $\mathbb{G} \equiv (\mathbb{R}^d, *)$ of step strictly larger than two are not necessarily h-convex, it is a priori not clear whether there exist h-convex, bounded open sets with a regular boundary in arbitrary stratified groups. This problem is settled by the following

THEOREM 3.6. *Let \mathbb{G} be a Carnot group. There exists a countable basis of the topology consisting of h -convex, bounded open sets with smooth boundary.*

PROOF. The theorem is a straightforward consequence of Proposition 3.7. \square

PROPOSITION 3.7. *Let $\mathbb{G} \equiv (\mathbb{R}^d, *)$ be a Carnot group. There exists a constant $r_0 > 0$ such that, whenever $0 < r < r_0$ and $\gamma : \mathbb{R} \rightarrow \mathbb{R}^d$ is an integral curve of some left invariant, horizontal vector field on $(\mathbb{R}^d, *)$ which satisfies the initial condition $\gamma(0) = x \in B_E(0, r)$, then there exist exactly one positive time $t_+ > 0$ and one negative time $t_- < 0$ such that $\gamma(t_+), \gamma(t_-) \in \partial B_E(0, r)$. In particular, $B_E(0, r)$ is h -convex.*

PROOF. Let $b \geq 1$ such that $\|\gamma''(t)\| \leq b$ on $[-1, 1]$ whenever γ is an integral curve of some left invariant, horizontal vector field of sub-Riemannian length one, and γ satisfies the initial condition $\gamma(0) \in B_E(0, 1)$. Define $r_0 := (\sqrt{5} - 2)/(8b)$. Fix $0 < r < r_0$ and $x \in B_E(0, r)$. If $v \in \mathbb{R}^{d_1} \times \{0\} \subseteq \mathbb{R}^d$, $\|v\| = 1$, then $\gamma : \mathbb{R} \rightarrow \mathbb{R}^d$, $\gamma(t) := x * \delta_t(v)$ is the integral curve of the left invariant, horizontal vector field X uniquely determined by the condition $X(0) = \sum_{i=1}^{d_1} v_i \partial_i(0)$, which satisfies the initial condition $\gamma(0) = x$. If $\pi : \mathbb{R}^{d_1} \times \mathbb{R}^{d-d_1} \rightarrow \mathbb{R}^{d_1}$ is orthogonal projection, then

$$\|\gamma(t)\| \geq \|\pi(\gamma(t))\| = \|(x_1, \dots, x_{d_1}) + t(v_1, \dots, v_{d_1})\| \geq |t| - \|(x_1, \dots, x_{d_1})\|.$$

Hence $|t| \geq 2r$ implies $\gamma(t) \notin \bar{B}_E(0, r)$. Now let

$$t_+ := \inf\{t > 0 \mid \gamma(t) \notin B_E(0, r)\}.$$

Then $0 < t_+ < 2r$, $\gamma(t_+) \in \partial B_E(0, r)$, $\gamma(t) \in B_E(0, r)$ for $0 \leq t < t_+$, and if n denotes the unit outer normal to $\partial B_E(0, r)$ at $\gamma(t_+)$, then $(\gamma'(t), n) \geq 0$. We have to show that $\gamma(t) \notin \bar{B}_E(0, r)$ when $t_+ < t < 2r$. We compute

$$\begin{aligned} \|\gamma(t)\| &= \left\| \gamma(t_+) + (t - t_+)\gamma'(t_+) + (t - t_+)^2 \int_0^1 (1-s)\gamma''(t_+ + s(t - t_+)) ds \right\| \\ &\geq \|\gamma(t_+) + (t - t_+)\gamma'(t_+)\| - (t - t_+)^2 \int_0^1 \|(1-s)\gamma''(t_+ + s(t - t_+))\| ds \\ &\geq \left(r^2 + (t - t_+)^2 \|\gamma'(t_+)\|^2 \right)^{\frac{1}{2}} - (t - t_+)^2 b \\ &\geq \left(r^2 + (t - t_+)^2 \right)^{\frac{1}{2}} - (t - t_+)^2 b \\ &= r + \frac{(t - t_+)^2}{2r} + (t - t_+)^4 \int_0^1 \frac{(-1)(1-s)}{4(r^2 + s(t - t_+)^2)^{\frac{3}{2}}} ds - (t - t_+)^2 b. \end{aligned}$$

If $t - t_+ \leq r/2$, then

$$\|\gamma(t)\| \geq r + \frac{(t - t_+)^2}{2r} - \frac{(t - t_+)^4}{4r^3} - (t - t_+)^2 b \geq r + \frac{(t - t_+)^2}{2r} - \frac{(t - t_+)^2}{16r} - \frac{(t - t_+)^2}{8r} > r.$$

On the other hand, if $t - t_+ > r/2$, then

$$\|\gamma(t)\| \geq \left(r^2 + (t - t_+)^2 \right)^{\frac{1}{2}} - (t - t_+)^2 b > \frac{\sqrt{5}r}{2} - (2r)^2 b > \frac{\sqrt{5}r}{2} - \frac{(\sqrt{5} - 2)r}{2} = r.$$

The proof for the negative time $t_- < 0$ is analogous. \square

The intuitive idea behind Proposition 3.7 is that a set which is ‘‘sufficiently strongly E-convex’’ must be h -convex. Similarly, the intuitive idea behind Proposition 3.8 is that a function which is ‘‘sufficiently strongly E-convex’’ must be h -convex:

PROPOSITION 3.8. Let $\mathbb{G} \equiv (\mathbb{R}^d, *)$ be a Carnot group, b and $r_0 = (\sqrt{5} - 2)/(8b)$ the constants appearing in the proof of Proposition 3.7, $u \in C^2(B_E(0, r_0))$ a uniformly strictly E -convex function. Suppose that there exist a lower bound $\lambda > 0$ for the eigenvalues of D^2u in $B_E(0, r_0)$ and an upper bound $0 < M < +\infty$ for $\|\nabla u\|$ in $B_E(0, r_0)$. Then there exists $0 < r_1 = r_1(\lambda, M) < r_0$ such that

$$v : B_E(0, r) \rightarrow \mathbb{R}, \quad v(x) := u\left(\frac{r_0}{r}x\right)$$

is h -convex whenever $0 < r < r_1$.

PROOF. By Proposition 3.7, $B_E(0, r)$ is h -convex for $0 < r < r_0$. Let X be a left invariant, horizontal vector field of sub-Riemannian length one and $\gamma : [t_-, t_+] \rightarrow B_E(0, r)$ be a segment of an integral curve of X . We can assume that $-1 \leq t_- < t_+ \leq 1$ and that $\|\gamma''(t)\| \leq b$ for all $t \in [t_-, t_+]$. Then, for each $t \in [t_-, t_+]$, we get

$$\begin{aligned} (v \circ \gamma)''(t) &= (D^2v(\gamma(t))\gamma'(t), \gamma'(t)) + (\nabla v(\gamma(t)), \gamma''(t)) \\ &= \left(\frac{r_0}{r}\right)^2 \left(D^2u\left(\frac{r_0}{r}\gamma(t)\right)\gamma'(t), \gamma'(t)\right) + \frac{r_0}{r} \left(\nabla u\left(\frac{r_0}{r}\gamma(t)\right), \gamma''(t)\right) \\ &\geq \left(\frac{r_0}{r}\right)^2 \lambda \|\gamma'(t)\|^2 - \frac{r_0}{r} \left\| \nabla u\left(\frac{r_0}{r}\gamma(t)\right) \right\| \|\gamma''(t)\| \\ &\geq \left(\frac{r_0}{r}\right)^2 \lambda - \frac{r_0}{r} Mb. \end{aligned}$$

□

From now on, we confine ourselves to the first Heisenberg group

$$\mathbb{H} = \mathbb{H}_1 = (\mathbb{R}^3, *) = (\{(x, y, t) \mid x, y, t \in \mathbb{R}\}, *)$$

(cf. §2 in the first chapter or §1 in the second). Recall that the group operation is given by

$$(x, y, t) * (x', y', t') = (x + x', y + y', t + t' + 2(x'y - xy')).$$

The differential structure on \mathbb{H} is determined by the left invariant vector fields

$$X = \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial t}, \quad Y = \frac{\partial}{\partial y} - 2x \frac{\partial}{\partial t} \quad \text{and} \quad T = \frac{\partial}{\partial t}.$$

If we let $V_1 := \text{span}_{\mathbb{R}}\{X, Y\}$ and $V_2 := \text{span}_{\mathbb{R}}\{T\}$, then $\mathfrak{h} = V_1 \oplus V_2$ is a stratification of the Lie algebra \mathfrak{h} of left invariant vector fields on \mathbb{H} . The dilation $\delta_\lambda : \mathbb{H} \rightarrow \mathbb{H}$ induced by this stratification is given by

$$\delta_\lambda(x, y, t) = (\lambda x, \lambda y, \lambda^2 t).$$

We will now construct h -convex functions which have a highly irregular pointwise behaviour in the vertical direction. Here and in the following, “vertical axis” means t -axis and “vertical direction” means in the direction of the t -axis. The first step consists in exhibiting an h -convex function whose restriction to the vertical axis is periodic. As a starting point, we consider functions of the type

$$h(x, y, t) = \left((x^2 + y^2)^2 + g(t) \right)^{\frac{1}{4}},$$

where $g : \mathbb{R} \rightarrow \mathbb{R}$ is assumed to be twice continuously differentiable and positive, and try to obtain conditions on g which ensure that the symmetrized horizontal Hessian $D_{\mathbb{H}}^2 h$ of h is positive semidefinite. Recall that we have defined the symmetrized horizontal Hessian in the first section of this chapter and that h is h -convex provided $D_{\mathbb{H}}^2 h$ is positive semidefinite (cf. Proposition 3.3). Observe that $D_{\mathbb{H}}^2 h(p)$ is positive semidefinite at some $p \in \mathbb{H}$ if and only if the determinant and the trace of its matrix representation (which

we denote again $D_{\mathbb{H}}^2 h(p)$ with respect to the basis $(X(p), Y(p))$ of $H_p\mathbb{H}$ are non-negative. After some rather lengthy calculations with partial derivatives, we obtain

$$\operatorname{tr}(D_{\mathbb{H}}^2 h) = h^{-7} \left((1 + g'') (x^2 + y^2)^3 + (4g - 3g'^2/4 + gg'') (x^2 + y^2) \right)$$

and

$$\det(D_{\mathbb{H}}^2 h) = 3h^{-10} \left((x^2 + y^2)^2 g (1 + g'') - 3(x^2 + y^2)^2 g'^2/4 \right).$$

Consequently, a sufficient condition for $\operatorname{tr}(D_{\mathbb{H}}^2 h) \geq 0$ to hold is that

$$(3.1) \quad 1 + g'' \geq 0 \quad \text{and} \quad 4g(4 + g'') \geq 3g'^2,$$

and a necessary and sufficient condition for $\det(D_{\mathbb{H}}^2 h) \geq 0$ to hold is that

$$(3.2) \quad 4g(1 + g'') \geq 3g'^2.$$

Summing up, we see that the following conditions on g are sufficient to guarantee that h is h-convex:

- (i) $g \in C^2(\mathbb{R})$, $g > 0$ on \mathbb{R} ,
- (ii) $1 + g'' \geq 0$ on \mathbb{R} and
- (iii) $4g(1 + g'') \geq 3g'^2$ on \mathbb{R} .

It is easy to check that the periodic function $g : \mathbb{R} \rightarrow \mathbb{R}$ with

$$g(t) := 2 + \frac{\sin(t)}{2} \quad \forall t \in \mathbb{R}$$

satisfies these conditions.

In the following, we use the h-convex function $h : \mathbb{H} \rightarrow \mathbb{R}$ with

$$h(x, y, t) = \left((x^2 + y^2)^2 + 2 + \frac{\sin(t)}{2} \right)^{\frac{1}{4}} \quad \forall (x, y, t) \in \mathbb{H}$$

as building block for our constructions. Here is our first result:

PROPOSITION 3.9. *There exists an h-convex function $w : \mathbb{H} \rightarrow \mathbb{R}$ which is invariant with respect to rotations that fix the vertical axis and whose restriction to the vertical axis is nowhere differentiable.*

PROOF. The idea is to perform a Weierstrass-type construction as described e.g. in §1, chapter 11 of [31]. For fixed $1/2 < \beta < 1$, choose $\lambda > 2$ in such a way that

$$(3.3) \quad \frac{\lambda^{\beta-1}}{1 - \lambda^{\beta-1}} + \frac{\lambda^{-\beta}}{1 - \lambda^{-\beta}} < \epsilon,$$

where $\epsilon > 0$ is to be specified later. Let

$$f_k(x, y, t) := h \circ \delta_{\lambda^{k/2}}(x, y, t) = \left(\lambda^{2k} (x^2 + y^2)^2 + 2 + \frac{\sin(\lambda^k t)}{2} \right)^{\frac{1}{4}}.$$

The function w is defined by the formula

$$w(x, y, t) := \sum_{k \in \mathbb{N}} \lambda^{-k\beta} f_k(x, y, t).$$

It follows from the h-convexity of h and from Lemma 3.1 that w is h-convex. In order to prove that w is nowhere differentiable on the vertical axis, we estimate the modulus of continuity of w there. The calculation is similar to the one in [31]. Given $t \in \mathbb{R}$, $0 < \tau < 1/\lambda$, let $N \in \mathbb{N}$ such that

$$\lambda^{-(N+1)} \leq \tau < \lambda^{-N}.$$

Then

$$\begin{aligned}
& \left| w(0, 0, t + \tau) - w(0, 0, t) - \lambda^{-N\beta}(f_N(0, 0, t + \tau) - f_N(0, 0, t)) \right| \\
& \leq \sum_{k=1}^{N-1} \lambda^{-k\beta} |f_k(0, 0, t + \tau) - f_k(0, 0, t)| + \sum_{k=N+1}^{\infty} \lambda^{-k\beta} |f_k(0, 0, t + \tau) - f_k(0, 0, t)| \\
& \leq \sum_{k=1}^{N-1} \lambda^{-k\beta} \lambda^k \tau + \sum_{k=N+1}^{\infty} \lambda^{-k\beta} \leq \tau \frac{(\lambda^{1-\beta})^N}{\lambda^{1-\beta} - 1} + \frac{(\lambda^{-\beta})^{N+1}}{1 - \lambda^{-\beta}} \\
& \leq \lambda^{-N\beta} \left(\frac{\lambda^{\beta-1}}{1 - \lambda^{\beta-1}} + \frac{\lambda^{-\beta}}{1 - \lambda^{-\beta}} \right) \leq \lambda^{-N\beta} \epsilon
\end{aligned}$$

by (3.3). On the other hand, we have

$$|f_N(0, 0, t + \tau) - f_N(0, 0, t)| \geq c |\sin(\lambda^N(t + \tau)) - \sin(\lambda^N t)|$$

for some $c > 0$ not depending on N or t . Since $1 - 1/\lambda \geq 1/2$ and $\lambda^{-(N+1)} \leq \tau < \lambda^{-N}$, we can find a τ in this interval such that

$$|f_N(0, 0, t + \tau) - f_N(0, 0, t)| \geq c/10.$$

Let $\epsilon := c/20$. Then, given $\lambda^{-N} \leq \delta < \lambda^{-N+1}$, we can choose $\lambda^{-(N+1)} \leq \tau < \lambda^{-N}$ in such a way that

$$|w(0, 0, t + \tau) - w(0, 0, t)| \geq \epsilon \lambda^{-N\beta} > \epsilon \lambda^{-\beta} \delta^\beta > C \delta^\beta$$

with some $C > 0$ independent of t and δ . In particular, the derivative of $w(0, 0, \cdot)$ does not exist at any t . \square

REMARK 3.2. One can verify that

$$\sum_{k \in \mathbb{N}} \lambda^{-k\beta} \partial_t f_k(x, y, t)$$

is locally uniformly convergent away from the vertical axis. This implies that the function $w(x, y, \cdot)$ is in $C^1(\mathbb{R})$ for any $(x, y) \neq (0, 0)$.

Our second result reads as follows:

THEOREM 3.10.

- (i) *There exists an h-convex function $u : \mathbb{H} \rightarrow \mathbb{R}$ and a set of vertical lines whose orthogonal projection to the (x, y) -plane is dense in the unit square, such that the restriction of u to any of these lines is nowhere differentiable.*
- (ii) *For any $0 < s < 1$, there exists an h-convex function u_s and a set of vertical lines whose orthogonal projection to the (x, y) -plane has positive s -dimensional Hausdorff measure, such that the restriction of u_s to any of these lines is nowhere differentiable.*

PROOF. (i) Let w be as in Proposition 3.9. For each $k \in \mathbb{N}$, consider the partition of the closed unit square $Q = \{(x, y, 0) \in \mathbb{R}^3 \mid |x|, |y| \leq 1/2\}$ in the (x, y) -plane in 2^{2k} closed squares $Q_{k,l}$ of side length $1/2^k$ each. Let $p_{k,l} = (x_{k,l}, y_{k,l}, 0)$ denote the center of $Q_{k,l}$. Clearly

$$\left\{ p_{k,l} \mid k \in \mathbb{N}, l \in \{1, \dots, 2^{2k}\} \right\}$$

is dense in the unit square. Let $g_{k,l} : \mathbb{H} \rightarrow \mathbb{R}$ be given by

$$g_{k,l}(x, y, t) := c_{k,l} \|(x - x_{k,l}, y - y_{k,l})\| \quad \forall (x, y, t) \in \mathbb{H},$$

where $c_{k,l} > 0$ is a constant chosen in order to ensure that $g_{k,l}(x, y, t) \geq \|w \circ l_{-p_{k,l}}\|_{L^\infty(Q)}$ when $(x, y) \in Q$ and $\|(x - x_{k,l}, y - y_{k,l})\| \geq 1/2^{k+1}$. Finally define

$$f_{k,l}(x, y, t) := \sup \{w \circ l_{-p_{k,l}}, g_{k,l}\} = \frac{|w \circ l_{-p_{k,l}} - g_{k,l}| + w \circ l_{-p_{k,l}} + g_{k,l}}{2}.$$

By definition of $g_{k,l}$, $f_{k,l} = g_{k,l}$ in $Q \setminus \text{int}(Q_{k,l})$.

Define u by

$$u(x, y, t) := \sum_{k \in \mathbb{N}} \frac{1}{k^2 2^{2k}} \sum_{l=1}^{2^{2k}} \frac{f_{k,l}(x, y, t)}{c_{k,l}}.$$

For fixed $K \in \mathbb{N}$ and $L \in \{1, \dots, 2^{2K}\}$, we have

$$u(x_{K,L}, y_{K,L}, t) = \sum_{k \leq K} \frac{1}{k^2 2^{2k}} \sum_{l=1}^{2^{2k}} \frac{f_{k,l}(x_{K,L}, y_{K,L}, t)}{c_{k,l}} + \sum_{k \geq K+1} \frac{1}{k^2 2^{2k}} \sum_{l=1}^{2^{2k}} \frac{f_{k,l}(x_{K,L}, y_{K,L}, t)}{c_{k,l}}.$$

For $k \leq K$, $l \in \{1, \dots, 2^{2k}\}$, $l \neq L$, the one-sided derivatives of $f_{k,l}(x_{K,L}, y_{K,L}, \cdot)$ exist everywhere. The second sum does not depend on t , because $(x_{K,L}, y_{K,L}, 0)$ is always outside of $\text{int}(Q_{k,l})$. Finally, the derivative of $f_{K,L}(x_{K,L}, y_{K,L}, \cdot)$ from the right does not exist anywhere since $f_{K,L}(x_{K,L}, y_{K,L}, \cdot)$ coincides with $w(0, 0, \cdot)$. This shows that the restriction of u to $\{(x_{K,L}, y_{K,L}, t) \mid t \in \mathbb{R}\}$ is nowhere differentiable for $K \in \mathbb{N}$, $L \in \{1, \dots, 2^{2K}\}$.

(ii) In order to obtain the family of functions $\{u_s\}_{0 < s < 1}$ appearing in (ii), we proceed in the following way: We define a Cantor set of positive s -dimensional Hausdorff measure as a countable intersection of finite unions of closed squares. We then use left translations to the centers of these squares together with dilations to perform a Weierstrass-type construction on the whole Cantor set. The argument involves substantially more technicalities than the one used in the proof of Proposition 3.9. Let us indicate the main steps:

Let $0 < s < 1$, $\alpha := s/2$ and $\alpha + 1/2 < \beta < 1$. Choose $\lambda > 2$ in such a way that

- (i) $\lambda^{\alpha/2} \in \mathbb{N}$ and
- (ii) $\lambda^{\beta-1} / (1 - \lambda^{\beta-1}) + \lambda^{-\beta} / (1 - \lambda^{-\beta}) < 10^{-3}$.

Suppose that for $k \in \mathbb{N}$ we have $\lambda^{k\alpha}$ pairwise disjoint closed squares $Q_{k,l}$, of side length $\lambda^{-k/2}$ each, distributed in the unit square. For fixed $1 \leq l \leq \lambda^{\alpha k}$, distribute λ^α closed squares $Q_{k+1,l'}$, of side length $\lambda^{-(k+1)/2}$ each, in $Q_{k,l}$. Clearly, when λ is sufficiently big, this can be done in such a way that the squares with the same centers as the $Q_{k+1,l'}$ and twice their side length are pairwise disjoint. Write

$$\mathcal{C}_k := \bigcup_{l=1}^{\lambda^{k\alpha}} Q_{k,l} \quad \text{and} \quad \mathcal{C} := \bigcap_{k \in \mathbb{N}} \mathcal{C}_k.$$

Using standard arguments (cf. §12 in the fourth chapter of [67]), one can prove that the s -dimensional Hausdorff measure of \mathcal{C} is positive and finite.

Let $p_{k,l} = (x_{k,l}, y_{k,l}, 0)$ denote the center of each of the squares $Q_{k,l}$. Let

$$f(x, y, t) := \max \left\{ \left(\|(x, y)\|^4 + 2 + \frac{\sin(t)}{2} \right)^{\frac{1}{4}}, c \|(x, y)\| \right\},$$

$$f_{k,l}(x, y, t) := f \circ \delta_{\lambda^{k/2}} \circ l_{-p_{k,l}}(x, y, t).$$

Here $c > 0$ is chosen in order to ensure that

$$f_{k,l}(x, y, t) = \left(\lambda^{2k} \|(x - x_{k,l}, y - y_{k,l})\|^4 + 2 + \frac{\sin(\lambda^k (t + 2x_{k,l}y - 2xy_{k,l}))}{2} \right)^{\frac{1}{4}}$$

in $Q_{k,l}$ and

$$f_{k,l}(x, y, t) = \lambda^{k/2} c \|(x - x_{k,l}, y - y_{k,l})\|$$

outside of the square of side length $2\lambda^{-k/2}$ with center $p_{k,l}$. We can take $c = (7/2)^{\frac{1}{4}}$ for instance. Thus $f_{k,l}(x, y, t) = \lambda^{k/2} c \|(x - x_{k,l}, y - y_{k,l})\|$ on $Q_{k,l'}$ for $l' \neq l$ by choice of the $Q_{k,l}$.

The function u_s is defined by the formula

$$u_s(x, y, t) := \sum_{k \in \mathbb{N}} \sum_{l=1}^{\lambda^{k\alpha}} \lambda^{-k\beta} f_{k,l}(x, y, t).$$

We show that for a given $p = (x, y, 0) \in \mathcal{C}$, the restriction of u_s to the vertical line $\{(x, y, t) \mid t \in \mathbb{R}\}$ is nowhere differentiable: Given $t \in \mathbb{R}$, $p = (x, y, 0) \in \mathcal{C}$ and $0 < \tau < 1/\lambda$, let $N \in \mathbb{N}$ such that

$$\lambda^{-(N+1)} \leq \tau < \lambda^{-N}.$$

Then

$$\begin{aligned} & \left| u_s(x, y, t + \tau) - u_s(x, y, t) - \sum_{l=1}^{\lambda^{N\alpha}} \lambda^{-N\beta} (f_{N,l}(x, y, t + \tau) - f_{N,l}(x, y, t)) \right| \\ & \leq \sum_{k=1}^{N-1} \sum_{l=1}^{\lambda^{k\alpha}} \lambda^{-k\beta} |f_{k,l}(x, y, t + \tau) - f_{k,l}(x, y, t)| + \\ & \qquad \sum_{k=N+1}^{\infty} \sum_{l=1}^{\lambda^{k\alpha}} \lambda^{-k\beta} |f_{k,l}(x, y, t + \tau) - f_{k,l}(x, y, t)|. \end{aligned}$$

Notice now that for fixed $k \in \mathbb{N}$, by construction,

$$|f_{k,l}(x, y, t + \tau) - f_{k,l}(x, y, t)| \neq 0$$

precisely for one $l \in \{1, \dots, \lambda^{k\alpha}\}$. A calculation shows that this expression is bounded by 1 for $k \geq N + 1$ while for $1 \leq k \leq N - 1$ we obtain the bound $\lambda^k \tau$ from the mean value theorem. It follows that

$$\begin{aligned} & \left| u_s(x, y, t + \tau) - u_s(x, y, t) - \sum_{l=1}^{\lambda^{N\alpha}} \lambda^{-N\beta} (f_{k,l}(x, y, t + \tau) - f_{k,l}(x, y, t)) \right| \\ & \leq \sum_{k=1}^{N-1} \lambda^{-k\beta} \lambda^k \tau + \sum_{k=N+1}^{\infty} \lambda^{-k\beta} \leq \tau \frac{(\lambda^{1-\beta})^N}{\lambda^{1-\beta} - 1} + \frac{(\lambda^{-\beta})^{N+1}}{1 - \lambda^{-\beta}} \leq \\ & \qquad \lambda^{-N\beta} \left(\frac{\lambda^{\beta-1}}{1 - \lambda^{\beta-1}} + \frac{\lambda^{-\beta}}{1 - \lambda^{-\beta}} \right) < \lambda^{-N\beta} 10^{-3} \end{aligned}$$

by choice of λ . Finally, we have

$$\sum_{l=1}^{\lambda^{N\alpha}} \lambda^{-N\beta} (f_{N,l}(x, y, t + \tau) - f_{N,l}(x, y, t)) = \lambda^{-N\beta} (f_{N,L}(x, y, t + \tau) - f_{N,L}(x, y, t))$$

for some $L \in \{1, \dots, \lambda^{N\alpha}\}$, and a computation gives the estimate

$$\begin{aligned} & \lambda^{-N\beta} |f_{N,L}(x, y, t + \tau) - f_{N,L}(x, y, t)| \\ & \geq \frac{\lambda^{-N\beta} |(\sin(\lambda^N ((t + \tau) + 2x_{N,L}y - 2xy_{N,L})) - \sin(\lambda^N (t + 2x_{N,L}y - 2xy_{N,L})))|}{52}. \end{aligned}$$

Since $1 - 1/\lambda \geq 1/2$ and $\lambda^{-(N+1)} \leq \tau < \lambda^{-N}$, we can find a τ in this interval so that

$$\sin(\lambda^N((t + \tau) + 2x_{N,L}y - 2xy_{N,L})) - \sin(\lambda^N(t + 2x_{N,L}y - 2xy_{N,L})) \geq \frac{1}{10}.$$

This yields

$$\sum_{l=1}^{\lambda^{N\alpha}} \lambda^{-N\beta} (f_{N,l}(x, y, t + \tau) - f_{N,l}(x, y, t)) > 2 \cdot 10^{-3} \lambda^{-N\beta}.$$

Hence, given $\lambda^{-N} \leq \delta < \lambda^{-N+1}$, we can choose $\lambda^{-(N+1)} \leq \tau < \lambda^{-N}$ in such a way that

$$|u_s(x, y, t + \tau) - u_s(x, y, t)| \geq 10^{-3} \lambda^{-N\beta} > 10^{-3} \lambda^{-\beta} \delta^\beta.$$

In particular, the derivative of $u_s(x, y, \cdot)$ does not exist at any t . \square

3. Equivalence of h-convexity and v-convexity

We start with three preparatory lemmata that will be used in the proofs of Proposition 3.14 and Theorem 3.15.

LEMMA 3.11. *Let $I \subseteq \mathbb{R}$ be an open interval, let $u : I \rightarrow \mathbb{R}$ satisfy*

$$u\left(\frac{x_1 + x_2}{2}\right) \leq \frac{u(x_1) + u(x_2)}{2} \quad \forall x_1, x_2 \in I,$$

and suppose that $x_0 \in I$ is a point of upper semicontinuity for u . Then x_0 is a point of continuity for u .

PROOF. Suppose not. Then there exists $\epsilon > 0$ and a sequence $\{x_n\}_{n \in \mathbb{N}}$ such that $|x_0 - x_n| \downarrow 0$ as $n \rightarrow \infty$ and $u(x_n) + \epsilon \leq u(x_0)$ for all $n \in \mathbb{N}$. Choose $\delta > 0$ such that $|x_0 - x| < \delta$ implies $u(x) \leq u(x_0) + \epsilon/2$. Let $N \in \mathbb{N}$ with $|x_0 - x_N| < \delta$. Then

$$\begin{aligned} u\left(\frac{x_N + (2x_0 - x_N)}{2}\right) &\leq \frac{u(x_N) + u(2x_0 - x_N)}{2} \\ &\leq \frac{u(x_0) - \epsilon}{2} + \frac{u(x_0) + \epsilon/2}{2} \\ &= u(x_0) - \epsilon/4 \\ &< u(x_0), \end{aligned}$$

a contradiction. \square

LEMMA 3.12. *Let $I \subseteq \mathbb{R}$ be an open interval. Suppose that $u : I \rightarrow \mathbb{R}$ is upper semicontinuous and that*

$$u\left(\frac{x_1 + x_2}{2}\right) \leq \frac{u(x_1) + u(x_2)}{2} \quad \forall x_1, x_2 \in I.$$

Then u is convex on I .

PROOF. We have to show that

$$(3.4) \quad u((1 - \lambda)x_1 + \lambda x_2) \leq (1 - \lambda)u(x_1) + \lambda u(x_2) \quad \forall x_1, x_2 \in I \quad \forall \lambda \in [0, 1].$$

By induction, we show that

$$(3.5) \quad u((1 - \lambda)x_1 + \lambda x_2) \leq (1 - \lambda)u(x_1) + \lambda u(x_2) \quad \forall x_1, x_2 \in I \quad \forall \lambda \in A_K$$

holds for all $K \in \mathbb{N}$, where

$$A_K := \left\{ \sum_{k=1}^K a_k 2^{-k} \mid a_1, \dots, a_K \in \{0, 1\} \right\} \quad \forall K \in \mathbb{N}.$$

This will imply (3.4) and conclude the proof, since $A := \bigcup_{K \in \mathbb{N}} A_K$ is dense in $[0, 1]$ and u is continuous by Lemma 3.11.

If $K = 1$ and $\lambda_K = 0$, then

$$u((1 - \lambda_K)x_1 + \lambda_K x_2) = u(x_1) = (1 - \lambda_K)u(x_1) + \lambda_K u(x_2).$$

If $K = 1$ and $\lambda_K = 1/2$, then

$$u((1 - \lambda_K)x_1 + \lambda_K x_2) = u\left(\frac{x_1 + x_2}{2}\right) \leq \frac{u(x_1) + u(x_2)}{2} = (1 - \lambda_K)u(x_1) + \lambda_K u(x_2).$$

Suppose that (3.5) holds for some $K \in \mathbb{N}$. Let $x_1, x_2 \in I$ and $\lambda = \sum_{k=1}^{K+1} a_k 2^{-k}$. If $a_0 = 0$, then $\lambda = \lambda_K/2$ for some $\lambda_K \in A_K$ and

$$(1 - \lambda)x_1 + \lambda x_2 = \frac{(2 - \lambda_K)x_1}{2} + \frac{\lambda_K x_2}{2} = \frac{x_1 + ((1 - \lambda_K)x_1 + \lambda_K x_2)}{2}.$$

Hence, by hypothesis and by inductive assumption,

$$\begin{aligned} u((1 - \lambda)x_1 + \lambda x_2) &= u\left(\frac{x_1 + ((1 - \lambda_K)x_1 + \lambda_K x_2)}{2}\right) \\ &\leq \frac{u(x_1) + u((1 - \lambda_K)x_1 + \lambda_K x_2)}{2} \\ &\leq \frac{u(x_1) + ((1 - \lambda_K)u(x_1) + \lambda_K u(x_2))}{2} \\ &= (1 - \lambda)u(x_1) + \lambda u(x_2). \end{aligned}$$

If $a_0 = 1$, then $\lambda = (\lambda_K + 1)/2$ for some $\lambda_K \in A_K$ and

$$(1 - \lambda)x_1 + \lambda x_2 = \frac{(1 - \lambda_K)x_1}{2} + \frac{\lambda_K x_2}{2} + \frac{x_2}{2} = \frac{((1 - \lambda_K)x_1 + \lambda_K x_2) + x_2}{2}.$$

Thus, by hypothesis and by inductive assumption,

$$\begin{aligned} u((1 - \lambda)x_1 + \lambda x_2) &= u\left(\frac{((1 - \lambda_K)x_1 + \lambda_K x_2) + x_2}{2}\right) \\ &\leq \frac{u((1 - \lambda_K)x_1 + \lambda_K x_2) + u(x_2)}{2} \\ &\leq \frac{((1 - \lambda_K)u(x_1) + \lambda_K u(x_2)) + u(x_2)}{2} \\ &= (1 - \lambda)u(x_1) + \lambda u(x_2). \end{aligned}$$

□

We will now show that h-convex, upper semicontinuous functions are v-convex. We start by proving a Taylor expansion formula of second order with integral remainder for the horizontal directions:

LEMMA 3.13. *Let $\varphi \in C_{\mathbb{H}}^2(\Omega)$, where Ω is an open subset of the stratified group \mathbb{G} , $g \in \Omega$, $X \in V_1$. Then*

$$(3.6) \quad \varphi(g \exp(tX)) = \varphi(g) + X\varphi(g) \cdot t + \int_0^1 (1 - s) D_{\mathbb{H}}^2 \varphi(h(s)) (X(h(s)), X(h(s))) ds \cdot t^2$$

for all t in some neighbourhood of 0, where $h(s) = g \exp(stX)$ for all $s \in [0, 1]$.

PROOF. Assume first that $\varphi \in C^\infty(U)$ where $U \subseteq \Omega$ is an open neighbourhood of g . Then (3.6) follows by direct computation from the Taylor formula with integral remainder for smooth, real valued functions of a real variable. In the general case, the claim follows via regularization from the above observation together with Lemma 1.6. □

Notice that we have to assume upper semicontinuity in the proposition below, since it is a priori not clear whether h-convex functions are continuous. We discuss the first order regularity of h-convex functions in chapters four and six.

PROPOSITION 3.14. *Let $\Omega \subseteq \mathbb{G}$ be an h -convex, open subset. Suppose that $u : \Omega \rightarrow \mathbb{R}$ is h -convex and upper semicontinuous. Then u is v -convex.*

PROOF. Let $g \in \Omega$. Assume that $\varphi \in C_{\mathbb{H}}^2(U)$, where $U \subseteq \Omega$ is an open neighbourhood of g in Ω , touches u from above at g . We have to show that

$$D_{\mathbb{H}}^2 \varphi(g)(X(g), X(g)) \geq 0$$

whenever $X \in V_1$. By Lemma 3.13, we have

$$\varphi(g \exp(tX)) = \varphi(g) + X\varphi(g) \cdot t + \int_0^1 (1-s) D_{\mathbb{H}}^2 \varphi(h(s))(X(h(s)), X(h(s))) ds \cdot t^2$$

for all t in a neighbourhood of 0. Thus

$$\frac{\varphi(g \exp(-tX)) + \varphi(g \exp(tX))}{2} = \varphi(g) + \int_0^1 (1-s) (R_1(s) + R_2(s)) ds \cdot t^2$$

for all sufficiently small $t > 0$, where

$$R_1(s) = \frac{1}{2} D_{\mathbb{H}}^2 \varphi(g \exp(-stX))(X(g \exp(-stX)), X(g \exp(-stX))) \quad \forall s \in [0, 1]$$

and

$$R_2(s) = \frac{1}{2} D_{\mathbb{H}}^2 \varphi(g \exp(stX))(X(g \exp(stX)), X(g \exp(stX))) \quad \forall s \in [0, 1].$$

Since

$$\varphi(g) = u(g) \leq \frac{u(g \exp(-tX)) + u(g \exp(tX))}{2} \leq \frac{\varphi(g \exp(-tX)) + \varphi(g \exp(tX))}{2},$$

for all sufficiently small $t > 0$, we obtain

$$\int_0^1 (1-s) (R_1(s) + R_2(s)) ds \cdot t^2 \geq 0$$

for all sufficiently small $t > 0$. The desired result follows. \square

The proof of the converse implication (v -convex implies h -convex) is substantially more difficult:

THEOREM 3.15. *Let $\Omega \subseteq \mathbb{G}$ be an h -convex, open subset and $u : \Omega \rightarrow \mathbb{R}$ upper semicontinuous. Suppose that $D_{\mathbb{H}}^2 \varphi(g_0)$ is positive semidefinite whenever $g_0 \in \Omega$, U is an open neighbourhood of g_0 in Ω and $\varphi \in C^\infty(U)$ touches u from above at g_0 . Then u is h -convex. In particular, if u is v -convex, then u is h -convex.*

PROOF. Assume by contradiction that there exist $g \in \Omega$ and $X \in V_1 \setminus \{0\}$ such that

$$u(g) > \frac{u(g \exp(-tX)) + u(g \exp(tX))}{2}$$

for some $t > 0$ (cf. Lemma 3.12). We have to exhibit a smooth function φ , defined on an open neighbourhood U of some $g_0 \in \Omega$, which touches u from above at g_0 and whose symmetrized horizontal Hessian at g_0 is not positive semidefinite.

After a left translation and a dilation, we can assume that $g = e$, $t = 1$ and

$$\{\exp(tX) \mid t \in [-1, 1]\} \subseteq U \subseteq \Omega.$$

By adding a v -convex function of the form

$$g \mapsto \alpha \langle X, (\exp^{-1}(g))_1 \rangle + \beta$$

(cf. Lemma 3.1, Lemma 3.4 and Proposition 3.14) with suitably chosen $\alpha, \beta \in \mathbb{R}$ to u and multiplying the sum with some appropriate $\gamma > 0$, we can also assume $u(e) = 0$,

$u(\exp(-X)) \leq -2$ and $u(\exp(X)) \leq -2$. Since \bar{U} is compact and u is upper semicontinuous, u attains a maximum M on \bar{U} . Let $\{X_{i,j}\}_{1 \leq i \leq s, 1 \leq j \leq d_i}$ be an adapted basis of \mathfrak{g} such that $X_{1,1} = X$. For $a \geq 0$, $\epsilon > 0$, consider the function $\varphi_{a,\epsilon} : \mathbb{G} \rightarrow \mathbb{R}$ defined by

$$\varphi_{a,\epsilon} \left(\exp \left(\sum_{i=1}^s \sum_{j=1}^{d_i} x_{i,j} X_{i,j} \right) \right) = a - x_{1,1}^2 + \sum_{j=2}^{d_1} \frac{x_{1,j}^2}{\epsilon^{2s}} + \sum_{i=2}^s \sum_{j=1}^{d_i} \frac{x_{i,j}^2}{\epsilon^{2(s-i+1)}}.$$

We let

$$D_\epsilon := \left\{ \exp \left(\sum_{i=1}^s \sum_{j=1}^{d_i} x_{i,j} X_{i,j} \right) \mid \sum_{j=2}^{d_1} \frac{x_{1,j}^2}{\epsilon^{2s}} + \sum_{i=2}^s \sum_{j=1}^{d_i} \frac{x_{i,j}^2}{\epsilon^{2(s-i+1)}} < M + 2, |x_1| < 1 \right\}.$$

Observe that there exists $\epsilon_0 > 0$ such that $\bar{D}_\epsilon \subseteq U$ when $0 < \epsilon \leq \epsilon_0$.

Let us first show that $\varphi_{a,\epsilon}$ fails to be positive semidefinite everywhere in D_ϵ whenever $a \geq 0$ and ϵ is sufficiently small: Given $0 < \epsilon \leq \epsilon_0$ and

$$g = \exp \left(\sum_{i=1}^s \sum_{j=1}^{d_i} x_{i,j} X_{i,j} \right) \in D_\epsilon,$$

we have

$$\begin{aligned} D_{\mathbb{H}}^2 \varphi_{a,\epsilon}(g)(X, X) &= D_{\mathbb{H}}^2 \varphi_{a,\epsilon}(g)(X_{1,1}, X_{1,1}) \\ &= \frac{d^2}{dt^2} \varphi_{a,\epsilon} \left(\exp \left(\left(\sum_{i=1}^s \sum_{j=1}^{d_i} x_{i,j} X_{i,j} \right) * (tX_{1,1}) \right) \right) \Big|_{t=0}, \end{aligned}$$

where $*$ is given by (1.2). We have

$$\begin{aligned} \left(\sum_{i=1}^s \sum_{j=1}^{d_i} x_{i,j} X_{i,j} \right) * (tX_{1,1}) &= \\ &= (x_{1,1} + t)X_{1,1} + \sum_{j=2}^{d_1} x_{1,j} X_{1,j} + \sum_{i=2}^s \sum_{j=1}^{d_i} (x_{i,j} + P_{i,j}(t)) X_{i,j}, \end{aligned}$$

where $P_{i,j}$ is a polynomial in t whose coefficients are bounded by a constant (which does not depend on ϵ) times $\epsilon^{2(s-i+2)}$. It follows immediately that there exists $0 < \epsilon_1 \leq \epsilon_0$ such that $0 < \epsilon \leq \epsilon_1$ and $a \geq 0$ implies

$$D_{\mathbb{H}}^2 \varphi_{a,\epsilon}(g)(X, X) \leq -1 \quad \forall g \in D_\epsilon.$$

Let us now show that there exists $0 < \epsilon_2 \leq \epsilon_1$ such that

$$\varphi_{a,\epsilon_2}(g) > u(g) \quad \forall g \in \partial D_{\epsilon_2}$$

whenever $a \geq 0$. We divide ∂D_ϵ in two parts:

$$\partial_1 D_\epsilon := \left\{ \exp \left(\sum_{i=1}^s \sum_{j=1}^{d_i} x_{i,j} X_{i,j} \right) \mid \sum_{j=2}^{d_1} \frac{x_{1,j}^2}{\epsilon^{2s}} + \sum_{i=2}^s \sum_{j=1}^{d_i} \frac{x_{i,j}^2}{\epsilon^{2(s-i+1)}} \leq M + 2, |x_1| = 1 \right\}$$

and

$$\partial_2 D_\epsilon = \left\{ \exp \left(\sum_{i=1}^s \sum_{j=1}^{d_i} x_{i,j} X_{i,j} \right) \mid \sum_{j=2}^{d_1} \frac{x_{1,j}^2}{\epsilon^{2s}} + \sum_{i=2}^s \sum_{j=1}^{d_i} \frac{x_{i,j}^2}{\epsilon^{2(s-i+1)}} = M + 2, |x_1| < 1 \right\}.$$

Note that $\exp(-X_{1,1}), \exp(X_{1,1}) \in \partial_1 D_\epsilon$, $\varphi_{a,\epsilon} \geq a - 1 \geq -1$ on $\partial_1 D_\epsilon$ independently of ϵ and a , while $u(\exp(-X_{1,1})) \leq -2$ and $u(\exp(X_{1,1})) \leq -2$. By upper semicontinuity of

u , we have $\varphi_{a,\epsilon} > u$ on $\partial_1 D_\epsilon$ when ϵ is sufficiently small, independently of a . On the other hand, on $\partial_2 D_\epsilon$, we have

$$\varphi_{a,\epsilon} \left(\sum_{i=1}^s \sum_{j=1}^{d_i} x_{i,j} X_{i,j} \right) = a - x_{1,1}^2 + M + 2 > a + M + 1 \geq M + 1 \geq u \left(\sum_{i=1}^s \sum_{j=1}^{d_i} x_{i,j} X_{i,j} \right) + 1,$$

independently of a .

Let us now observe that $\varphi_{0,\epsilon_2}(e) = u(e) = 0$, while $\varphi_{a,\epsilon_2} > u$ on $\overline{D}_{\epsilon_2}$ if $a > M + 1$. Define

$$a_0 := \inf \{ a > 0 \mid \varphi_{a,\epsilon_2}(g) > u(g) \quad \forall g \in \overline{D}_{\epsilon_2} \} \geq 0.$$

Then $\varphi_{a_0,\epsilon_2}(g) \geq u(g)$ for all $g \in \overline{D}_{\epsilon_2}$, and $\varphi_{a_0,\epsilon_2}(g_0) = u(g_0)$ for some $g_0 \in \overline{D}_{\epsilon_2}$ by upper semicontinuity of u . Since $\varphi_{a_0,\epsilon_2} > u$ on the boundary of D_{ϵ_2} , it follows that $g_0 \in D_{\epsilon_2}$. \square

First order regularity of h-convex functions in step two

In this chapter, we begin our investigation of the first order regularity of h-convex functions. In the first section, we show that h-convex functions which are locally bounded above are locally Lipschitz continuous with respect to an intrinsic metric. In the second section, we introduce a sufficient geometric condition –h-convex finiteness– for the local upper boundedness of h-convex functions on Carnot groups which satisfy this condition. We prove that any Carnot group of step at most two is finitely h-convex. Eventually, in the third section, we show that the Engel group (compare §2 of the first chapter for a definition) is not finitely h-convex. The existence of a stratified group of step three which is not finitely h-convex shows that a new strategy is needed in order to obtain first order regularity for h-convex functions on general stratified groups. We will address this issue again in the first section of chapter six.

1. Local Lipschitz continuity of bounded h-convex functions

Let \mathbb{G} be a Carnot group, $\oplus_{i=1}^s V_i$ a stratification of its Lie algebra \mathfrak{g} of left invariant vector fields, ρ the sub-Riemannian distance induced by an inner product $\langle \cdot, \cdot \rangle$ on V_1 and $\Omega \subseteq \mathbb{G}$ an h-convex, open subset.

We start by proving that an h-convex function which is locally bounded above is also locally bounded below.

LEMMA 4.1. *Let $u : \Omega \rightarrow \mathbb{R}$ be h-convex. Suppose that u is locally bounded above. Then u is locally bounded below.*

PROOF. Let $g_0 \in \Omega$ and $r > 0$ such that $B(g_0, 4r) \Subset \Omega$ and u is bounded above in $\overline{B}(g_0, 4r)$, say $u \leq M$ in $\overline{B}(g_0, 4r)$ for some $M \geq 0$. We claim that there exist $l = l(\mathbb{G}) \in \mathbb{N}$ and $n = n(\mathbb{G}) \in \mathbb{N}$ such that $g_1 \in B(g_0, r)$ and $u(g_1) \leq -4^n m$ implies $u \leq -m$ in $B(g_1, r/(l \cdot n))$ for all $m \geq 2M$. Notice that this gives the lemma since then, on the one hand,

$$\mathcal{H}^Q(\{g \in B(g_0, 4r) \mid u(g) > -m\}) \leq \mathcal{H}^Q(B(g_0, 4r)) - \mathcal{H}^Q(B(g_1, r/(l \cdot n))),$$

while, on the other hand,

$$\mathcal{H}^Q(\{g \in B(g_0, 4r) \mid u(g) \geq -m\}) > \mathcal{H}^Q(B(g_0, 4r)) - \mathcal{H}^Q(B(g_1, r/(l \cdot n)))$$

when m is sufficiently large (cf. Theorem 2 in the first section of the first chapter of [30]). This contradiction forces $u \geq -4^n m$ in $B(g_0, r)$ for sufficiently large m .

Let $\gamma : \mathbb{R} \rightarrow \mathbb{G}$ be an integral curve of some left invariant, horizontal vector field of sub-Riemannian length one, and suppose that γ satisfies the initial condition $\gamma(0) \in B(g_0, 2r)$. Define

$$t_- := \max \{t < 0 \mid \gamma(t) \in \partial B(g_0, 4r)\}$$

and

$$t_+ := \min \{t > 0 \mid \gamma(t) \in \partial B(g_0, 4r)\}.$$

We have $t_- \geq -6r$ and $t_+ \leq 6r$. Let $t \in [t_-, t_+]$ such that $\gamma(t) \in B(g_0, 2r)$. If $t \geq 0$, then $t = (1 - \lambda)0 + \lambda t_+$ with $\lambda \leq \frac{2}{3}$. Suppose that $u(\gamma(0)) \leq -4^{j+1}m$. Then the convexity of

$u \circ \gamma : [t_-, t_+] \rightarrow \mathbb{G}$ implies

$$u(\gamma(t)) \leq (1 - \lambda)u(\gamma(0)) + \lambda u(\gamma(t_+)) \leq -\frac{1}{3}4^{j+1}m + \frac{2}{3}M \leq \left(-\frac{4}{3} + \frac{4^{-j}}{3}\right)4^j m \leq -4^j m.$$

Similarly, $u(\gamma(t)) \leq -4^j m$ if $t \leq 0$ and $u(\gamma(0)) \leq -4^{j+1}m$.

This shows that if $S \subseteq B(g_0, 2r)$ is a segment of an integral curve of some left invariant, horizontal vector field and if $u(g_1) \leq -4^{j+1}m$ for some $g_1 \in S$, then $u \leq -4^j m$ on the whole segment ($j \in \mathbb{N} \cup \{0\}$, $m \geq 2M$). By Proposition 1.2, there exist constants $l = l(\mathbb{G}) \in \mathbb{N}$ and $n = n(\mathbb{G}) \in \mathbb{N}$ with the following property: Any pair of points $g_1, g_2 \in \mathbb{G}$ can be connected by a path consisting of at most n segments of integral curves of left invariant, horizontal vector fields, such that each segment has length at most $l \cdot \rho(g_1, g_2)$. Thus, if $g_1 \in B(g_0, r)$ and $u(g_1) \leq -4^n m$, then $u(g) \leq -m$ for each $g \in B(g_1, r/(l \cdot n))$. \square

REMARK 4.1. Note that if u is integrable on $B(g_0, 4r)$, then

$$(4.1) \quad u(g_1) \geq -4^n \max \left\{ 2M, (4l \cdot n)^Q \int_{B(g_0, 4r)} |u(g)| d\mathcal{H}^Q(g) \right\} \quad \forall g_1 \in B(g_0, r).$$

Otherwise $u(g_1) \leq -4^n m$ for some $g_1 \in B(g_0, r)$ and some

$$m > \max \left\{ 2M, (4l \cdot n)^Q \int_{B(g_0, 4r)} |u(g)| d\mathcal{H}^Q(g) \right\},$$

whence $u \leq -m$ on $B(g_1, r/(l \cdot n))$ by the proof of Lemma 4.1, whence

$$\int_{B(g_0, 4r)} |u(g)| d\mathcal{H}^Q(g) \geq \frac{m}{(4l \cdot n)^Q},$$

a contradiction. We will need the explicit lower bound (4.1) in our proof of the L^∞ - L^1 estimates (Theorem 6.2).

We now state and prove the main result of this section:

PROPOSITION 4.2. *Let $u : \Omega \rightarrow \mathbb{R}$ be h-convex. Suppose that u is locally bounded. Then u is locally Lipschitz continuous with respect to ρ .*

PROOF. Let $g_0 \in \Omega$ and $r > 0$ such that $B(g_0, 2r) \Subset \Omega$ and u is bounded in $\overline{B}(g_0, 2r)$, say $|u| \leq M$ in $\overline{B}(g_0, 2r)$ for some $M \geq 0$. Let $\gamma : \mathbb{R} \rightarrow \mathbb{G}$ be an integral curve of some left invariant, horizontal vector field of sub-Riemannian length one, and suppose that γ satisfies the initial condition $\gamma(0) \in B(g_0, r)$. Define

$$t_- := \max \{t < 0 \mid \gamma(t) \in \partial B(g_0, 2r)\}$$

and

$$t_+ := \min \{t > 0 \mid \gamma(t) \in \partial B(g_0, 2r)\}.$$

We have $2r \leq t_+ - t_- \leq 4r$, and if $t \in [t_-, t_+]$ and $\gamma(t) \in B(g_0, r)$, then $t - t_- \geq r$ and $t_+ - t \geq r$. Hence $t = (1 - \lambda)t_- + \lambda t_+$ where $\lambda \in [1/4, 3/4]$. Now let $t_1, t_2 \in [t_-, t_+]$ such that $t_1 < t_2$ and $\gamma(t_1), \gamma(t_2) \in B(g_0, r)$. Then

$$t_1 = (1 - \lambda_1)t_- + \lambda_1 t_+ \quad \text{and} \quad t_2 = (1 - \lambda_2)t_- + \lambda_2 t_+,$$

where $\lambda_1, \lambda_2 \in [1/4, 3/4]$ and $\lambda_1 < \lambda_2$. Thus

$$t_1 = \frac{\lambda_2 - \lambda_1}{\lambda_2} t_- + \frac{\lambda_1}{\lambda_2} t_+ \quad \text{and} \quad t_2 = \frac{1 - \lambda_2}{1 - \lambda_1} t_1 + \frac{\lambda_2 - \lambda_1}{1 - \lambda_1} t_+.$$

The convexity of $u \circ \gamma : [t_-, t_+] \rightarrow \mathbb{R}$ implies

$$\begin{aligned} u(\gamma(t_1)) - u(\gamma(t_2)) &\leq \frac{\lambda_2 - \lambda_1}{\lambda_2} u(\gamma(t_-)) + \frac{\lambda_1 - \lambda_2}{\lambda_2} u(\gamma(t_2)) \\ &= \frac{\rho(\gamma(t_1), \gamma(t_2))}{\lambda_2(t_+ - t_-)} u(\gamma(t_-)) - \frac{\rho(\gamma(t_1), \gamma(t_2))}{\lambda_2(t_+ - t_-)} u(\gamma(t_2)) \\ &\leq \frac{8M}{r} \rho(\gamma(t_1), \gamma(t_2)) \end{aligned}$$

and

$$\begin{aligned} u(\gamma(t_2)) - u(\gamma(t_1)) &\leq \frac{\lambda_1 - \lambda_2}{1 - \lambda_1} u(\gamma(t_1)) + \frac{\lambda_2 - \lambda_1}{1 - \lambda_1} u(\gamma(t_+)) \\ &= -\frac{\rho(\gamma(t_1), \gamma(t_2))}{(1 - \lambda_1)(t_+ - t_-)} u(\gamma(t_1)) + \frac{\rho(\gamma(t_1), \gamma(t_2))}{(1 - \lambda_1)(t_+ - t_-)} u(\gamma(t_+)) \\ &\leq \frac{8M}{r} \rho(\gamma(t_1), \gamma(t_2)). \end{aligned}$$

We have shown that

$$(4.2) \quad |u(g_1) - u(g_2)| \leq \frac{8M}{r} \rho(g_1, g_2) \quad \forall g_1, g_2 \in S$$

whenever $S \subseteq B(g_0, r)$ is a segment of an integral curve of some left invariant vector field. By Proposition 1.2, there exist constants $l = l(\mathbb{G}) \in \mathbb{N}$ and $n = n(\mathbb{G}) \in \mathbb{N}$ with the following property: Any pair of points $g_1, g_2 \in \mathbb{G}$ can be connected by a path consisting of at most n segments of integral curves of left invariant, horizontal vector fields, such that each segment has length at most $l \rho(g_1, g_2)$. In particular, (4.2) gives

$$(4.3) \quad |u(g_1) - u(g_2)| \leq \frac{8M \cdot l \cdot n}{r} \cdot \rho(g_1, g_2) \quad \forall g_1, g_2 \in B(g_0, r/(2l \cdot n + 1)).$$

□

2. Boundedness of h-convex functions in step two

LEMMA 4.3. *Let \mathbb{G} be a stratified group. Suppose there exists a finite subset $F \subseteq \mathbb{G}$ whose h-convex closure $\mathcal{C}(F)$ has non-empty interior. Then any h-convex function defined on an h-convex, open subset of \mathbb{G} is locally bounded above.*

PROOF. Let Γ denote the set of integral curves $\gamma : \mathbb{R} \rightarrow \mathbb{G}$ of left invariant, horizontal vector fields on \mathbb{G} . Given $A \subseteq \mathbb{G}$ we let

$$\begin{aligned} \mathbf{H}(A) &:= \{\gamma(t) \mid \gamma \in \Gamma, t \in [0, 1], \gamma(0), \gamma(1) \in A\}, \\ \mathbf{H}^0(A) &:= A, \\ \mathbf{H}^{k+1}(A) &:= \mathbf{H}(\mathbf{H}^k(A)) \quad \forall k \in \mathbb{N}_0 \quad \text{and} \\ \mathbf{H}^\infty(A) &:= \bigcup_{k \in \mathbb{N}_0} \mathbf{H}^k(A). \end{aligned}$$

Clearly, $\mathcal{C}(A) = \mathbf{H}^\infty(A)$, $l_g(\mathbf{H}(A)) = \mathbf{H}(l_g(A))$ for all $g \in \mathbb{G}$, $\delta_\lambda(\mathbf{H}(A)) = \mathbf{H}(\delta_\lambda(A))$ for all $\lambda > 0$, and $\mathbf{H}(A)$ is compact if A is.

The compactness property of the operator \mathbf{H} and the theorem of Baire imply that $\mathbf{H}^k(F)$ has non-empty interior for some $k = k(\mathbb{G}) \in \mathbb{N}$. Since \mathbf{H} –and thus \mathbf{H}^k – commutes with left translations and dilations, it follows that if $\Omega \subseteq \mathbb{G}$ is any h-convex, open subset, then for each $g_0 \in \Omega$ there exists a finite subset $F(g_0) \subseteq \Omega$ such that g_0 is contained in the interior of $\mathbf{H}^k(F(g_0))$. Finally, if $u : \Omega \rightarrow \mathbb{R}$ is h-convex, then $u \leq \max\{u(g) \mid g \in F(g_0)\}$ in $\mathbf{H}^j(F(g_0))$ for all $1 \leq j \leq k$ by induction, using the convexity of $u \circ \gamma$ for $\gamma \in \Gamma$. In particular, $u \leq \max\{u(g) \mid g \in F(g_0)\}$ in the interior of $\mathbf{H}^k(F(g_0))$. □

Lemma 4.3, Lemma 4.1 and Proposition 4.2 motivate the following

DEFINITION 4.1. We say that a Carnot group \mathbb{G} is *finitely h-convex* if it contains a finite subset $F \subseteq \mathbb{G}$ whose h-convex closure $\mathcal{C}(F)$ has non-empty interior.

LEMMA 4.4. Let \mathbb{G} be a Carnot group of step two and let $(X_1, \dots, X_{d_1}, Y_1, \dots, Y_{d_2})$ be a basis of the Lie algebra \mathfrak{g} of left invariant vector fields on \mathbb{G} adapted to the given stratification $\mathfrak{g} = V_1 \oplus V_2$. We identify \mathbb{G} with $(\mathbb{R}^d, *) \equiv (\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}, *)$ in the usual way with respect to this basis. Suppose that for some $2 \leq k \leq d_1$, the following hypotheses are verified:

(i) The Lie subalgebra generated by X_1, \dots, X_k is contained in

$$\text{span}_{\mathbb{R}}\{X_1, \dots, X_k\} \oplus \text{span}_{\mathbb{R}}\{Y_1, \dots, Y_l\}.$$

(ii) There exists a finite set $A_0 \subseteq \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ and a constant $K_0 > 0$ such that the set B_0 consisting of pairs $(0, y) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ with $|y_j| \leq K_0$ for $1 \leq j \leq l$ and $y_j = 0$ for $l+1 \leq j \leq d_2$ is contained in $\mathcal{C}(A_0)$.

Then there exists a finite set $A_k \subseteq \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ and a constant $K_k > 0$ such that the set B_k consisting of pairs $(x, y) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ with $|x_i| \leq K_k$ for $1 \leq i \leq k$, $x_i = 0$ for $k+1 \leq i \leq d_1$, $|y_j| \leq K_k$ for $1 \leq j \leq l$ and $y_j = 0$ for $l+1 \leq j \leq d_2$ is contained in $\mathcal{C}(A_k)$.

PROOF. Let $K_1 := K_0$ and $g := (x, 0)$ with $x_1 := K_1$ and $x_i = 0$ for $2 \leq i \leq d_1$. Define $A_1 := l_{-g}(A_0) \cup l_g(A_0)$. By assumption,

$$l_{-g}(B_0) \subseteq l_{-g}(\mathcal{C}(A_0)) = \mathcal{C}(l_{-g}(A_0)) \quad \text{and} \quad l_g(B_0) \subseteq l_g(\mathcal{C}(A_0)) = \mathcal{C}(l_g(A_0)),$$

whence

$$l_{-g}(B_0) \cup l_g(B_0) \subseteq \mathcal{C}(l_{-g}(A_0)) \cup \mathcal{C}(l_g(A_0)) \subseteq \mathcal{C}(l_{-g}(A_0) \cup l_g(A_0)) = \mathcal{C}(A_1).$$

For fixed $y_1, \dots, y_l \in \mathbb{R}$ with $|y_j| \leq K_1$,

$$\begin{aligned} S &= \{(0, \dots, 0, y_1, \dots, y_l, 0, \dots, 0) * \delta_\lambda(g) \mid \lambda \in [-1, 1]\} \\ &= \{(\lambda, 0, \dots, 0, y_1, \dots, y_l, 0, \dots, 0) \mid \lambda \in [-K_1, K_1]\} \end{aligned}$$

is a segment of an integral curve of a left invariant, horizontal vector field, and the endpoints of S are contained in $l_{-g}(B_0) \cup l_g(B_0)$. Thus the set B_1 consisting of pairs $(x, y) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ with $|x_1| \leq K_1$, $x_i = 0$ for $2 \leq i \leq d_1$, $|y_j| \leq K_1$ for $1 \leq j \leq l$ and $y_j = 0$ for $l+1 \leq j \leq d_2$ is contained in $\mathcal{C}(A_1)$.

Let $1 \leq \tilde{k} < k$ and suppose that there exist a finite set $A_{\tilde{k}} \subseteq \mathbb{G}$ and a constant $K_{\tilde{k}} > 0$ such that the set $B_{\tilde{k}}$ consisting of pairs $(x, y) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ with $|x_i| \leq K_{\tilde{k}}$ for $1 \leq i \leq \tilde{k}$, $x_i = 0$ for $\tilde{k}+1 \leq i \leq d_1$, $|y_j| \leq K_{\tilde{k}}$ for $1 \leq j \leq l$ and $y_j = 0$ for $l+1 \leq j \leq d_2$ is contained in $\mathcal{C}(A_{\tilde{k}})$. Let $0 < \epsilon \leq K_{\tilde{k}}$ and $g := (x, 0)$ with $x_{\tilde{k}+1} := \epsilon$ and $x_i = 0$ for $1 \leq i \leq d_1$ and $i \neq \tilde{k}+1$. Define $A_{\tilde{k}+1} := l_{-g}(A_{\tilde{k}}) \cup l_g(A_{\tilde{k}})$. By inductive hypothesis,

$$l_{-g}(B_{\tilde{k}}) \subseteq l_{-g}(\mathcal{C}(A_{\tilde{k}})) = \mathcal{C}(l_{-g}(A_{\tilde{k}})) \quad \text{and} \quad l_g(B_{\tilde{k}}) \subseteq l_g(\mathcal{C}(A_{\tilde{k}})) = \mathcal{C}(l_g(A_{\tilde{k}})),$$

whence

$$l_{-g}(B_{\tilde{k}}) \cup l_g(B_{\tilde{k}}) \subseteq \mathcal{C}(l_{-g}(A_{\tilde{k}})) \cup \mathcal{C}(l_g(A_{\tilde{k}})) \subseteq \mathcal{C}(l_{-g}(A_{\tilde{k}}) \cup l_g(A_{\tilde{k}})) = \mathcal{C}(A_{\tilde{k}+1}).$$

Let

$$h = (x_1, \dots, x_{\tilde{k}}, 0, \dots, 0, y_1, \dots, y_l, 0, \dots, 0) \in B_{\tilde{k}}.$$

In view of hypothesis (i), we have

$$(-g) * h = (x_1, \dots, x_{\tilde{k}}, -\epsilon, 0, \dots, 0, y_1, \dots, y_l, 0, \dots, 0) - R(x_1, \dots, x_{\tilde{k}}, \epsilon)$$

and

$$g * h = (x_1, \dots, x_{\tilde{k}}, \epsilon, 0, \dots, 0, y_1, \dots, y_l, 0, \dots, 0) + R(x_1, \dots, x_{\tilde{k}}, \epsilon),$$

where the first d_1 and last $d_2 - l$ coordinates of $R(x_1, \dots, x_{\tilde{k}}, \epsilon)$ vanish, and

$$\|R(x_1, \dots, x_{\tilde{k}}, \epsilon)\| \leq \beta K_{\tilde{k}} \epsilon$$

for some constant $\beta = \beta(\mathbb{G})$. Hence if we choose ϵ small enough, the pairs $(x, y) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ with $|x_i| \leq K_{\tilde{k}}$ for $1 \leq i \leq \tilde{k}$, $x_{\tilde{k}+1} = -\epsilon$, $x_i = 0$ for $\tilde{k} + 1 \leq i \leq d_1$, $|y_j| \leq K_{\tilde{k}}/2$ for $1 \leq j \leq l$ and $y_j = 0$ for $l + 1 \leq j \leq d_2$ are contained in $l_{-g}(B_{\tilde{k}})$. Similarly, the pairs $(x, y) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ with $|x_i| \leq K_{\tilde{k}}$ for $1 \leq i \leq \tilde{k}$, $x_{\tilde{k}+1} = \epsilon$, $x_i = 0$ for $\tilde{k} + 1 \leq i \leq d_1$, $|y_j| \leq K_{\tilde{k}}/2$ for $1 \leq j \leq l$ and $y_j = 0$ for $l + 1 \leq j \leq d_2$ are contained in $l_g(B_{\tilde{k}})$. For fixed

$$h = (x_1, \dots, x_{\tilde{k}}, 0, \dots, 0, y_1, \dots, y_l, 0, \dots, 0)$$

with $|x_i| \leq \eta$ for $1 \leq i \leq \tilde{k}$ and $|y_j| \leq K_{\tilde{k}}/4$ for $1 \leq j \leq l$,

$$S = \{h * \delta_\lambda(g) \mid \lambda \in [-1, 1]\}$$

is a segment of an integral curve of a left invariant, horizontal vector field. In view of hypothesis (i), we have

$$h * \delta_\lambda(g) = (x_1, \dots, x_{\tilde{k}}, \lambda\epsilon, 0, \dots, 0, y_1, \dots, y_l, 0, \dots, 0) + R(x_1, \dots, x_{\tilde{k}}, \lambda),$$

where the first d_1 and last $d_2 - l$ coordinates of $R(x_1, \dots, x_{\tilde{k}}, \lambda)$ vanish, and

$$\|R(x_1, \dots, x_{\tilde{k}}, \lambda)\| \leq \beta \eta \epsilon.$$

Hence, if η is sufficiently small, the endpoints of S are contained in $l_{-g}(B_{\tilde{k}}) \cup l_g(B_{\tilde{k}})$, whence $S \subseteq \mathcal{C}(A_{\tilde{k}+1})$, and the union of such segments contains the set consisting of pairs $(x, y) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ with

- (i) $|x_i| \leq \eta$ for $1 \leq i \leq \tilde{k}$,
- (ii) $|x_{\tilde{k}+1}| \leq \epsilon$,
- (iii) $x_i = 0$ for $\tilde{k} + 2 \leq i \leq d_1$,
- (iv) $|y_j| \leq \eta$ for $1 \leq j \leq l$ and
- (v) $y_j = 0$ for $l + 1 \leq j \leq d_2$.

Thus if $K_{\tilde{k}+1} = \min\{\eta, \epsilon\}$, then the set $B_{\tilde{k}+1}$ consisting of pairs $(x, y) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ with

- (i) $|x_i| \leq K_{\tilde{k}+1}$ for $1 \leq i \leq \tilde{k} + 1$,
- (ii) $x_i = 0$ for $\tilde{k} + 2 \leq i \leq d_1$,
- (iii) $|y_j| \leq K_{\tilde{k}+1}$ for $1 \leq j \leq l$ and
- (iv) $y_j = 0$ for $l + 1 \leq j \leq d_2$

is contained in $\mathcal{C}(A_{\tilde{k}+1})$. This concludes the induction step and the proof. \square

THEOREM 4.5. *Let \mathbb{G} be a stratified group of step two. Then \mathbb{G} is finitely h-convex.*

PROOF. Let $\mathfrak{g} = V_1 \oplus V_2$ be the given stratification of the Lie algebra of left invariant vector fields on \mathbb{G} . Let (X_1, \dots, X_{d_1}) be a basis of V_1 such that $[X_1, X_2] \neq 0$. Set $l_1 := 0$. Clearly, we can find a basis (Y_1, \dots, Y_{d_2}) of V_2 with the following properties:

- (i) There exist integers $1 = l_2 \leq \dots \leq l_{d_1} = d_2$ such that (Y_1, \dots, Y_{l_k}) is a basis of $\text{span}_{\mathbb{R}}\{[X_i, X_j] \mid 1 \leq i, j \leq k\}$ for each $2 \leq k \leq d_1$.
- (ii) If $2 \leq k \leq d_1$, $l_{k-1} < l_k$ and $l_{k-1} < j \leq l_k$, there is $i_j \in \{1, \dots, k-1\}$ such that $[X_k, X_{i_j}] = Y_j$, and $l_{k-1} < j_1 < j_2 \leq l_k$ implies $i_{j_1} < i_{j_2}$.

We identify \mathbb{G} with $(\mathbb{R}^d, \text{ast}) \equiv (\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}, *)$ in the usual way with respect to the basis $(X_1, \dots, X_{d_1}, Y_1, \dots, Y_{d_2})$. We claim that for each $2 \leq k \leq d_1$ there exists a finite set $F_k \subseteq \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ and a constant $\kappa_k > 0$ such that the set

$$\left\{ (0, y) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \mid |y_j| \leq \kappa_k \text{ for } 1 \leq j \leq l_k, y_j = 0 \text{ for } l_k + 1 \leq j \leq d_2 \right\}$$

is contained in $\mathcal{C}(F_k)$. The theorem then follows from the claim in the case $k = d_1$ via Lemma 4.4.

Let $k = 2$, $\kappa_2 := 1$. The sets

$$S_1 = \{(-2, 0, \dots, 0) * (0, \lambda, 0, \dots, 0) \mid \lambda \in [-1, 1]\}$$

and

$$S_2 = \{(2, 0, \dots, 0) * (0, \lambda, 0, \dots, 0) \mid \lambda \in [-1, 1]\}$$

are segments of integral curves of left invariant, horizontal vector fields which are contained in the h-convex closure of

$$F_2 = \{g_1, g_2, g_3, g_4\},$$

where

$$\begin{aligned} g_1 &= (-2, 0, \dots, 0) * (0, -1, 0, \dots, 0), & g_2 &= (-2, 0, \dots, 0) * (0, 1, 0, \dots, 0), \\ g_3 &= (2, 0, \dots, 0) * (0, -1, 0, \dots, 0), & g_4 &= (2, 0, \dots, 0) * (0, 1, 0, \dots, 0). \end{aligned}$$

For each $y_1 \in \mathbb{R}$ with $|y_1| \leq \kappa_2$,

$$\begin{aligned} S &= \{(0, \dots, 0, y_1, 0, \dots, 0) * \delta_\lambda(2, y_1, 0, \dots, 0) \mid \lambda \in [-1, 1]\} \\ &= \{(\lambda 2, \lambda y_1, 0, \dots, 0, y_1, 0, \dots, 0) \mid \lambda \in [-1, 1]\} \end{aligned}$$

is a segment of an integral curve of a left invariant, horizontal vector field, and the endpoints of S are contained in $S_1 \cup S_2 \subseteq \mathcal{C}(F_2)$. Thus $S \subseteq \mathcal{C}(F_2)$. Since $(0, \dots, 0, y_1, 0, \dots, 0)$ belongs to S , it follows that the set

$$\{(0, \dots, 0, y_1, 0, \dots, 0) \mid |y_1| \leq \kappa_2\}$$

is contained in $\mathcal{C}(F_2)$, which verifies the claim in the case $k = 2$.

Let $2 \leq k < d_1$. Suppose that there exists a finite set $F_k \subseteq \mathbb{G}$ and a constant $\kappa_k > 0$ such that the set

$$\left\{ (0, y) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \mid |y_j| \leq \kappa_k \text{ for } 1 \leq j \leq l_k, y_j = 0 \text{ for } l_k + 1 \leq j \leq d_2 \right\}$$

is contained in $\mathcal{C}(F_k)$. If $l_{k+1} = l_k$, the claim is also verified for $k + 1$ and there is nothing to show. Assume therefore $\Delta := l_{k+1} - l_k > 0$. By choice of Y_1, \dots, Y_{d_2} , there exist $1 \leq i_1 < \dots < i_\Delta \leq k$ such that $[X_{i_j}, X_{k+1}] = Y_{l_k+j}$ for $1 \leq j \leq \Delta$. In view of Lemma 4.4, there exists a finite set $A_k \subseteq \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ and a constant $0 < K_k \leq \kappa_k$ such that the set B consisting of the pairs $(x, y) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ with

- (i) $|x_i| \leq K_k$ for $i \in \{i_1, \dots, i_\Delta\}$,
- (ii) $x_i = 0$ for $i \in \{1, \dots, d_1\} \setminus \{i_1, \dots, i_\Delta\}$,
- (iii) $|y_j| \leq K_k$ for $1 \leq j \leq l_k$ and
- (iv) $y_j = 0$ for $l_k + 1 \leq j \leq d_2$

is contained in $\mathcal{C}(A_k)$. Let $\kappa_{k+1} := K_k$, $g = (0, \dots, 0, x_{k+1}, 0, \dots, 0)$ with $x_{k+1} = 2$ and define $F_{k+1} := l_{-g}(A_k) \cup l_g(A_k)$. Then $l_{-g}(B) \cup l_g(B)$ is contained in

$$l_{-g}(\mathcal{C}(A_k)) \cup l_g(\mathcal{C}(A_k)) = \mathcal{C}(l_{-g}(A_k)) \cup \mathcal{C}(l_g(A_k)) \subseteq \mathcal{C}(l_{-g}(A_k) \cup l_g(A_k)) = \mathcal{C}(F_{k+1}).$$

Notice that the set $l_{-g}(B)$ consists of the pairs $(x, y) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ with

- (i) $|x_i| \leq \kappa_{k+1}$ for $i \in \{i_1, \dots, i_\Delta\}$,
- (ii) $x_{k+1} = -2$,
- (iii) $x_i = 0$ for $i \in \{1, \dots, d_1\} \setminus \{i_1, \dots, i_\Delta, k+1\}$,
- (iv) $|y_j| \leq \kappa_{k+1}$ for $1 \leq j \leq l_k$,
- (v) $y_j = -x_{i_j}$ for $l_k + 1 \leq j \leq l_{k+1}$ and
- (vi) $y_j = 0$ for $l_{k+1} + 1 \leq j \leq d_2$.

Similarly, the set $l_g(B)$ consists of the pairs $(x, y) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ with

- (i) $|x_i| \leq \kappa_{k+1}$ for $i \in \{i_1, \dots, i_\Delta\}$,

- (ii) $x_{k+1} = 2$,
- (iii) $x_i = 0$ for $i \in \{1, \dots, d_1\} \setminus \{i_1, \dots, i_\Delta, k+1\}$,
- (iv) $|y_j| \leq \kappa_{k+1}$ for $1 \leq j \leq l_k$,
- (v) $y_j = x_{i_j}$ for $l_k + 1 \leq j \leq l_{k+1}$ and
- (vi) $y_j = 0$ for $l_{k+1} + 1 \leq j \leq d_2$.

For $j = 1, \dots, l_{k+1}$, fix $y_j \in \mathbb{R}$ with $|y_j| \leq \kappa_{k+1}$. The set

$$\begin{aligned} S &= \{(0, \dots, 0, y_1, \dots, y_{l_{k+1}}, 0, \dots, 0) * \delta_\lambda(x_1, \dots, x_{d_1}, 0, \dots, 0) \mid \lambda \in [-1, 1]\} \\ &= \{(\lambda x_1, \dots, \lambda x_{d_1}, y_1, \dots, y_{l_{k+1}}, 0, \dots, 0) \mid \lambda \in [-1, 1]\}, \end{aligned}$$

where $x_i = y_j$ if $i = i_j$ for some $l_k + 1 \leq j \leq l_{k+1}$, $x_{k+1} = 2$ and $x_i = 0$ otherwise, is a segment of an integral curve of a left invariant, horizontal vector field, and the endpoints of S belong to $l_{-g}(B) \cup l_g(B)$. Thus $S \subseteq \mathcal{C}(F_{k+1})$. Since $(0, \dots, 0, y_1, \dots, y_{l_{k+1}}, 0, \dots, 0)$ belongs to S , the set

$$\{(0, y) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \mid |y_j| \leq \kappa_{k+1}, 1 \leq j \leq l_{k+1}, y_j = 0, l_{k+1} + 1 \leq j \leq d_2\}$$

is contained in $\mathcal{C}(F_{k+1})$. This concludes the induction step and the proof. \square

The main result of this chapter is an immediate consequence of Theorem 4.5, Lemma 4.3, Lemma 4.1 and Proposition 4.2:

THEOREM 4.6. *If \mathbb{G} is a stratified group of step two and $\Omega \subseteq \mathbb{G}$ is an h-convex, open subset, then every h-convex function $u : \Omega \rightarrow \mathbb{R}$ is locally Lipschitz continuous with respect to an intrinsic metric on \mathbb{G} .*

3. The Engel group is not finitely h-convex

We have introduced the Engel group

$$\mathbb{E} = (\mathbb{R}^4, *) = (\{(x_1, x_2, y, z) \mid x_1, x_2, y, z \in \mathbb{R}\}, *)$$

in §2 of the first chapter. Recall that the group law is given by the formula

$$(x_1, x_2, y, z) * (x'_1, x'_2, y', z') = (x_1 + x'_1, x_2 + x'_2, y + y', z + z') + P$$

for all $(x_1, x_2, y, z), (x'_1, x'_2, y', z') \in \mathbb{R}^4$, where

$$P = \left(0, 0, \frac{(x_1 x'_2 - x_2 x'_1)}{2}, \frac{(x_1 y' - y x'_1)}{2} + \frac{(x_1 - x'_1)(x_1 x'_2 - x_2 x'_1)}{12}\right).$$

If X_1, X_2, Y, Z denote the left invariant vector fields uniquely determined by the conditions

$$X_1(0) = \partial_{x_1}(0), X_2(0) = \partial_{x_2}(0), Y(0) = \partial_y(0), Z(0) = \partial_z(0),$$

then

$$\text{span}_{\mathbb{R}}\{X_1, X_2\} \oplus \text{span}_{\mathbb{R}}\{Y\} \oplus \text{span}_{\mathbb{R}}\{Z\}$$

is a stratification of the Lie algebra of left invariant vector fields on \mathbb{E} . Notice that

$$\gamma : \mathbb{R} \rightarrow \mathbb{R}^4, \gamma(t) = g * (tx_1, tx_2, 0, 0)$$

is the integral curve of the left invariant, horizontal vector field $X = x_1 X_1 + x_2 X_2$ which passes through $g \in \mathbb{R}^4$ at time 0.

LEMMA 4.7. *Let Γ_1, Γ_2 be integral curves of left invariant, horizontal vector fields on $(\mathbb{R}^4, *)$. Define $M_1 := (\bigcup \Gamma) \setminus \Gamma_1$, where the union is taken over all integral curves of left invariant, horizontal vector fields which intersect Γ_1 . If Γ_2 has more than two distinct intersections with M_1 , then Γ_2 intersects Γ_1 . Consequently $\text{card}(\Gamma_2 \cap (M_1 \cup \Gamma_1)) \leq 2$ if $\Gamma_1 \cap \Gamma_2 = \emptyset$.*

PROOF. Notice first that by left translation, it suffices to prove the statement in the case where Γ_1 passes through 0. We will only consider the case $\Gamma_1 = \{(\mu, \alpha\mu, 0, 0) \mid \mu \in \mathbb{R}\}$ for some $\alpha \in \mathbb{R}$. The computations in the case $\Gamma_1 = \{(0, \mu, 0, 0) \mid \mu \in \mathbb{R}\}$ are similar (but easier). We have

$$\begin{aligned} (\mu, \alpha\mu, 0, 0) * (x_1, x_2, 0, 0) &= \left(\mu + x_1, \alpha\mu + x_2, \frac{\mu(x_2 - \alpha x_1)}{2}, \frac{(\mu - x_1)\mu(x_2 - \alpha x_1)}{12} \right) \\ &= (u, v, w, f(u, v, w)), \end{aligned}$$

with $u = \mu + x_1$, $v = \alpha\mu + x_2$, $w = \mu(x_2 - \alpha x_1)/2$. A short computation gives

$$M_1 = \left\{ \left(u, v, w, \frac{w}{6} \left(\frac{4w}{v - \alpha u} - u \right) \right) \mid u, v, w \in \mathbb{R}, v - \alpha u \neq 0 \right\}.$$

Let $(x_1, x_2, y, z) \in \mathbb{R}^4$ and suppose that Γ_2 passes through (x_1, x_2, y, z) . As above, we will only consider the case $\Gamma_2 = \{(x_1, x_2, y, z) * (\lambda, \beta\lambda, 0, 0) \mid \lambda \in \mathbb{R}\}$ for some $\beta \in \mathbb{R}$. The computations in the case $\Gamma_2 = \{(x_1, x_2, y, z) * (0, \lambda, 0, 0) \mid \lambda \in \mathbb{R}\}$ are again similar and easier. We have

$$\begin{aligned} \Gamma_2 &= \{(x_1, x_2, y, z) * (\lambda, \beta\lambda, 0, 0) \mid \lambda \in \mathbb{R}\} \\ &= \left\{ \left(x_1 + \lambda, x_2 + \beta\lambda, y + \frac{\lambda(x_1\beta - x_2)}{2}, z + \frac{(-y)\lambda}{2} + \frac{(x_1 - \lambda)\lambda(x_1\beta - x_2)}{12} \right) \mid \lambda \in \mathbb{R} \right\}. \end{aligned}$$

Suppose first that $x_1\beta - x_2 = 0$. By hypothesis,

$$\frac{y}{6} \left(\frac{4y}{x_2 + \beta\lambda - \alpha x_1 - \alpha\lambda} - (x_1 + \lambda) \right) = z + \frac{(-y)\lambda}{2}$$

holds for at least three distinct values of λ . After simplification of this expression, we obtain

$$a_2\lambda^2 + a_1\lambda + a_0 = 0$$

for at least three distinct values of λ , where

$$\begin{aligned} a_0 &= 4y^2 + (\alpha x_1 - x_2)(x_1 y + 6z) \\ a_1 &= 2y(x_2 - \alpha x_1) + (\alpha - \beta)(x_1 y + 6z) \\ a_2 &= 2y(\beta - \alpha). \end{aligned}$$

If $\alpha - \beta = 0$, then $\alpha x_1 - x_2 = 0$, $y = 0$, and thus

$$\Gamma_2 = \{(x_1 + \lambda, \alpha x_1 + \alpha\lambda, 0, z) \mid \lambda \in \mathbb{R}\}.$$

Hence $\Gamma_2 \cap M_1 = \emptyset$, a contradiction. This forces $\alpha - \beta \neq 0$, which implies $y = z = 0$ and

$$\Gamma_2 = \{(x_1, \beta x_1, 0, 0) * (\lambda, \beta\lambda, 0, 0) \mid \lambda \in \mathbb{R}\} = \{(x_1 + \lambda, \beta(x_1 + \lambda), 0, 0) \mid \lambda \in \mathbb{R}\}.$$

Thus $0 \in \Gamma_1 \cap \Gamma_2$, and the claim follows.

Suppose now that $x_1\beta - x_2 \neq 0$. Then Γ_2 intersects the hyperplane

$$\{(x'_1, x'_2, y', z') \in \mathbb{R}^4 \mid y' = 0\}$$

at some point $(x'_1, x'_2, 0, z')$, and we can write

$$\begin{aligned} \Gamma_2 &= \{(x'_1, x'_2, 0, z') * (\lambda, \beta\lambda, 0, 0) \mid \lambda \in \mathbb{R}\} \\ &= \left\{ \left(x'_1 + \lambda, x'_2 + \beta\lambda, \frac{\lambda(x'_1\beta - x'_2)}{2}, z' + \frac{(x'_1 - \lambda)\lambda(x'_1\beta - x'_2)}{12} \right) \mid \lambda \in \mathbb{R} \right\} \end{aligned}$$

with $x'_1\beta - x'_2 \neq 0$. By hypothesis,

$$\frac{\lambda(x'_1\beta - x'_2)}{12} \left(\frac{2\lambda(x'_1\beta - x'_2)}{x'_2 + \beta\lambda - \alpha x'_1 - \alpha\lambda} - (x'_1 + \lambda) \right) = z' + \frac{(x'_1 - \lambda)\lambda(x'_1\beta - x'_2)}{12},$$

holds for at least three distinct values of λ . After simplification of this expression, we obtain

$$a_2\lambda^2 + a_1\lambda + a_0 = 0$$

for at least three distinct values of λ , where

$$\begin{aligned} a_0 &= 12z'(x'_2 - \alpha x'_1) \\ a_1 &= 2(x'_1\beta - x'_2)x'_1(\alpha x'_1 - x'_2) + 12z'(\alpha - \beta) \\ a_2 &= 2(x'_1\beta - x'_2)^2 + 2(x'_1\beta - x'_2)x'_1(\alpha - \beta). \end{aligned}$$

If $z' \neq 0$, then $x'_2 - \alpha x'_1 = \beta - \alpha = 0$, and thus

$$\Gamma_2 = \{(x'_1 + \lambda, \alpha x'_1 + \alpha\lambda, 0, z') \mid \lambda \in \mathbb{R}\}.$$

Hence $\Gamma_2 \cap M_1 = \emptyset$, a contradiction. This forces $z' = 0$ and thus

$$(x'_1\beta - x'_2)x'_1(\alpha x'_1 - x'_2) = 0.$$

If $x'_1 = 0$, then $x'_2 = 0$, whence $x'_1\beta - x'_2 = 0$. This contradiction forces $\alpha x'_1 - x'_2 = 0$, which implies

$$\Gamma_2 = \{(x'_1, \alpha x'_1, 0, 0) * (\lambda, \beta\lambda, 0, 0) \mid \lambda \in \mathbb{R}\}.$$

Thus $(x'_1, \alpha x'_1, 0, 0) \in \Gamma_1 \cap \Gamma_2$, and the claim follows. \square

LEMMA 4.8. *Let S_1 and S_2 be bounded, closed, intersecting segments (possibly points) of distinct integral curves Γ_1, Γ_2 of left invariant, horizontal vector fields. Let S be a bounded, closed segment of an integral curve of a left invariant, horizontal vector field, and suppose that one endpoint g_1 of S belongs to S_1 and the other endpoint g_2 belongs to S_2 . Then $S \subseteq S_1$ or $S \subseteq S_2$.*

PROOF. After a left translation, we can assume

$$0 \in S_1 \cap S_2, \quad \Gamma_1 = \{(\lambda x_1, \lambda x_2, 0, 0) \mid \lambda \in \mathbb{R}\} \quad \text{and} \quad \Gamma_2 = \{(\lambda x'_1, \lambda x'_2, 0, 0) \mid \lambda \in \mathbb{R}\}$$

for suitable, linearly independent $(x_1, x_2), (x'_1, x'_2) \in \mathbb{R}^2$ with $(x_1^2 + x_2^2) = (x'_1{}^2 + x'_2{}^2) = 1$. We have

$$g_1 = (\lambda x_1, \lambda x_2, 0, 0)$$

for some $\lambda \in \mathbb{R}$ and

$$\begin{aligned} g_2 &= g_1 * (u_1, u_2, 0, 0) = (\lambda x_1, \lambda x_2, 0, 0) * (u_1, u_2, 0, 0) \\ &= \left(\lambda x_1 + u_1, \lambda x_2 + u_2, \frac{\lambda(x_1 u_2 - x_2 u_1)}{2}, \frac{(\lambda x_1 - u_1)\lambda(x_1 u_2 - x_2 u_1)}{12} \right) \end{aligned}$$

for some $(u_1, u_2) \in \mathbb{R}^2$. $g_2 \in \Gamma_2$ implies that $\lambda = 0$ or that (x_1, x_2) and (u_1, u_2) are linearly dependent. If $\lambda = 0$, then $g_1 = 0 \in S_2$. If (x_1, x_2) and (u_1, u_2) are linearly dependent, then $g_2 \in \Gamma_1$, forcing $g_2 = 0 \in S_1$ since $\Gamma_1 \cap \Gamma_2 = \{0\}$. \square

LEMMA 4.9. *If $A \subseteq \mathbb{E}$ is a finite union of bounded, closed segments S_1, \dots, S_k (possibly points) of integral curves $\Gamma_1, \dots, \Gamma_k$ of left invariant, horizontal vector fields, then $H(A)$ is contained in a finite union of bounded, closed segments (possibly points) of integral curves of left invariant, horizontal vector fields.*

PROOF. Enlarging A if necessary, we can assume that

- (i) $\Gamma_1, \dots, \Gamma_k$ are all distinct and
- (ii) $\Gamma_i \cap \Gamma_j \neq \emptyset$ for some $i, j \in \{1, \dots, k\}$ implies $S_i \cap S_j \neq \emptyset$.

Note that $H(A)$ is the union of all bounded, closed segments of integral curves of left invariant, horizontal vector fields with endpoints in A . Hence a bounded, closed segment S of an integral curve of some left invariant, horizontal vector field is contained in $H(A)$ if and only if there exist $i, j \in \{1, \dots, k\}$ such that one endpoint g_i of S belongs to S_i and

the other endpoint g_j belongs to S_j . We can assume $i \neq j$, for otherwise $S \subseteq S_i = S_j$ since S is determined by its endpoints.

If $\Gamma_i \cap \Gamma_j \neq \emptyset$, then $S_i \cap S_j \neq \emptyset$ and $S \subseteq S_i$ or $S \subseteq S_j$ by Lemma 4.8.

If $\Gamma_i \cap \Gamma_j = \emptyset$, then S is one out of four (at most) possible segments. Indeed, by virtue of Lemma 4.7, g_i is one out of two (at most) possible intersection points of Γ_i with $M_j := (\bigcup \Gamma) \setminus \Gamma_j$. Similarly, g_j is one out of two (at most) possible intersection points of Γ_j with $M_i := (\bigcup \Gamma) \setminus \Gamma_i$. Since S is determined by its endpoints, the claim follows. \square

In view of $\mathcal{C}(A) = H^\infty(A) = \bigcup_{k \in \mathbb{N}_0} H^k(A)$, the following theorem is an immediate consequence of Lemma 4.9.

THEOREM 4.10. *If $F \subseteq \mathbb{E}$ is finite, then the h-convex closure $\mathcal{C}(F)$ of F is contained in a countable union of bounded, closed segments (possibly points) of integral curves of left invariant, horizontal vector fields.*

Geometric and measure-theoretic properties of h-convex sets

In the first section of this chapter, we prove that the upper density of an h-convex, measurable set at boundary points is uniformly bounded away from one. This fundamental estimate has several interesting consequences, which will be discussed in chapter six. In the second section, we show that the horizontal perimeter of h-convex, measurable sets is locally finite. Hence the rectifiability theory of Franchi, Serapioni and Serra Cassano can be applied to h-convex, measurable subsets of stratified groups of step two. In the last section, we exhibit an h-convex subset of the first Heisenberg group which is not measurable. This shows that the measurability assumptions in the preceding results cannot be removed.

1. Upper density bound at the boundary of h-convex sets

This section is entirely devoted to the proof of the estimate (5.5) at boundary points of h-convex, measurable sets. We start with preparatory considerations about the size of the characteristic set of a smooth submanifold of \mathbb{R}^d with respect to given smooth vector fields. We use Theorem 5.3 in order to prove Lemma 5.4. The main result of this chapter, Theorem 5.5, is a straightforward consequence of this lemma.

DEFINITION 5.1. Given smooth vector fields X_1, \dots, X_n on \mathbb{R}^d and a smooth, imbedded, m -dimensional submanifold M^m ($1 \leq m < d$), define the *characteristic set* of M^m with respect to the vector fields X_1, \dots, X_n to be the set

$$C(M^m) = \{p \in M^m \mid X_i(p) \in T_p M^m, i = 1, \dots, n\}.$$

Characteristic points have been extensively studied because of their fundamental importance in several problems of geometry and analysis related to systems of vector fields satisfying Hörmander's condition. However, in our setting, the most basic estimate of the size of the characteristic locus suffices: if X_1, \dots, X_n and their commutators of order at most s span $T_x \mathbb{R}^d$ at each $x \in \mathbb{R}^d$, then $\mathcal{H}_E^m(C(M^m)) = 0$ (recall that \mathcal{H}_E^m denotes m -dimensional Hausdorff measure with respect to the Euclidean metric on \mathbb{R}^d). The idea of the proof is borrowed from Derridj (cf. [26]), who proves the above statement for smooth submanifolds of codimension 1.

LEMMA 5.1. *Let $Y = \sum_{i=1}^d a_i \partial_i$, $Z = \sum_{j=1}^d b_j \partial_j$ be smooth vector fields on \mathbb{R}^d . Let $1 \leq m < d$, $\mathbb{R}^m = \{x \in \mathbb{R}^d \mid x_i = 0, m+1 \leq i \leq d\}$, and write $[Y, Z] = \sum_{k=1}^d c_k \partial_k$. The set C consisting of the points $x \in \mathbb{R}^m$ such that $a_i(x) = b_j(x) = 0$ for all $m < i, j \leq d$ and $c_k(x) \neq 0$ for some $m < k \leq d$ has vanishing \mathcal{H}_E^m measure.*

PROOF. We compute

$$\begin{aligned} [Y, Z] &= \sum_{i=1}^d \sum_{j=1}^d a_i \partial_i (b_j \partial_j) - b_j \partial_j (a_i \partial_i) = \sum_{i=1}^d \sum_{j=1}^d a_i (\partial_i b_j) \partial_j - b_j (\partial_j a_i) \partial_i \\ &= \sum_{k=1}^d \left(\sum_{l=1}^d a_l (\partial_l b_k) - b_l (\partial_l a_k) \right) \partial_k. \end{aligned}$$

Thus $c_k = \sum_{l=1}^d a_l(\partial_l b_k) - b_l(\partial_l a_k)$. For $m < k \leq d$, $1 \leq l \leq m$, let us consider the sets

$$A_{k,l} = \{x \in \mathbb{R}^m \mid a_k(x) = 0, \partial_l a_k(x) \neq 0\}$$

and

$$B_{k,l} = \{x \in \mathbb{R}^m \mid b_k(x) = 0, \partial_l b_k(x) \neq 0\}.$$

Since $C \subseteq \bigcup_{m < k \leq d, 1 \leq l \leq m} (A_{k,l} \cup B_{k,l})$, it suffices to show that $\mathcal{H}_E^m(A_{k,l})$ and $\mathcal{H}_E^m(B_{k,l})$ vanish for $m < k \leq d$ and $1 \leq l \leq m$. Let us prove $\mathcal{H}_E^m(A_{d,m}) = 0$ for instance. Consider $\mathbb{R}^{m-1} = \{x \in \mathbb{R}^d \mid x_i = 0, m \leq i \leq d\}$. By Fubini's theorem,

$$\mathcal{H}_E^m(A_{d,m}) = \int_{\mathbb{R}^{m-1}} \mathcal{H}_E^1(\{x + te_m \mid t \in \mathbb{R}\} \cap A_{d,m}) d\mathcal{H}_E^{m-1}(x).$$

Fix $x \in \mathbb{R}^{m-1}$. The set

$$\{t \in \mathbb{R} \mid a_d(x + te_m) = 0, \partial_m a_d(x + te_m) \neq 0\}$$

consists of isolated points. Therefore

$$\mathcal{H}_E^1(\{x + te_m \mid t \in \mathbb{R}\} \cap A_{d,m}) = 0,$$

whence $\mathcal{H}_E^m(A_{d,m}) = 0$. □

Given smooth vector fields X_1, \dots, X_n in \mathbb{R}^d and a multiindex I of length $|I| = l$, i.e. $I \in \{1, \dots, n\}^l$, we let $X_I := X_i$ if $l = 1$ and $I = (i)$, $X_I := [X_{(i_1, \dots, i_{l-1})}, X_{i_l}]$ if $l \geq 2$ and $I = (i_1, \dots, i_l)$, and we write $X_I = \sum_{j=1}^d a_{I,j} \partial_j$.

LEMMA 5.2. *Let X_1, \dots, X_n be smooth vector fields on \mathbb{R}^d . Fix $1 \leq m < d$ and consider $\mathbb{R}^m = \{x \in \mathbb{R}^d \mid x_i = 0, m+1 \leq i \leq d\}$. Then, for all $s \in \mathbb{N}$ with $s \geq 2$, the set C consisting of the points $x \in \mathbb{R}^m$ such that*

- (i) $a_{(i),j}(x) = 0$ for all $i = 1, \dots, n$ and $m < j \leq d$ and
- (ii) $a_{I_0, j_0}(x) \neq 0$ for some I_0 with $|I_0| \leq s$ and some $m < j_0 \leq d$

has vanishing \mathcal{H}_E^m measure.

PROOF. Let $s \in \mathbb{N}$, $s \geq 2$. For $l = 1, \dots, s-1$, let C_l be the set of points $x \in \mathbb{R}^m$ such that

- (i) $a_{I,j}(x) = 0$ for all I with $|I| \leq l$ and all $m < j \leq d$ and
- (ii) $a_{I_0, j_0}(x) \neq 0$ for some I_0 with $|I_0| = l+1$ and some $m < j_0 \leq d$.

Clearly $C = \bigcup_{l=1}^{s-1} C_l$. Hence it is enough to show $\mathcal{H}_E^m(C_l) = 0$ for $1 \leq l \leq s-1$.

For each $1 \leq l \leq s-1$, there exists $N = N(l) \in \mathbb{N}$ and smooth vector fields Y_1, \dots, Y_N on \mathbb{R}^d such that $\{Y_k \mid k = 1, \dots, N\} = \{X_I \mid |I| \leq l\}$ and $(Y_1, \dots, Y_N) = (X_1, \dots, X_n)$. Let us write $Y_k = \sum_{j=1}^d b_{k,j} \partial_j$ and $[Y_{k_1}, Y_{k_2}] = \sum_{j=1}^d c_{k_1, k_2, j} \partial_j$. Observe that

$$C_l \subseteq \bigcup_{1 \leq k_1 \leq N, 1 \leq k_2 \leq N} C_{k_1, k_2},$$

where C_{k_1, k_2} is the set of points $x \in \mathbb{R}^m$ such that

- (i) $b_{k_1, j}(x) = b_{k_2, j}(x) = 0$ for $m < j \leq d$ and
- (ii) $c_{k_1, k_2, j_0}(x) \neq 0$ for some $m < j_0 \leq d$.

We have $\mathcal{H}_E^m(C_{k_1, k_2}) = 0$ for all $1 \leq k_1 \leq N$ and $1 \leq k_2 \leq N$ by Lemma 5.1, whence $\mathcal{H}_E^m(C_l) = 0$, as required. □

THEOREM 5.3. *Let M^m be a smooth, m -dimensional, imbedded submanifold of \mathbb{R}^d . Let X_1, \dots, X_n be smooth vector fields on \mathbb{R}^d such that the subspace of $T_x \mathbb{R}^d$ spanned by the commutators of order at most s has dimension d at each $x \in \mathbb{R}^d$. Then $\mathcal{H}_E^m(C(M^m)) = 0$.*

PROOF. Clearly, it suffices to show that for each $p \in M^m$, there exists an open neighbourhood U of p in M^m such that $\mathcal{H}_E^m(C(M^m) \cap U) = 0$. Fix $p \in M^m$. Let V be an open neighbourhood of p in \mathbb{R}^d and let $\varphi : V \rightarrow \mathbb{R}^d$ be a diffeomorphism such that $\varphi(M^m \cap V) = \mathbb{R}^m = \{x \in \mathbb{R}^d \mid x_i = 0, m < i \leq d\}$. The existence of such charts is a consequence of the implicit function theorem (see e.g. [90, Proposition 1.35]).

Let us write $\tilde{X}_i = d\varphi(X_i)$, and observe that for each multiindex $I \in \{1, \dots, n\}^l$ of length $|I| = l$, we have $\tilde{X}_I = d\varphi(X_I)$ ([90, Proposition 1.55]). In particular,

$$\text{span}_{\mathbb{R}} \left\{ \tilde{X}_I(x) \mid |I| \leq s \right\} = T_x \mathbb{R}^d$$

at each $x \in \mathbb{R}^d$. Hence $\varphi(C(M^m) \cap V)$ is the set of points $x \in \varphi(M^m \cap V)$ such that $\tilde{X}_i(x) \in T_x \mathbb{R}^m$ for all $i = 1, \dots, n$ and $\tilde{X}_I(x) \notin T_x \mathbb{R}^m$ for some I with $|I| \leq s$. By Lemma 5.2, we have $\mathcal{H}_E^m(\varphi(C(M^m) \cap V)) = 0$ and consequently $\mathcal{H}_E^m(C(M^m) \cap V) = 0$. \square

LEMMA 5.4. *Let $\mathbb{G} \equiv (\mathbb{R}^d, *)$ be a stratified group. Let $C \subseteq \mathbb{R}^d$ be h-convex with $0 \in \mathbb{R}^d \setminus C$. There exist smooth, imbedded submanifolds M^1, M^2, \dots, M^d such that*

- (i) M^m is m -dimensional,
- (ii) $M^m \subseteq B(0, m/d)$,
- (iii) $\mathcal{H}_E^m(M^m) < +\infty$ and
- (iv) $\mathcal{H}_E^m(M^m \setminus C) \geq c_m$

for $m = 1, \dots, d$, where $0 < c_m = c_m(\mathbb{G}) \leq \mathcal{H}_E^m(M^m)$ does not depend on C .

PROOF. Let (X_1, \dots, X_{d_1}) be a basis of the first layer in the given stratification of the Lie algebra of left invariant vector fields on $(\mathbb{R}^d, *)$. Given $x_0 \in \mathbb{R}^d \setminus C$ and $1 \leq j \leq d_1$, let $\gamma : \mathbb{R} \rightarrow \mathbb{R}^d$ denote the integral curve of the vector field X_j which passes through x_0 at time 0. By definition of h-convexity,

$$(5.1) \quad \gamma((-\infty, 0]) \cap C = \emptyset \quad \text{or} \quad \gamma([0, +\infty)) \cap C = \emptyset.$$

Now let $\gamma : \mathbb{R} \rightarrow \mathbb{R}^d$ denote the integral curve of the vector field X_1 which passes through 0 at time 0. Choose $\eta > 0$ such that $\gamma((- \eta, \eta)) \subseteq B(0, 1/d)$. By (5.1), we have

$$\gamma((- \eta, 0]) \cap C = \emptyset \quad \text{or} \quad \gamma([0, \eta)) \cap C = \emptyset.$$

We let $c_1 := \mathcal{H}_E^1(\gamma((- \eta, 0)))$ in the first case and $c_1 := \mathcal{H}_E^1(\gamma([0, \eta)))$ in the second. Clearly, $M^1 := \gamma((- \eta, \eta))$ has the desired properties.

Let $1 \leq m < d$ and suppose we had already constructed smooth, imbedded submanifolds M^1, M^2, \dots, M^m which satisfy our claims. Define $A^m := M^m \setminus C$. For $1 \leq j \leq d_1$, $k \in \mathbb{N}$, denote by $F_{j,k}^m$ the closed set consisting of points $p \in M^m$ such that

$$\max \left\{ \left(\frac{X_j(p)}{(X_j(p), X_j(p))^{\frac{1}{2}}}, Y(p) \right) \mid Y(p) \in (T_p M^m)^\perp, (Y(p), Y(p)) = 1 \right\} \leq \frac{1}{k}.$$

Let $F_k^m := \bigcap_{j=1}^{d_1} F_{j,k}^m$. By Theorem 5.3, we have

$$\lim_{k \rightarrow +\infty} \mathcal{H}_E^m(F_k^m) = \mathcal{H}_E^m \left(\bigcap_{k \in \mathbb{N}} F_k^m \right) = \mathcal{H}_E^m(C(M^m)) = 0.$$

Choose

$$\frac{\mathcal{H}_E^m(M^m)}{\mathcal{H}_E^m(M^m) + c_m/d_1} < q < 1.$$

We can pick $k \in \mathbb{N}$ such that

$$\mathcal{H}_E^m(F_k^m) \leq (1 - q)c_m,$$

whence

$$\mathcal{H}_E^m(A^m \setminus F_k^m) \geq qc_m,$$

which implies that

$$(5.2) \quad \mathcal{H}_E^m (A^m \setminus F_{j,k}^m) \geq qc_m/d_1$$

for some $1 \leq j \leq d_1$. There exists $\Omega_j^m \in M^m \setminus F_{j,k}^m$ such that

$$\mathcal{H}_E^m (\Omega_j^m) \geq q\mathcal{H}_E^m (M^m \setminus F_{j,k}^m),$$

whence

$$\mathcal{H}_E^m ((M^m \setminus F_{j,k}^m) \setminus \Omega_j^m) \leq (1-q)\mathcal{H}_E^m (M^m \setminus F_{j,k}^m).$$

It follows that

$$(5.3) \quad \mathcal{H}_E^m (A^m \cap \Omega_j^m) \geq \mathcal{H}_E^m (A^m \setminus F_{j,k}^m) - (1-q)\mathcal{H}_E^m (M^m \setminus F_{j,k}^m).$$

Given $p \in \Omega_j^m$, let $\gamma_{j,p} : \mathbb{R} \rightarrow \mathbb{R}^d$ be the integral curve of X_j which passes through p at time 0. Since $M^m \setminus F_{j,k}^m$ is imbedded and $\overline{\Omega_j^m} \subseteq M^m \setminus F_{j,k}^m$ is compact, it follows by standard reasoning that there exists $\epsilon_j^m > 0$ such that the mapping

$$\Phi_j^m : \Omega_j^m \times (-\epsilon_j^m, \epsilon_j^m) \rightarrow \mathbb{R}^d, \quad \Phi_j^m(p, t) := \gamma_{j,p}(t)$$

is a smooth imbedding and is bi-Lipschitz for some constant $0 < L_j^m < +\infty$. In particular,

$$M_j^{m+1} := \Phi_j^m (\Omega_j^m \times (-\epsilon_j^m, \epsilon_j^m))$$

is a smooth, imbedded, $(m+1)$ -dimensional submanifold of \mathbb{R}^d .

With the help of Φ_j^m , using (5.1) and the estimate

$$\mathcal{H}_E^{m+1}(A \times I) \geq \frac{\alpha(m+1)}{\alpha(m)\alpha(1)} \mathcal{H}_E^m(A) \mathcal{H}_E^1(I)$$

(see [32, 2.10.27]), valid if $I \subseteq \mathbb{R}$ is an interval and $A \subseteq \mathbb{R}^m$ is an arbitrary subset, it is not difficult to show that there exists a constant $\lambda_j^m > 0$ such that

$$(5.4) \quad \mathcal{H}_E^{m+1}(\Phi_j^m (\Omega_j^m \times (-\epsilon_j^m, \epsilon_j^m)) \setminus C) \geq \lambda_j^m \mathcal{H}_E^m (A^m \cap \Omega_j^m).$$

Combining (5.3) and (5.4), we obtain

$$\begin{aligned} \mathcal{H}_E^{m+1} (M_j^{m+1} \setminus C) &\geq \lambda_j^m \mathcal{H}_E^m (A^m \cap \Omega_j^m) \\ &\geq \lambda_j^m (\mathcal{H}_E^m (A^m \setminus F_{j,k}^m) - (1-q) (\mathcal{H}_E^m (M^m \setminus F_{j,k}^m))). \end{aligned}$$

Let $M^{m+1} := M_j^{m+1}$. Then, by (5.2), we obtain

$$\mathcal{H}_E^{m+1} (M^{m+1} \setminus C) \geq \lambda_j^m (qc_m/d_1 - (1-q)\mathcal{H}_E^m (M^m)) \geq c_{m+1} > 0,$$

where c_{m+1} depends on c_m and on the choice of q , but not on C . \square

Theorem 5.5 below is the main result of this chapter. Roughly, it says that h-convex, measurable subsets of a Carnot group do not admit inward cusps.

THEOREM 5.5. *Let $\mathbb{G} \equiv (\mathbb{R}^d, *)$ be a Carnot group. There exists $0 \leq c = c(\mathbb{G}) < 1$ such that*

$$(5.5) \quad \frac{\mathcal{H}^Q(B(x_0, r) \cap C)}{\mathcal{H}^Q(B(x_0, r))} \leq c \quad \forall 0 < r < +\infty$$

whenever $C \subseteq \mathbb{R}^d$ is an h-convex, measurable subset and $x_0 \in \partial C$ is a point on its boundary. In particular, the upper density

$$\limsup_{r \downarrow 0} \frac{\mathcal{H}^Q(B(x_0, r) \cap C)}{\mathcal{H}^Q(B(x_0, r))}$$

of C at boundary points $x_0 \in \partial C$ is uniformly bounded away from 1.

PROOF. Fix $1 < \lambda < (\mathcal{H}_E^d(B(0, 1)) / (\mathcal{H}_E^d(B(0, 1)) - c_d))^{1/Q}$, where c_d is the constant appearing in Lemma 5.4. Pick $x \in \mathbb{R}^d \setminus C$ sufficiently close to x_0 , in such a way that $B(x_0, r) \subseteq B(x, \lambda r)$. The set $\tilde{C} := \delta_{1/(\lambda r)} \circ l_{-x}(C)$ is h-convex and measurable, and $0 \in \mathbb{R}^d \setminus \tilde{C}$. By Lemma 5.4, there exists a smooth, imbedded, d -dimensional submanifold $M^d \subseteq B(0, 1)$ such that $\mathcal{H}_E^d(M^d \setminus \tilde{C}) \geq c_d$. Thus

$$\begin{aligned} \frac{\mathcal{H}^Q(B(x_0, r) \cap C)}{\mathcal{H}^Q(B(x_0, r))} &\leq \frac{\mathcal{H}^Q(l_x \circ \delta_{\lambda r}(B(0, 1) \cap \tilde{C}))}{\mathcal{H}^Q(l_x \circ \delta_r(B(0, 1)))} = \frac{\lambda^Q \mathcal{H}^Q(B(0, 1) \cap \tilde{C})}{\mathcal{H}^Q(B(0, 1))} \\ &= \lambda^Q \left(1 - \frac{\mathcal{H}^Q(B(0, 1) \setminus \tilde{C})}{\mathcal{H}^Q(B(0, 1))} \right) = \lambda^Q \left(1 - \frac{\mathcal{H}_E^d(B(0, 1) \setminus \tilde{C})}{\mathcal{H}_E^d(B(0, 1))} \right), \end{aligned}$$

and the claim follows with $c := \lambda^Q (1 - c_d / \mathcal{H}_E^d(B(0, 1)))$. \square

2. Local finiteness of the horizontal perimeter of h-convex sets

DEFINITION 5.2. Let \mathbb{G} be a Carnot group, $\oplus_{i=1}^s V_i$ a stratification of its Lie algebra \mathfrak{g} of left invariant vector fields, $\langle \cdot, \cdot \rangle$ an inner product on V_1 , (X_1, \dots, X_{d_1}) an orthonormal basis of V_1 with respect to $\langle \cdot, \cdot \rangle$, ρ the sub-Riemannian distance induced by an inner product $\langle \cdot, \cdot \rangle$ on V_1 and \mathcal{H}^Q the Q -dimensional Hausdorff measure induced by ρ . Given an open subset $\Omega \subseteq \mathbb{G}$, we denote $\mathcal{F}(\Omega)$ the set of C^1 smooth sections of the horizontal bundle with compact support in Ω and $\sum_{i=1}^{d_1} \langle \varphi, X_i \rangle^2 \leq 1$ in Ω . If $E \subseteq \mathbb{G}$ is measurable, the *horizontal perimeter of E in Ω* is

$$(5.6) \quad P(E, \Omega) := \sup_{\varphi \in \mathcal{F}(\Omega)} \int_E \operatorname{div}_H \varphi(g) d\mathcal{H}^Q(g).$$

(Recall that $\operatorname{div}_H \varphi$ is the horizontal divergence of φ , see Definition 1.8). E has *finite horizontal perimeter in Ω* if $P(E, \Omega) < +\infty$. E has *locally finite horizontal perimeter in Ω* if $P(E, \Omega') < +\infty$ for each $\Omega' \Subset \Omega$.

Sets of locally finite horizontal perimeter are the natural generalization to the setting of Carnot-Carathéodory spaces of sets of locally finite perimeter in Euclidean spaces, which were introduced by Caccioppoli and De Giorgi. They have been studied by several authors, see for instance [39], [37], [3], [76], [61], [4], [38], [6].

THEOREM 5.6. *Let $\mathbb{G} \equiv (\mathbb{R}^d, *)$ be a stratified group and let $C \subseteq \mathbb{R}^d$ be measurable and h-convex. Then C has locally finite horizontal perimeter in $(\mathbb{R}^d, *)$.*

PROOF. For $1 \leq i \leq d_1$, let X_i be the left invariant, horizontal vector field uniquely determined by the condition $X_i(0) = \partial_i(0)$ and let $\Phi_{X_i} : \mathbb{R}^{d-1} \times \mathbb{R} \rightarrow \mathbb{R}^d$ be a measure-preserving diffeomorphism, as in Proposition 1.3. Given $\Omega \Subset \mathbb{R}^d$ and $\varphi \in \mathcal{F}(\Omega)$, we have

$$\int_{\mathbb{R}^d} \chi_C(x) \operatorname{div}_H \varphi(x) d\mathcal{H}_E^d(x) = \sum_{i=1}^{d_1} \int_{\mathbb{R}^d} \chi_C(x) X_i \varphi_i(x) d\mathcal{H}_E^d(x),$$

where χ_C denotes the characteristic function of C . We fix $1 \leq i \leq d_1$ and compute

$$\begin{aligned} \int_{\mathbb{R}^d} \chi_C(x) X_i \varphi_i(x) d\mathcal{H}_E^d(x) &= \int_{\mathbb{R}^{d-1} \times \mathbb{R}} \chi_C(\Phi_{X_i}(y, t)) X_i \varphi_i(\Phi_{X_i}(y, t)) d\mathcal{H}_E^d(y, t) \\ &= \int_{K_i} \int_a^b \chi_C(\Phi_{X_i}(y, t)) X_i \varphi_i(\Phi_{X_i}(y, t)) dt d\mathcal{H}_E^{d-1}(y), \end{aligned}$$

where $K_i \subseteq \mathbb{R}^{d-1}$ is compact, $a, b \in \mathbb{R}$ with $a < b$ and $\Phi_{X_i}^{-1}(\operatorname{spt}(\varphi)) \subseteq K_i \times [a, b]$.

Given $y \in K_i$, let

$$t_-(y) := \inf \{t \in [a, b] \mid \Phi_{X_i}(y, t) \in C\}, \quad t_+(y) := \sup \{t \in [a, b] \mid \Phi_{X_i}(y, t) \in C\}$$

(let $t_-(y) = t_+(y) = 0$ if the set $\{t \in [a, b] \mid \Phi_{X_i}(y, t) \in C\}$ is empty). The h-convexity of C implies $\Phi_{X_i}(y, t) \in C$ whenever $t_-(y) < t < t_+(y)$. Thus

$$\begin{aligned} \int_{K_i} \int_a^b \chi_C(\Phi_{X_i}(y, t)) X_i \varphi_i(\Phi_{X_i}(y, t)) dt d\mathcal{H}_E^{d-1}(y) &= \int_{K_i} \varphi_i(\Phi_{X_i}(y, t)) \Big|_{t_-(y)}^{t_+(y)} d\mathcal{H}_E^{d-1}(y) \\ &\leq 2\mathcal{H}_E^{d-1}(K_i). \end{aligned}$$

□

REMARK 5.1. By the previous theorem, the rectifiability theory for sets of locally finite perimeter in stratified groups of step two due to Franchi, Serapioni and Serra Cassano (cf. [38]) applies to measurable, h-convex subsets of such groups.

3. A non-measurable h-convex subset of the first Heisenberg group

Recall that every convex subset $C \subseteq (\mathbb{R}^d, +)$ is measurable since its boundary has vanishing d -dimensional Lebesgue measure (see for instance [60]). By contrast, we have the following

THEOREM 5.7. *There exists a non-measurable, h-convex subset of the first Heisenberg group.*

PROOF. We have met the first Heisenberg group \mathbb{H} more than once in this work. Let us recall once again that

$$\mathbb{H} = \mathbb{H}_1 = (\mathbb{R}^3, *) = (\{(x, y, t) \mid x, y, t \in \mathbb{R}\}, *)$$

with the group law

$$(x, y, t) * (x', y', t') = (x + x', y + y', t + t' + 2(x'y - xy')).$$

The left invariant vector fields

$$X = \partial_x + 2y\partial_t, \quad Y = \partial_y - 2x\partial_t \quad \text{and} \quad T = \partial_t$$

form a basis of the Lie algebra \mathfrak{h} of left invariant vector fields on \mathbb{H} . If we define

$$V_1 := \text{span}_{\mathbb{R}}\{X, Y\}, \quad V_2 := \text{span}_{\mathbb{R}}\{T\},$$

then $V_1 \oplus V_2$ is a stratification of \mathfrak{h} . Recall that $\aleph_0 = \text{card}(\mathbb{N})$, $2^{\aleph_0} = \text{card}(\mathbb{R})$. Lower case Greek letters such as $\alpha, \beta, \gamma, \dots$ denote ordinal numbers and \mathcal{L}^d is d -dimensional (outer) Lebesgue measure on \mathbb{R}^d .

Observe that there are at most 2^{\aleph_0} open sets in \mathbb{R}^3 . Hence the set

$$\mathcal{K} := \{K \subseteq \mathbb{R}^3 \text{ compact} \mid \mathcal{L}^3(K) > 0\}$$

has cardinality 2^{\aleph_0} , and we can write

$$\mathcal{K} = \left\{ K_\alpha \mid \alpha < 2^{\aleph_0} \right\}.$$

Fix $p_0 = (x_0, y_0, t_0) \in K_0$. We let $f : \{0\} \rightarrow \mathbb{H}$ be given by $f(0) = p_0$. We consider the set E of extensions $\tilde{f} : \alpha \rightarrow \mathbb{H}$ of f such that

- (i) $\alpha \leq 2^{\aleph_0}$,
- (ii) $f(\beta) = p_\beta = (x_\beta, y_\beta, t_\beta) \in K_\beta$ for all $\beta < \alpha$,
- (iii) $(x_\beta, y_\beta) \neq (x_\gamma, y_\gamma)$ whenever $\beta, \gamma < \alpha$, $\beta \neq \gamma$ and
- (iv) $p_\beta \notin p_\gamma * \exp(V_1)$ (equivalently $p_\gamma \notin p_\beta * \exp(V_1)$) whenever $\beta, \gamma < \alpha$, $\beta \neq \gamma$.

Graph inclusion defines a partial ordering on E . Clearly, if $C \subseteq E$ is any chain with respect to this partial ordering, then $\bigcup C \in E$ is an upper bound for C . By Zorn's lemma, there exists a maximal extension $\bar{f} : \alpha \rightarrow \mathbb{H}$ which enjoys the properties (i) through (iv). We claim that $\alpha = 2^{\aleph_0}$. By contradiction, assume that $\alpha < 2^{\aleph_0}$. For $\beta < \alpha$, let us denote $(x_\beta, y_\beta, t_\beta) = p_\beta = \bar{f}(\beta)$. Consider the set

$$\pi(K_\alpha) = \{(x, y) \in \mathbb{R}^2 \mid \mathcal{L}^1(\{(x, y, t) \mid t \in \mathbb{R}\} \cap K_\alpha) > 0\}.$$

We have $\mathcal{L}^2(\pi(K_\alpha)) > 0$, whence

$$\text{card}(\pi(K_\alpha)) > \text{card}(\{(x_\beta, y_\beta) \mid \beta < \alpha\})$$

by Lemma 5.8 below. Pick $(x_\alpha, y_\alpha) \in \pi(K_\alpha) \setminus \{(x_\beta, y_\beta) \mid \beta < \alpha\}$ and denote

$$M_\alpha := \{(x_\alpha, y_\alpha, t) \mid t \in \mathbb{R}\} \cap K_\alpha.$$

Since $\mathcal{L}^1(M_\alpha) > 0$, we have $\text{card}(M_\alpha) = 2^{\aleph_0}$ by Lemma 5.8. On the other hand, each $p_\beta * \exp(V_1)$ intersects M_α in at most one point, and it follows that

$$\text{card}(M_\alpha) > \text{card}\left(\left(\bigcup_{\beta < \alpha} p_\beta * \exp(V_1)\right) \cap M_\alpha\right).$$

Consequently, we can find

$$t_\alpha \in M_\alpha \setminus \left(\left(\bigcup_{\beta < \alpha} p_\beta * \exp(V_1)\right) \cap M_\alpha\right).$$

We let $\bar{g}(\beta) := \bar{f}(\beta)$ if $\beta < \alpha$ and $\bar{g}(\alpha) := (x_\alpha, y_\alpha, t_\alpha)$. Then $\bar{g} \in E$, contradicting the maximality of \bar{f} .

Define $C := \bar{f}(2^{\aleph_0})$. Then C is h-convex by property (iv). It remains to show that it is not measurable. Suppose by contradiction that C is measurable. Then property (iii) together with the theorem of Fubini imply that $\mathcal{L}^3(C) = 0$. Thus $\mathcal{L}^3(B(0, 1) \setminus C) > 0$, and there exists a compact set $K \subseteq B(0, 1) \setminus C$ with $\mathcal{L}^3(K) > 0$. But this is impossible since property (ii) implies $C \cap K \neq \emptyset$. \square

In the proof of Theorem 5.7, we have used the following

LEMMA 5.8. *Suppose that $M \subseteq \mathbb{R}^d$ is \mathcal{L}^d -measurable with $\mathcal{L}^d(M) > 0$. Then*

$$\text{card}(M) = 2^{\aleph_0}.$$

PROOF. Let $K \subseteq M$ be a compact set with $\mathcal{L}^d(K) > 0$. Let K_1 denote the set of condensation points of K (i.e. the set of points $x \in \mathbb{R}^d$ such that each neighbourhood of x contains uncountably many points of K) and let K_2 denote the complement of K_1 in K . By the theorem of Cantor–Bendixson (cf. [49, Theorem 6.66]), K_1 is perfect (closed and without isolated points), K_2 is countable and $K = K_1 \cup K_2$. Since $\mathcal{L}^d(K) > 0$, it follows that $K_1 \neq \emptyset$. Hence $\text{card}(K_1) = 2^{\aleph_0}$ (cf. [49, Theorem 6.65]). \square

Consequences of the upper density bound for h-convex sets

In this short chapter, we present some interesting consequences of the estimate (5.5). First, we prove that h-convex, measurable functions are locally Lipschitz continuous. Second, we give a concise alternative proof of the L^∞ - L^1 estimates due to Danielli, Garofalo and Nhieu (cf. Theorem 9.2 in [23]). Finally, we show how (5.5) can be combined with sufficient conditions proved by Danielli in [21] (see also [10]), in order to demonstrate that boundary points of an h-convex, bounded open subset Ω of a Carnot group are regular and Hölder regular for weak solutions of the Dirichlet problem for the subelliptic p -Laplacian.

1. Local Lipschitz continuity of measurable h-convex functions

THEOREM 6.1. *Let \mathbb{G} be a Carnot group, $\Omega \subseteq \mathbb{G}$ an h-convex, open set and $u : \Omega \rightarrow \mathbb{R}$ an h-convex function. Suppose there exists a sequence $\{b_k\}_{k \in \mathbb{N}}$ of real numbers such that $b_k \rightarrow +\infty$ and $C_k := \{g \in \Omega \mid u(g) < b_k\}$ is measurable for all $k \in \mathbb{N}$. Then u is locally Lipschitz continuous.*

PROOF. Fix $g_0 \in \Omega$ and $r > 0$ such that $B(g_0, 2r) \subseteq \Omega$. Each C_k is h-convex and measurable. Take $k \in \mathbb{N}$ large enough, in order to guarantee $u(g_0) < b_k$ (whence $g_0 \in C_k$), and suppose that $B(g_0, r) \setminus C_k \neq \emptyset$. By connectedness of $B(g_0, r)$, we can find $g \in B(g_0, r) \cap \partial C_k$. We have

$$\frac{\mathcal{H}^Q(B(g, r) \setminus C_k)}{\mathcal{H}^Q(B(g, r))} \geq 1 - c$$

by Theorem 5.5, where $0 \leq c < 1$ depends only on \mathbb{G} . Hence

$$\frac{\mathcal{H}^Q(B(g_0, 2r) \setminus C_k)}{\mathcal{H}^Q(B(g_0, 2r))} \geq \frac{1 - c}{2^Q}.$$

On the other hand, by hypothesis,

$$\frac{\mathcal{H}^Q(B(g_0, 2r) \setminus C_k)}{\mathcal{H}^Q(B(g_0, 2r))} \rightarrow 0 \quad \text{as } k \rightarrow +\infty.$$

It follows that $B(g_0, r) \setminus C_k = \emptyset$ for large enough $k \in \mathbb{N}$. Thus u is locally bounded above in Ω . The local Lipschitz continuity now follows from Lemma 4.1 and Proposition 4.2. \square

REMARK 6.1. We point out that if $\{u_k\}_{k \in \mathbb{N}}$ is a sequence of measurable h-convex functions $u_k : \Omega \rightarrow \mathbb{R}$ (where Ω is an h-convex, open subset of some Carnot group \mathbb{G}) that admits pointwise upper bounds, respectively that converges pointwise, then $\sup_{k \in \mathbb{N}} u_k$, respectively $\lim_{k \rightarrow +\infty} u_k$, is again h-convex and measurable. In particular, if the sequence admits pointwise upper bounds, then $\limsup_{k \rightarrow +\infty} u_k$ is an h-convex, measurable function.

2. L^∞ - L^1 estimates

THEOREM 6.2. *Let \mathbb{G} be a Carnot group, $\oplus_{i=1}^s V_i$ a stratification of its Lie algebra of left invariant vector fields, ρ the sub-Riemannian distance induced by an inner product $\langle \cdot, \cdot \rangle$*

on V_1 , $\Omega \subseteq \mathbb{G}$ an h-convex, open set and $u : \Omega \rightarrow \mathbb{R}$ an h-convex, measurable function. There exist constants $c_1 = c_1(\mathbb{G})$ and $c_2 = c_2(\mathbb{G})$ such that

$$(6.1) \quad \sup \{|u(g)| \mid g \in B(g_0, r)\} \leq c_1 \int_{B(g_0, 8r)} |u(g)| d\mathcal{H}^Q(g)$$

whenever $B(g_0, 8r) \Subset \Omega$, and

$$(6.2) \quad \text{ess sup} \{|\nabla_{\mathbb{H}} u(g)| \mid g \in B(g_0, r)\} \leq \frac{c_2}{r} \int_{B(g_0, 32r)} |u(g)| d\mathcal{H}^Q(g)$$

whenever $B(g_0, 32r) \Subset \Omega$.

PROOF. Let us first prove (6.1). Assume $B(g_0, 2r) \Subset \Omega$. Fix $0 < \epsilon < +\infty$ and let $c_0 := 2^Q/(1-c)$, where c is the constant from Theorem 5.5. The set

$$C := \left\{ g \in \Omega \mid u(g) < c_0 \int_{B(g_0, 2r)} |u(g)| d\mathcal{H}^Q(g) + \epsilon \right\}$$

is h-convex. We have $B(g_0, r) \cap C \neq \emptyset$, since otherwise

$$\begin{aligned} \int_{B(g_0, 2r)} |u(g)| d\mathcal{H}^Q(g) &\geq \frac{1}{2^Q} \int_{B(g_0, r)} |u(g)| d\mathcal{H}^Q(g) \\ &\geq \frac{1}{2^Q} \left(c_0 \int_{B(g_0, 2r)} |u(g)| d\mathcal{H}^Q(g) + \epsilon \right) \\ &\geq \int_{B(g_0, 2r)} |u(g)| d\mathcal{H}^Q(g) + \frac{\epsilon}{2^Q}. \end{aligned}$$

Suppose now that $B(g_0, r) \setminus C \neq \emptyset$. Then we can find $g_1 \in B(g_0, r) \cap \partial C$ since $B(g_0, r)$ is connected. We have

$$\frac{\mathcal{H}^Q(B(g_1, r) \setminus C)}{\mathcal{H}^Q(B(g_1, r))} \geq 1 - c$$

by Theorem 5.5. Hence

$$\frac{\mathcal{H}^Q(B(g_1, r) \setminus C)}{\mathcal{H}^Q(B(g_0, 2r))} \geq \frac{1-c}{2^Q}$$

and

$$\begin{aligned} \int_{B(g_0, 2r)} |u(g)| d\mathcal{H}^Q(g) &\geq \frac{1-c}{2^Q} \left(c_0 \int_{B(g_0, 2r)} |u(g)| d\mathcal{H}^Q(g) + \epsilon \right) \\ &= \int_{B(g_0, 2r)} |u(g)| d\mathcal{H}^Q(g) + \frac{(1-c)\epsilon}{2^Q}. \end{aligned}$$

This contradiction implies

$$u(g_1) < c_0 \int_{B(g_0, 2r)} |u(g)| d\mathcal{H}^Q(g) + \epsilon$$

for all $g_1 \in B(g_0, r)$ and all $\epsilon > 0$. Now if $B(g_0, 8r) \Subset \Omega$, then the above reasoning and the continuity of u give

$$u(g_1) \leq M := c_0 \int_{B(g_0, 8r)} |u(g)| d\mathcal{H}^Q(g)$$

in $\overline{B}(g_0, 4r)$. In view of (4.1), (6.1) follows with $c_1 := 4^n (8l \cdot n)^Q c_0$, where $l, n \in \mathbb{N}$ depend only on \mathbb{G} .

To prove (6.2), it is now enough to show

$$\text{ess sup} \{|\nabla_{\mathbb{H}} u(g)| \mid g \in B(g_0, r)\} \leq \frac{8\sqrt{d_1} \sup \{|u(g)| \mid g \in B(g_0, 4r)\}}{r},$$

since the claim then follows from (6.1) with $c_2 := 8\sqrt{d_1}c_1$.

Let (X_1, \dots, X_{d_1}) be an orthonormal basis of V_1 with respect to $\langle \cdot, \cdot \rangle$. By Lemma 1.7, the weak derivatives $X_1u, X_2u, \dots, X_{d_1}u$ of u in Ω exist, and

$$X_i u(g) = \lim_{t \downarrow 0} \frac{u(g \exp(tX_i)) - u(g)}{t}$$

for $1 \leq i \leq d_1$ and almost every $g \in B(g_0, r)$. By (4.2), we have

$$\begin{aligned} \left| \lim_{t \downarrow 0} \frac{u(g \exp(tX_i)) - u(g)}{t} \right| &\leq \lim_{t \downarrow 0} \left(\frac{8 \sup \{|u(g)| \mid g \in \overline{B}(g_0, 4r)\}}{r} \frac{\rho(g, g \exp(tX_i))}{t} \right) \\ &= \frac{8 \sup \{|u(g)| \mid g \in \overline{B}(g_0, 4r)\}}{r} \\ &= \frac{8 \sup \{|u(g)| \mid g \in B(g_0, 4r)\}}{r}. \end{aligned}$$

□

3. Regularity of p -harmonic functions at the boundary of h -convex sets

In this section, \mathbb{G} is a stratified group of homogeneous dimension Q , $\oplus_{i=1}^s V_i$ is a stratification of its Lie algebra of left invariant vector fields, ρ is the sub-Riemannian distance induced by an inner product $\langle \cdot, \cdot \rangle$ on V_1 , (X_1, \dots, X_{d_1}) is an orthonormal basis of V_1 with respect to $\langle \cdot, \cdot \rangle$ and $\Omega \subseteq \mathbb{G}$ is a bounded, open set.

We start with a reminder on horizontal Sobolev spaces: Given $1 \leq p < +\infty$, we denote $W^{1,p}(\Omega)$ the vector space of functions $f \in L^p(\Omega)$ whose weak horizontal derivatives in the directions X_1, \dots, X_{d_1} exist and belong to $L^p(\Omega)$. For $f \in W^{1,p}(\Omega)$, we define

$$\|f\|_{W^{1,p}(\Omega)} := \|f\|_{L^p(\Omega)} + \|\nabla_H f\|_{L^p(\Omega)}.$$

Clearly, $\|\cdot\|_{W^{1,p}(\Omega)}$ is a norm on $W^{1,p}(\Omega)$ and $(W^{1,p}(\Omega), \|\cdot\|_{W^{1,p}(\Omega)})$ is complete. We denote by $W_0^{1,p}(\Omega)$ the completion of $C_c^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{W^{1,p}(\Omega)}$. Intuitively, $W_0^{1,p}(\Omega)$ is the closed subspace of trace zero Sobolev functions. Using Lemma 1.6 together with a partition of unity argument and Lemma 1.7, it can be shown that the spaces $W^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$ coincide with the spaces of functions considered in [21].

Given $1 < p < +\infty$, we call a function $u \in W^{1,p}(\Omega)$ p -harmonic if it is a weak solution of the *subelliptic p -Laplace equation*

$$(6.3) \quad \operatorname{div}_H (|\nabla_H u|^{p-2} \nabla_H u) \equiv 0 \quad \text{in } \Omega,$$

which is the Euler–Lagrange equation of the p -energy integral

$$\frac{1}{p} \int_\Omega |\nabla_H u(g)|^p d\mathcal{H}^Q(g).$$

Using the convexity of $\xi \mapsto |\xi|^p$, it is easy to show that a p -harmonic function $u \in W^{1,p}(\Omega)$ has the minimizing property

$$\int_\Omega |\nabla_H u(g)|^p d\mathcal{H}^Q(g) \leq \int_\Omega |\nabla_H (u + \varphi)(g)|^p d\mathcal{H}^Q(g) \quad \forall \varphi \in W_0^{1,p}(\Omega).$$

Let $1 < p \leq Q$ and $f \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$. It can be shown that the Dirichlet problem

$$(6.4) \quad \begin{cases} \operatorname{div}_H (|\nabla_H u|^{p-2} \nabla_H u) \equiv 0 & \text{in } \Omega \\ u - f \in W_0^{1,p}(\Omega) \end{cases}$$

admits a unique weak solution $u \in W^{1,p}(\Omega)$, and that the precise representative of this solution is locally Hölder continuous in Ω . In the following, when we speak of the solution of (6.4), we refer to this precise representative.

In [21] (cf. Theorem 3.1 and Theorem 3.8), Danielli obtained the following criteria for the regularity, respectively the Hölder regularity, of a boundary point:

THEOREM 6.3. *Let $g_0 \in \partial\Omega$. If*

$$\int_0^1 \left(\frac{\text{cap}_p(U(g_0, r) \cap \Omega^c, U(g_0, 2r))}{\text{cap}_p(U(g_0, r), U(g_0, 2r))} \right)^{1/(p-1)} \frac{1}{r} dr = +\infty$$

(where $\Omega^c = \mathbb{G} \setminus \Omega$), then g_0 is a regular boundary point, i.e.

$$\lim_{g \in \Omega, g \rightarrow g_0} u(g) = f(g_0)$$

whenever $f \in W^{1,p}(\Omega) \cap C(\bar{\Omega})$ and u is the solution of the corresponding Dirichlet problem (6.4).

THEOREM 6.4. *Let $g_0 \in \partial\Omega$, $r_0 > 0$ and $0 < r_1 \leq r_2 \leq r_0/2$. There exists a constant $\beta > 0$ depending only on Ω and p such that*

$$\begin{aligned} \text{osc}(u, \Omega \cap U(g_0, r_1)) &\leq \text{osc}(f, \partial\Omega \cap \bar{U}(g_0, 2r_2)) \\ &\quad + \text{osc}(f, \partial\Omega) \exp\left(-\beta \int_{r_1}^{r_2} \psi(p, g_0, r) \frac{1}{r} dr\right), \end{aligned}$$

whenever $f \in W^{1,p}(\Omega) \cap C(\bar{\Omega})$ and u is the solution of the corresponding Dirichlet problem (6.4), where

$$\psi(p, g_0, r) := \left(\frac{\text{cap}_p(U(g_0, r) \cap \Omega^c, U(g_0, 2r))}{\text{cap}_p(U(g_0, r), U(g_0, 2r))} \right)^{1/(p-1)}.$$

Some words of explanation are in order: The sets $U(g, r)$ ($g \in \mathbb{G}$, $r > 0$) are open subsets of \mathbb{G} which are obtained as appropriate level sets of the fundamental solution of the subelliptic Laplace equation $\sum_{i=1}^{d_1} X_i^2 u \equiv 0$, and which enjoy the following properties:

- (i) There exist constants $r_0 > 0$ and $a \geq 1$ depending on Ω such that

$$B(g, r/a) \subseteq U(g, r) \subseteq B(g, ar) \quad \forall g \in \bar{\Omega} \quad \forall 0 < r \leq r_0.$$

- (ii) For each $1 \leq p < +\infty$, there exists a constant $0 < c_p < +\infty$ depending on p and Ω , such that the Poincaré–Sobolev type inequality

$$\left(\int_{U(g_0, r)} |\varphi(g)|^p d\mathcal{H}^Q(g) \right)^{1/p} \leq c_p r \left(\int_{U(g_0, r)} |\nabla_{\mathbb{H}} \varphi(g)|^p d\mathcal{H}^Q(g) \right)^{1/p}$$

holds for all $g_0 \in \bar{\Omega}$, $0 < r \leq r_0$ (r_0 as above) and $\varphi \in C_c^\infty(U(g_0, r))$. (These inequalities follow via standard considerations from Theorem 2.2 in [21]).

The *subelliptic p -capacity* cap_p of a subset of Ω ($1 \leq p < +\infty$) is defined as follows: If $K \subseteq \Omega$ is compact, let

$$\text{cap}_p(K, \Omega) = \inf \left\{ \int_{\Omega} |\nabla_{\mathbb{H}} \varphi|^p d\mathcal{H}^Q(g) \mid \varphi \in C_c^\infty(\Omega), \varphi \geq 1 \text{ on } K \right\}.$$

For an arbitrary subset $E \subseteq \Omega$, let

$$\text{cap}_p(E, \Omega) = \inf_{E \subseteq \Omega' \subseteq \Omega} \sup_{K \subseteq \Omega'} \text{cap}_p(K, \Omega),$$

where Ω' is open and K is compact.

We easily deduce the following corollaries of Theorem 6.3 and Theorem 6.4:

COROLLARY 6.5. *Let $g_0 \in \partial\Omega$. Suppose that there exists $b > 0$ such that*

$$\frac{\text{cap}_p(U(g_0, r) \cap \Omega^c, U(g_0, 2r))}{\text{cap}_p(U(g_0, r), U(g_0, 2r))} \geq b$$

for all sufficiently small $r > 0$. Then g_0 is a regular boundary point.

PROOF. Immediate by Theorem 6.3. \square

COROLLARY 6.6. *Let $g_0 \in \partial\Omega$. Suppose there exist $b > 0$ and $0 < R \leq \min\{1, r_0/2\}$ such that*

$$\frac{\text{cap}_p(U(g_0, r) \cap \Omega^c, U(g_0, 2r))}{\text{cap}_p(U(g_0, r), U(g_0, 2r))} \geq b \quad \forall 0 < r \leq R.$$

Then g_0 is a Hölder regular boundary point, i.e. whenever $f \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ is Hölder continuous at g_0 , then the solution of the corresponding Dirichlet problem (6.4) is Hölder continuous at g_0 .

PROOF. Fix $0 < \delta \leq 1$. For $0 < r_1 \leq R^{1/\delta}$, let $r_2 := r_1^\delta$. Theorem 6.4 and a short computation show that

$$\text{osc}(u, \Omega \cap U(g_0, r_1)) \leq \text{osc}\left(f, \partial\Omega \cap \overline{U}\left(g_0, 2r_1^\delta\right)\right) + \text{osc}(f, \partial\Omega) r_1^\eta,$$

where $\eta = (1 - \delta)\beta b^{1/(p-1)}$. Hence

$$\text{osc}(u, \Omega \cap B(g_0, r_1/a)) \leq \text{osc}\left(f, \partial\Omega \cap \overline{B}\left(g_0, 2ar_1^\delta\right)\right) + \text{osc}(f, \partial\Omega) r_1^\eta$$

and the claim follows. \square

Let us now show that the hypotheses of Corollary 6.5 and Corollary 6.6 are satisfied when Ω is h-convex:

PROPOSITION 6.7. *Suppose that Ω is h-convex. Let $1 < p < +\infty$. Then there exist $b > 0$ and $0 < R \leq \min\{1, r_0/2\}$ such that*

$$\frac{\text{cap}_p(U(g_0, r) \cap \Omega^c, U(g_0, 2r))}{\text{cap}_p(U(g_0, r), U(g_0, 2r))} \geq b \quad \forall g_0 \in \partial\Omega \quad \forall 0 < r \leq R.$$

PROOF. In view of (5.5), it suffices to show that there exist constants $\gamma > 0$ and $0 < R \leq \min\{1, r_0/2\}$ such that

$$\frac{\text{cap}_p(U(g_0, r) \cap \Omega^c, U(g_0, 2r))}{\text{cap}_p(U(g_0, r), U(g_0, 2r))} \geq \gamma \frac{\mathcal{H}^Q(B(g_0, r/2a) \cap \Omega^c)}{\mathcal{H}^Q(B(g_0, 2ar))} \quad \forall g_0 \in \partial\Omega \quad \forall 0 < r \leq R.$$

We prove that there exist constants $0 < \gamma_1, \gamma_2 < +\infty$ and $0 < R \leq \min\{1, r_0/2\}$ such that

$$\text{cap}_p(U(g_0, r), U(g_0, 2r)) \leq \gamma_1 \frac{\mathcal{H}^Q(B(g_0, 2ar))}{r^p} \quad \forall g_0 \in \partial\Omega \quad \forall 0 < r \leq R$$

and

$$\text{cap}_p(U(g_0, r) \cap \Omega^c, U(g_0, 2r)) \geq \gamma_2 \frac{\mathcal{H}^Q(B(g_0, r/2a) \cap \Omega^c)}{r^p} \quad \forall g_0 \in \partial\Omega \quad \forall 0 < r \leq R.$$

By Lemma 2.1 in [21], there exist constants $0 < \gamma_1 < +\infty$ and $0 < R \leq \min\{1, r_0/2\}$ such that, for all $g_0 \in \partial\Omega$ and all $0 < r \leq R$, we can find a $\varphi \in C_c^\infty(U(g_0, 2r))$ with $\varphi \geq 1$ in $U(g_0, r)$ and $|\nabla_{\mathbb{H}} \varphi| \leq \gamma_1^{1/p}/r$ in $U(g_0, 2r)$. Hence, by definition of the subelliptic p -capacity, we get

$$\begin{aligned} \text{cap}_p(U(g_0, r), U(g_0, 2r)) &\leq \int_{U(g_0, 2r)} |\nabla_{\mathbb{H}} \varphi(g)|^p d\mathcal{H}^Q(g) \leq \gamma_1 \frac{\mathcal{H}^Q(U(g_0, 2r))}{r^p} \\ &\leq \gamma_1 \frac{\mathcal{H}^Q(B(g_0, 2ar))}{r^p} \end{aligned}$$

for all $g_0 \in \partial\Omega$ and all $0 < r \leq R$.

Now let $\gamma_2 = 1/c_p^p$ and let R be as above. The Poincaré–Sobolev inequality gives

$$\int_{U(g_0, 2r)} |\nabla_{\mathbb{H}} \varphi(g)|^p d\mathcal{H}^Q(g) \geq \frac{\gamma_2}{r^p} \int_{U(g_0, 2r)} |\varphi(g)|^p d\mathcal{H}^Q(g) \quad \forall g_0 \in \partial\Omega \quad \forall 0 < r \leq R$$

whenever $\varphi \in C_c^\infty(U(g_0, 2r))$. Hence

$$\int_{U(g_0, 2r)} |\nabla_{\mathbb{H}} \varphi(g)|^p d\mathcal{H}^Q(g) \geq \gamma_2 \frac{\mathcal{H}^Q(\overline{B}(g_0, r/2a) \cap \Omega^c)}{r^p} \quad \forall g_0 \in \partial\Omega \quad \forall 0 < r \leq R$$

whenever $\varphi \in C_c^\infty(U(g_0, 2r))$ satisfies $\varphi \geq 1$ on $\overline{B}(g_0, r/2a) \cap \Omega^c$. By definition of the subelliptic p -capacity, we obtain

$$\begin{aligned} \text{cap}_p(U(g_0, r) \cap \Omega^c, U(g_0, 2r)) &\geq \text{cap}_p(\overline{B}(g_0, r/2a) \cap \Omega^c, U(g_0, 2r)) \\ &\geq \gamma_2 \frac{\mathcal{H}^Q(\overline{B}(g_0, r/2a) \cap \Omega^c)}{r^p} \end{aligned}$$

for all $g_0 \in \partial\Omega$ and all $0 < r \leq R$. □

REMARK 6.2. Similar boundary regularity problems can be formulated and studied in a much more general setting. Indeed, the Newtonian spaces of Shanmugalingam ([85]) provide a framework within which the notion of quasiminimizer of the p -energy integral on bounded, open subsets Ω of general metric measure spaces can be defined. In [54], Kinnunen and Shanmugalingam prove that quasiminimizers of the p -energy integral are locally Hölder continuous in Ω and satisfy the Harnack inequality and the maximum principle, provided the space is doubling and admits a suitable Poincaré inequality. In [10], J. Björn studies the boundary regularity of quasiminimizers of the p -energy integral subject to appropriate boundary conditions on bounded, open subsets of a metric measure space. Assuming that the space is doubling and admits a suitable Poincaré inequality, she obtains results very similar to the ones presented in this section (cf. Theorems 2.11, 2.12 and 2.13 and Remark 2.15 in [10]).

CHAPTER 7

Second order regularity of h-convex functions

In this final chapter, we describe the main steps which lead to the generalization of the Aleksandrov theorem (pointwise second order differentiability almost everywhere) to (continuous) h-convex functions on Carnot groups of step two. In the first section, we present the second order approximate differentiability theorem for functions with bounded horizontal variation up to order two, due to Ambrosio and Magnani ([5]). In the second section, we prove that the symmetrized horizontal Hessian of an h-convex, measurable function exists in the sense of distributions. Then we state a result of Danielli, Garofalo, Nhieu and Tournier ([24]), which asserts that a (continuous) h-convex function on a general Carnot group of step two has bounded horizontal variation up to order two. This result generalizes the corresponding theorem for h-convex functions on the first Heisenberg group, which was obtained by Gutiérrez and Montanari ([45], [46]). Finally, in the last section, we show how to combine the results of the previous sections in order to obtain the Aleksandrov theorem in stratified groups of step two.

In the following, \mathbb{G} is a Carnot group, $\oplus_{i=1}^s V_i$ is a stratification of its Lie algebra \mathfrak{g} of left invariant vector fields, $\langle \cdot, \cdot \rangle$ is an inner product on V_1 , (X_1, \dots, X_d) is an adapted basis of \mathfrak{g} with respect to $\langle \cdot, \cdot \rangle$, ρ is the sub-Riemannian distance on \mathbb{G} induced by $\langle \cdot, \cdot \rangle$ and \mathcal{H}^Q is the Q -dimensional Hausdorff measure on \mathbb{G} induced by ρ .

1. A second order approximate differentiability result

We start with a brief reminder on polynomials on stratified groups: $P : \mathbb{G} \rightarrow \mathbb{R}$ is called a *polynomial* if $P \circ \exp$ is a polynomial on \mathfrak{g} . Let (ξ_1, \dots, ξ_d) be the basis for the linear forms on \mathfrak{g} dual to the basis (X_1, \dots, X_d) and define $\eta_j := \xi_j \circ \exp^{-1}$ for $j = 1, \dots, d$. Then every polynomial P on \mathbb{G} admits a unique representation

$$P = \sum_{I \in \mathbb{N}_0^d} a_I \eta^I$$

with $\eta^I = \eta_1^{i_1} \cdots \eta_d^{i_d}$, where all but finitely many of the coefficients a_I vanish. The *weighted degree* of the polynomial $P = \sum_{I \in \mathbb{N}_0^d} a_I \eta^I$ is defined to be

$$\deg(P) := \max \left\{ \sum_{j=1}^d \deg(j) i_j \mid I = (i_1, \dots, i_d) \in \mathbb{N}_0^d, a_I \neq 0 \right\}.$$

For $n \in \mathbb{N}_0$, we denote by \mathcal{P}_n the space of polynomials of weighted degree at most n . By Proposition 1.25 in [36], \mathcal{P}_n is invariant under left and right translation.

It follows from the theorem of Poincaré–Birkhoff–Witt that

$$\left\{ X^I = X_1^{i_1} \cdots X_d^{i_d} \mid I = (i_1, \dots, i_d) \in \mathbb{N}_0^d \right\}$$

is a basis for the algebra of left invariant differential operators on \mathbb{G} (see the third chapter of [87]). Let (X_1, \dots, X_d) be an adapted basis for \mathfrak{g} . The *weighted degree* of the left

invariant differential operator $D = \sum_{I \in \mathbb{N}_0^d} a_I X^I$ is defined to be

$$\deg(D) := \max \left\{ \sum_{j=1}^d \deg(j) i_j \mid I = (i_1, \dots, i_d) \in \mathbb{N}_0^d, a_I \neq 0 \right\}.$$

For $n \in \mathbb{N}_0$, let us denote by \mathcal{D}_n the space of left invariant differential operators of weighted degree at most n .

PROPOSITION 7.1. *Let $n \in \mathbb{N}_0$. Then the mapping $f : \mathcal{P}_n \rightarrow \mathcal{D}_n$ defined by*

$$f(P) := \sum_{\deg(X^I) \leq n} (X^I P)(e) X^I \quad \forall P \in \mathcal{P}_n$$

is an \mathbb{R} -linear isomorphism.

PROOF. An immediate consequence of Proposition 1.30 in [36]. \square

We now define functions of bounded horizontal variation:

DEFINITION 7.1. Let $\Omega \subseteq \mathbb{G}$ be an open subset. A function $f \in L^1(\Omega)$ has *bounded horizontal variation* if its weak horizontal derivatives exist in the sense of measure, that is if there exist signed Radon measures $X_1 f, \dots, X_{d_1} f$ on Ω with finite total variation such that

$$\int_{\Omega} f(g) X_i \varphi(g) d\mathcal{H}^Q(g) = - \int_{\Omega} \varphi(g) d(X_i f)(g) \quad \forall \varphi \in C_c^\infty(\Omega)$$

($i = 1, \dots, d_1$). The vector space of functions of bounded horizontal variation is denoted $BV_H(\Omega)$.

A function $f \in L^1(\Omega)$ has *bounded horizontal variation up to order two* if its weak horizontal derivatives $X_1 f, \dots, X_{d_1} f$ exist and are representable by functions which belong to $BV_H(\Omega)$. The vector space of functions of bounded horizontal variation up to order two is denoted $BV_H^2(\Omega)$.

We say that f has *locally bounded horizontal variation (up to order two)* and we denote $f \in BV_{H,loc}(\Omega)$ ($f \in BV_{H,loc}^2(\Omega)$) if $f \in BV_{H,loc}(\Omega')$ ($f \in BV_{H,loc}^2(\Omega')$) for each $\Omega' \Subset \Omega$.

REMARK 7.1. Observe that if $f \in BV_H^2(\Omega)$, then Df exists in the sense of measure for each $D \in \mathcal{D}_2$. That is, for each $D \in \mathcal{D}_2$ there exists a signed Radon measure Df with finite variation such that

$$\int_{\Omega} f(g) D\varphi(g) d\mathcal{H}^Q(g) = - \int_{\Omega} \varphi(g) d(Df)(g) \quad \forall \varphi \in C_c^\infty(\Omega).$$

Given $f \in BV_H^2(\Omega)$ and $D \in \mathcal{D}_2$, we denote by $(Df)_a$ and $(Df)_s$ respectively the absolutely continuous and the singular part of Df with respect to \mathcal{H}^Q .

The following second order approximate differentiability result for functions in $BV_H^2(\Omega)$ is due to Ambrosio and Magnani (see Theorem 3.9 in [5]):

THEOREM 7.2. *Let $\Omega \subseteq \mathbb{G}$ be an open subset and $f \in BV_H^2(\Omega)$. Then, for almost every $g_0 \in \Omega$, there exists a polynomial P_{g_0} of weighted degree at most two, such that*

$$(7.1) \quad \lim_{r \downarrow 0} \frac{1}{r^2} \int_{B(g_0, r)} |f(g) - P_{g_0}(g)| d\mathcal{H}^Q(g) = 0.$$

REMARK 7.2. It is an exercise to show that P_{g_0} –if it exists– is unique. In fact, for \mathcal{H}^Q almost every $g_0 \in \Omega$, P_{g_0} can be characterized as follows: consider the basis

$$B = \{Id\} \cup \{X_i f \mid 1 \leq i \leq d_1\} \cup \{X_i X_j f \mid 1 \leq i, j \leq d_1\}$$

of \mathcal{D}_2 . By standard measure theory, for \mathcal{H}^Q almost every $g_0 \in \Omega$, the limit

$$\lambda(D, g_0) := \lim_{r \downarrow 0} \int_{B(g_0, r)} \frac{d(Df)_a}{d\mathcal{H}^Q}(g) d\mathcal{H}^Q(g)$$

exists and

$$\lim_{r \downarrow 0} \int_{B(g_0, r)} \left| \frac{d(Df)_a}{d\mathcal{H}^Q}(g) - \lambda(D, g_0) \right| d\mathcal{H}^Q(g) = 0$$

holds for all $D \in B$, and

$$\lim_{r \downarrow 0} \frac{|(Df)_s|(B(g_0, r))}{\mathcal{H}^Q(B(g_0, r))} = 0$$

for all $D \in B$, where $|(Df)_s|$ is the variation measure associated with the singular part $(Df)_s$ of Df . By Proposition 7.1, there exists a unique polynomial P in \mathcal{P}_2 such that

$$DP(e) = \lim_{r \downarrow 0} \int_{B(g_0, r)} \frac{d(Df)_a}{d\mathcal{H}^Q}(g) d\mathcal{H}^Q(g) \quad \forall D \in B,$$

and (7.1) holds with $P_{g_0} := P$.

2. Horizontal variation of h-convex functions

In this section, $\Omega \subseteq \mathbb{G}$ is an h-convex, open subset and $u : \Omega \rightarrow \mathbb{R}$ is h-convex and measurable.

Proposition 7.3 shows that the symmetrized horizontal Hessian of an h-convex, measurable function exists in the sense of distributions (compare Theorem 8.1 in [23] and Theorem 4.2 in [64]):

PROPOSITION 7.3. *There exist unique signed Radon measures $\{\mu_{ij}\}_{1 \leq i, j \leq d_1}$ on Ω such that*

$$(7.2) \quad \int_{\Omega} \varphi(g) d\mu_{ij}(g) = \int_{\Omega} u(g) D_{\mathbb{H}}^2 \varphi(g)(X_i(g), X_j(g)) d\mathcal{H}^Q(g) \quad \forall \varphi \in C_c^\infty(\Omega).$$

The measures $\mu_{11}, \mu_{22}, \dots, \mu_{d_1 d_1}$ are non-negative.

PROOF. Let $X \in V_1$ such that $\langle X, X \rangle = 1$. Define

$$T_X(\varphi) := \int_{\Omega} u(g) X X \varphi(g) d\mathcal{H}^Q(g) \quad \forall \varphi \in C_c^\infty(\Omega).$$

Given a non-negative $\varphi \in C_c^\infty(\Omega)$, let $\Omega' \Subset \Omega$ such that the support of φ is contained in Ω' . By Theorem 6.1, u is continuous in Ω . The regularization u_ϵ is defined on Ω' for sufficiently small $\epsilon > 0$. Using Lemma 1.4, Lemma 1.6 and Lemma 3.2, we obtain

$$\int_{\Omega} u(g) X X \varphi(g) d\mathcal{H}^Q(g) = \lim_{\epsilon \downarrow 0} \int_{\Omega'} u_\epsilon(g) X X \varphi(g) d\mathcal{H}^Q(g) = \lim_{\epsilon \downarrow 0} \int_{\Omega'} X X u_\epsilon(g) \varphi(g) d\mathcal{H}^Q(g),$$

and the last expression is non-negative. It follows that T_X is a non-negative linear functional on $C_c^\infty(\Omega)$. By the Riesz representation theorem, see for instance Corollary 1 in §8 of the first chapter of [30], there exists a unique Radon measure μ_X on Ω such that

$$T_X(\varphi) = \int_{\Omega} \varphi(g) d\mu_X(g) \quad \forall \varphi \in C_c^\infty(\Omega).$$

Clearly, the signed Radon measures

$$\mu_{ij} := \mu_{X_{ij}} - \frac{1}{2}\mu_{X_i} - \frac{1}{2}\mu_{X_j}$$

with $X_{ij} := (X_i + X_j)/\sqrt{2}$ for $1 \leq i, j \leq d_1$ satisfy (7.2). \square

Proposition 7.3 implies that $u \in BV_{H,loc}^2(\Omega)$ if the weak derivatives $[X_i, X_j]u$ of u in the directions of the commutators exist as Radon measures:

COROLLARY 7.4. *Suppose that for all $1 \leq i, j \leq d_1$ there exists a signed Radon measure ν_{ij} such that*

$$\int_{\Omega} \varphi(g) d\nu_{ij}(g) = - \int_{\Omega} u(g) ([X_i, X_j]\varphi)(g) d\mathcal{H}^Q(g) \quad \forall \varphi \in C_c^\infty(\Omega).$$

Then $u \in BV_{H,loc}^2(\Omega)$.

PROOF. For $1 \leq i, j \leq d_1$, we define the signed Radon measures $m_{ij} := \mu_{ij} - \frac{1}{2}\nu_{ij}$. Then

$$\begin{aligned} \int_{\Omega} \varphi(g) dm_{ij}(g) &= \int_{\Omega} u(g) \left(D_H^2 \varphi(g)(X_i(g), X_j(g)) + \frac{1}{2} ([X_i, X_j]\varphi)(g) \right) d\mathcal{H}^Q(g) \\ &= \int_{\Omega} u(g) X_i X_j \varphi(g) d\mathcal{H}^Q(g) \end{aligned}$$

for all $\varphi \in C_c^\infty(\Omega)$. □

Let $\Omega' \Subset \Omega$. We have $\Omega' \Subset \Omega_\epsilon$ when $\epsilon > 0$ is sufficiently small and the regularization $u_\epsilon : \Omega_\epsilon \rightarrow \mathbb{R}$ is well-defined. It follows from Lemma 3.2, Proposition 3.3 and Theorem 3.6 that the symmetrized horizontal Hessian $D_H^2 u_\epsilon$ is positive semidefinite in Ω' . In particular, the eigenvalues $\lambda_1(D_H^2 u_\epsilon), \dots, \lambda_{d_1}(D_H^2 u_\epsilon)$ are non-negative in Ω' and thus

$$\mathcal{F}_2[u_\epsilon] := \sum_{1 \leq i < j \leq d_1} \lambda_i(D_H^2 u_\epsilon) \lambda_j(D_H^2 u_\epsilon) \geq 0$$

in Ω' .

Estimate (7.3) below, which can be derived from a monotonicity result for the operator

$$\mathcal{F}_2[u_\epsilon] + \frac{3}{4} \sum_{1 \leq i < j \leq d_1} ([X_i, X_j]u_\epsilon)^2$$

appearing in the integral on the left hand side of (7.3) via a covering argument, is due to Danielli, Garofalo, Nhieu and Tournier (cf. Theorem 4.3 in [24]). It is the crucial step on the way to the generalization of the Aleksandrov pointwise second order differentiability theorem to stratified groups of step two:

THEOREM 7.5. *Suppose that the step of \mathbb{G} is two. Let $\Omega' \Subset \Omega'' \Subset \Omega$ and $\epsilon > 0$ such that $\Omega'' \Subset \Omega_\epsilon$. There exists a constant c depending only on \mathbb{G} , Ω' and Ω'' such that*

$$(7.3) \quad \int_{\Omega'} \mathcal{F}_2[u_\epsilon] + \frac{3}{4} \sum_{1 \leq i < j \leq d_1} ([X_i, X_j]u_\epsilon)^2 d\mathcal{H}^Q(g) \leq c (\text{osc}(u_\epsilon, \Omega''))^2.$$

COROLLARY 7.6. *Suppose that the step of \mathbb{G} is two. Then $u \in BV_{H,loc}^2(\Omega)$.*

PROOF. Let $\Omega' \Subset \Omega$ arbitrary. It follows from (7.3), that $\{[X_i, X_j]u_{\epsilon_k}\}_{k \in \mathbb{N}}$ is a bounded sequence in $L^2(\Omega')$ for $1 \leq i, j \leq d_1$ whenever $\{\epsilon_k\}_{k \in \mathbb{N}}$ is a sequence of positive numbers decreasing to 0 (and ϵ_1 is small enough). Using the weak compactness of L^2 , it is easy to show that the weak derivatives $[X_i, X_j]u$ of u in Ω exist and belong to $L_{loc}^2(\Omega)$ ($1 \leq i, j \leq d_1$). The claim now follows from Corollary 7.4. □

3. Pointwise second order differentiability of h-convex functions in step two

Theorem 7.7 below tells us that h-convex functions with locally bounded horizontal variation up to order two are twice differentiable almost everywhere. The proof is a straightforward adaptation to the stratified group setting of the proof of the corresponding Euclidean statement (see Theorem 1 in §4 of the sixth chapter of [30]). The main tool is the second order approximate differentiability theorem for functions in $BV_H^2(\Omega)$ (cf. Theorem 7.2).

THEOREM 7.7. *Let $\Omega \subseteq \mathbb{G}$ be an h-convex, open subset and let $u \in BV_{H,loc}^2(\Omega)$ be h-convex. Then, for almost every $g_0 \in \Omega$, there exists a polynomial P_{g_0} of weighted degree at most two such that*

$$\lim_{g \rightarrow g_0} \frac{u(g) - P_{g_0}(g)}{\rho(g_0, g)^2} = 0.$$

PROOF. Let $g_0 \in \Omega$ and let P_{g_0} be a polynomial of weighted degree at most two, such that (7.1) holds. Let us write P instead of P_{g_0} for notational convenience. We have $P = P_1 - P_2$, where P_1 is a polynomial of weighted degree at most one and P_2 is either vanishing or a homogeneous polynomial of weighted degree two. Hence

$$u - P = (u - P_1) + P_2,$$

where $u - P_1 \in BV_{H,loc}^2(\Omega)$ is h-convex and P_2 is either vanishing or a homogeneous polynomial of weighted degree two.

It follows from the stratified mean value Theorem 1.41 in [36] that

$$|P_2(g_1) - P_2(g_2)| \leq c_0 r \rho(g_1, g_2) \quad \forall g_1, g_2 \in B(g_0, r)$$

whenever $B(g_0, r) \Subset \Omega$, where $1 \leq c_0 < +\infty$ is a constant which does not depend on r . Fix r_0 such that $B(g_0, R_0) \Subset \Omega$, where $R_0 = 16(2l \cdot n + 1)r_0$ and $l, n \in \mathbb{N}$ are the constants appearing in Proposition 1.2. By (4.3) and (6.1), we have

$$|(u - P_1)(g_1) - (u - P_1)(g_2)| \leq \left(\frac{8 \cdot l \cdot n \cdot c_1}{r} \int_{B(g_0, R)} |(u - P_1)(g)| d\mathcal{H}^Q(g) \right) \rho(g_1, g_2)$$

for all $g_1, g_2 \in B(g_0, r)$ whenever $0 < r \leq r_0$, with $R = 16(2l \cdot n + 1)r$. It follows that

$$|(u - P)(g_1) - (u - P)(g_2)| \leq \left(\frac{8 \cdot l \cdot n \cdot c_1}{r} \int_{B(g_0, R)} |(u - P_1)(g)| d\mathcal{H}^Q(g) + c_0 r \right) \rho(g_1, g_2)$$

for all $g_1, g_2 \in B(g_0, r)$ whenever $0 < r \leq r_0$, R as above.

Given $0 < \epsilon < 1/2$, let $\eta := (\epsilon/c_0)^Q$. By hypothesis, we can find $0 < r_1 = r_1(\epsilon) \leq r_0$ such that

$$(7.4) \quad \mathcal{H}^Q(\{g \in B(g_0, r) \mid |(u - P)(g)| \geq \epsilon r^2\}) \leq \frac{1}{\epsilon r^2} \int_{B(g_0, r)} |(u - P)(g)| d\mathcal{H}^Q(g) < \eta \mathcal{H}^Q(B(g_0, r))$$

and

$$\frac{8 \cdot l \cdot n \cdot c_1}{c_0 r^2} \int_{B(g_0, R)} |(u - P)(g)| d\mathcal{H}^Q(g) \leq 1$$

for all $0 < r \leq r_1$ with $R = 16(2l \cdot n + 1)r$. Given $0 < r \leq r_1$, $g_1 \in B(g_0, r/2)$, there exists $g_2 \in B(g_0, r)$ such that

$$|(u - P)(g_2)| \leq \epsilon r^2 \quad \text{and} \quad \rho(g_1, g_2) \leq \eta^{1/Q} r.$$

Otherwise

$$\begin{aligned} \mathcal{H}^Q(\{g \in B(g_0, r) \mid |(u - P)(g)| \geq \epsilon r^2\}) &\geq \mathcal{H}^Q(B(g_1, \eta^{1/Q} r)) \\ &= \eta \mathcal{H}^Q(B(g_0, r)), \end{aligned}$$

contradicting (7.4). Consequently,

$$\begin{aligned}
|(u - P)(g_1)| &\leq |(u - P)(g_2)| + |(u - P)(g_1) - (u - P)(g_2)| \\
&\leq \epsilon r^2 + \left(\frac{8 \cdot l \cdot n \cdot c_1}{r} \int_{B(g_0, R)} |(u - P)(g)| d\mathcal{H}^Q(g) + c_0 r \right) \rho(g_1, g_2) \\
&\leq \epsilon r^2 + \left(\frac{8 \cdot l \cdot n \cdot c_1}{c_0 r^2} \int_{B(g_0, R)} |(u - P)(g)| d\mathcal{H}^Q(g) + 1 \right) c_0 \eta^{1/Q} r^2 \\
&\leq 3\epsilon r^2
\end{aligned}$$

for all $0 < r \leq r_1$ and all $g_1 \in B(g_0, r/2)$. \square

From Corollary 7.6 and Theorem 7.7, we immediately obtain the main result of [24] (Theorem 1.1), that is the pointwise second order differentiability of h-convex functions on stratified groups of step two. In view of Theorem 4.6, we can remove the continuity assumption from the hypotheses of Theorem 1.1 in [24].

THEOREM 7.8. *Suppose that the step of \mathbb{G} is two. Let $\Omega \subseteq \mathbb{G}$ be an h-convex, open subset and let $u : \Omega \rightarrow \mathbb{R}$ be h-convex. Then, for almost every $g_0 \in \Omega$, there exists a polynomial P_{g_0} of weighted degree at most two, such that*

$$\lim_{g \rightarrow g_0} \frac{u(g) - P_{g_0}(g)}{\rho(g_0, g)^2} = 0.$$

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