ON THE CONSISTENCY STRENGTH OF THE STRICT Π^1_1 REFLECTION PRINCIPLE

INAUGURALDISSERTATION
DER PHILOSOPHISCH-NATURWISSENSCHAFTLICHEN FAKULTÄT
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VORGELEGT VON
VINCENZO SALIPANTE
VON ITALIEN

LEITER DER ARBEIT:
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CONTENTS

INTRODUCTION

Zermelo-Fraenkel set-theory (ZF) is based on the iterative conception of the set-theoretic universe V. Accordingly, a set is an object that appears in some stage of the cumulative hierarchy, $\bigcup_\alpha V_\alpha$, obtained from the empty set by transfinitely iterating the Power-Set operation through the ordinals. In accordance with the cumulative hierarchy's view of the set-theoretic universe, $V = \bigcup_\alpha V_\alpha$, is Zermelo's pivotal proposal [26] to consider initial segments as models for the set-theoretic axioms. For example:

$$V_\omega \models \mathsf{ZF} \setminus \text{Infinity},$$
$$V_{\omega+\omega} \models \mathsf{ZF} \setminus \text{Replacement}.$$

The question is: for which ordinals α do we have $V_\alpha \models \mathsf{ZF}$? From the examples above this reduces to asking why ω "satisfies" Replacement and why $\omega+\omega$ "satisfies" Infinity. In the former, it is that ω is a regular ordinal whereas the latter it is that $\omega+\omega$ is a limit ordinal greater than ω. Hence the question is: which regular limit ordinals greater than ω "satisfy" ZF?

Since any regular limit ordinal is a cardinal and Replacement is known to fail in V_α whenever α is a successor cardinal, we are led to consider regular limit cardinals greater than ω. However, if α is such a cardinal, then all we can conclude is that $L_\alpha \models \mathsf{ZF}$, where L_α is the α-th stage of the constructible hierarchy. In order to obtain $V_\alpha \models \mathsf{ZF}$, we need our cardinal to further be closed under cardinal exponentiation. Note that cardinals satisfying this property alone are limit cardinals and we call them *strong limit cardinals*. Note that ω is such a cardinal. Indeed Zermelo [26] proved that

$$\text{if } \alpha > \omega \text{ is a regular strong limit cardinal, then } V_\alpha \models \mathsf{ZF}.$$

Therefore, the existence of such cardinals entails the consistency of ZF. It follows, by Gödel's Second Incompleteness Theorem, that such cardinals cannot be proved to exist in ZF. This justifies that regular (strong) limit cardinals greater than ω are called *(strongly) weakly inaccessible*. Hence, inaccessible in the sense of going beyond all the ordinals that can be reached by Power-Set and Replacement in ZF.

It is worth noticing that, due to the Axiom of Infinity, ω is the only regular strong limit cardinal whose existence can be established in ZF. The existential postulation of an inaccessible cardinal is the first example of a *strong axiom of infinity* or otherwise known as a *large cardinal axiom*.

So far we have seen that if α is a strongly inaccessible cardinal then $V_\alpha \models \mathsf{ZF}$. However, the converse does not hold: a consequence of the Montague-Vaught Theorem [21]. In this sense, ZF does not characterize inaccessibility. To the aim of achieving such a characterization, all that is required is to formulate Replacement as a single axiom rather than a schema. Hence, we consider an axiomatization of class-set theory, as given, for example, by von Neumann and Bernays, (VNB) (see Bernays [3] and von Neumann [25]). Then under the standard interpretation of class-variables as ranging over arbitrary subsets of the domain V_α, we obtain

$$\alpha \text{ is strongly inaccessible if and only if } V_\alpha \models \mathsf{VNB}.$$

Since VNB is finitely axiomatizable, the existential postulation of a strongly inaccessible cardinal is equivalent to asserting that $\exists\alpha(\mathsf{VNB})^{V_\alpha}$ is true in V; where, $(\mathsf{VNB})^{V_\alpha}$ is the result of restricting bound set- and class-variables to V_α and $V_{\alpha+1}$, respectively. Under this interpretation, we talk about sets (as elements of V_α), classes (as elements of $V_{\alpha+1}$) and proper-classes (as elements of $V_{\alpha+1} \setminus V_\alpha$). Hence these are proper-classes only in this relative sense, since each *proper-class* of V_α will be coextensive with a *set* in $V_{\alpha+1}$.

Since $\mathsf{VNB} \not\vdash \exists\alpha(\mathsf{VNB})^{V_\alpha}$, it is natural to consider $\mathsf{VNB} + \exists\alpha(\mathsf{VNB})^{V_\alpha}$ which entails $\mathsf{VNB} \rightarrow \exists\alpha(\mathsf{VNB})^{V_\alpha}$. According to this implication, the closure of V under the axioms of VNB can be reasonably regarded as an existence condition for the strongly inaccessible cardinals. By generalizing the implication above to arbitrary properties φ then we obtain $\varphi \rightarrow \exists\alpha(\varphi)^{V_\alpha}$. Axioms of this form have been called *Reflection principles*, because they express the fact that V's possession of a certain property is reflected by V_α's possession of it, for some ordinal α. In other words, the whole universe of sets is beyond being captured by any closure condition on sets; so that, any closure property we think to be ascribable to the universe must already close off at some arbitrarily large initial segment of the universe itself, viewed as a kind of partial universe approximating the totality of all sets.

Reflection axiom schemata are classified according to the logical complexity of set-theoretical formulae expressing the reflected properties. Initially formulated by Lévy [18] for first-order set-theoretical properties, the principle was extended to include second-order properties and used as basis for an axiomatization of class-set theory by Bernays [4]. Further generalizations of the reflected properties to finite or even transfinite higher-orders languages have been postulated by Hanf and Scott [12]. Asserting this principle for Π^1_1 formulae entails the existence of arbitarily large Mahlo cardinals, see Gloede [9]. Hence by the reflection principles we are led to a hierarchy of cardinal existence axioms (inaccessible, hyper-inaccessible,..., Mahlo, hyper-Mahlo,...), which results in progressively axiomatizing increasingly large segments of the cumulative hierarchy. Hence, reflection principles formally capture the open-endedness character of the set-theoretic universe.

Over the standard structure of the natural numbers, as first observed by Kreisel, there exists a striking difference between predicates of the form

$$\forall f \in \mathbb{N}^{\mathbb{N}} \exists y \varphi$$

and of the form

$$\forall f \in \{0,1\}^{\mathbb{N}} \exists y \varphi,$$

where φ is a recursive predicate of natural numbers. Whereas every Π^1_1 set in the analytical hierarchy is definable by some formula of the first form, the sets defined by formulae of the second form (i.e. in terms of quantification over characteristic functions) are all recursively enumerable. The latter predicates were dubbed *strict* Π^1_1 by Barwise in [1] and [2]. Hence, over the standard structure of the natural numbers strict Π^1_1 and Σ_1 predicates coincide. When generalizing recursion theory to domains other than the natural numbers, to admissible sets for instance, then strict Π^1_1 predicates have been recognized as probably the most adequate analog of recursive enumerability. Indeed, over countable admissible sets, strict Π^1_1 predicates are equivalent to Σ_1 predicates. However, this is no longer the case for uncountable admissible sets. It was the context of generalized recursion theory on admissible sets that originated the formulation of the strict Π^1_1 reflection principle. The principle might be regarded as a set-theoretic version of König's Lemma. The reader is again referred to Barwise for a thorough introduction to the strict Π^1_1 reflection principle.

In the present contribution, following upon Bernays [4], we start off by introducing and proof-theoretically analyzing a second-order axiomatization of admissible sets based on the strict Π^1_1 reflection principle. We use as base theory Jäger's $\mathsf{KPu}^{\mathsf{r}}$, introduced in [14], with the adjunction of the strict Π^1_1 reflection principle and Δ^{C}_1-Comprehension (the superscript "C" is to indicate that class-parameters are allowed to appear in the defining formulae of the Comprehension schema). The resulting theory is denoted by $\mathsf{sKPu}^{\mathsf{r}}_2{\upharpoonright}$. In Chapter 1 we will show that $\mathsf{sKPu}^{\mathsf{r}}_2{\upharpoonright}$ is proof-theoretically reducible to Peano Arithmetic PA (i.e., a conservative extension of PA), *as long as* class parameters are not allowed in the defining formulae of the Separation schema.

It must be admitted, however, that in having such a restrictive condition on the Separation schema, only a slight interplay between classes and sets is attainable in $\mathsf{sKPu}^{\mathsf{r}}_2{\upharpoonright}$. Therefore such a restriction is unorthodox from a pure set-theoretic perspective. Accordingly, in Chapter 2, we strengthen the schema by permitting free class parameters to occur in its defining formulae. Hence the schema can then be reformulated as a single axiom which we call *Aussonderungsaxiom*. As for the Π^1_1 reflection principle, it will be shown that the strict Π^1_1 reflection principle along with the Aussonderungsaxiom implies the existence of the Power-Set axiom and admits a self-strengthening to a schema with a super-transitive reflecting set (that is, a reflecting transitive set closed under the subsets of its members). On the account of Aussonderungsaxiom the strict Π^1_1 reflection principle gains its actual "power" determining a significant increase in strength of the resulting theory, $\mathsf{sKPu}^{\mathsf{r}}_2$. Indeed the consistency of PA is derivable in $\mathsf{sKPu}^{\mathsf{r}}_2$. However, as we shall show, the existence of ω remains underivable in $\mathsf{sKPu}^{\mathsf{r}}_2$. Hence, contrary to the Π^1_1 reflection principle, we cannot regard the strict Π^1_1 reflection principle as a strong axiom of infinity. The exact consistency strength of $\mathsf{sKPu}^{\mathsf{r}}_2$ is established: $\mathsf{sKPu}^{\mathsf{r}}_2$ turns out to be conservative for set-theoretic Π_2 sentences over the power admissible set theory, as axiomatized by $\mathsf{KPu}^{\mathsf{r}}$ with the Power-Set axiom adjoined (see also Barwise [2] and Friedman [8]).

We conclude Chapter 2, by showing that the strict Π^1_1 reflection principle along with the Aussonderungsaxiom also makes the Δ^{c}_1-Comprehension redundant. This justifies the replacement of this axiom by the schema of Predicative Comprehension in Chapter 3. This results in a theory denoted by sBL_1. In the literature (see Gloede [9]), BL_1 denotes the Bernays-Lévy class-set theory corresponding to VNB augumented with any instance of the schema of Π^1_1 reflection. Hence sBL_1 should be VNB augumented with any instance of the schema of strict Π^1_1 reflection. Indeed, this makes sense since we will show that sBL_1 contains VNB as a subsystem and further the strict Π^1_1 reflection principle will be proved to be independent from VNB.

The theory sBL_1 comprises the following non-logical axioms: Predicative Comprehension, Infinity, Foundation, Aussonderungsaxiom and strict Π^1_1 reflection. It will be proved that both the axioms of Infinity and Predicative Comprehensions are independent from the remaining axioms of sBL_1. In particular, we will show that by striking out the axiom of Infinity from sBL_1, V_ω is a model of this theory, otherwise we need to make a "huge" jump to a *weakly compact cardinal*: a strongly inaccessible cardinal with the tree-property. We will also show that sBL_1 and BL_1 admit the same standard models. We conclude Chapter 3, by proving the relative consistency of Gödel's Axiom of Constructibility with sBL_1. The exact consistency strength of sBL_1 remains an open problem, see Appendix B. It is a conjecture of Sy Friedman that every instance of the schema of Π^1_1 reflection is derivable in sBL_1 plus some kind of Axiom of Choice. If so, then BL_1 would be a susbsystem of sBL_1+V=L. Hence, on the account of the above-mentioned equiconsistency result between sBL_1 and sBL_1+V=L, we would have that the Π^1_1 reflection principle is consistent with sBL_1. It would follow that for the consistency of the Π^1_1 reflection principle an external appeal to a weakly compact cardinal will be no longer necessary: the assumed consistency of sBL_1 would suffice.

A fruitful offshoot of the study of large cardinals has been the investigation of their various analogues in restricted contexts e.g., admissible set and recursion theory, constructive set theory and Explicit mathematics. The first substantive move in this direction was made in the early 1970's by Richter and Aczel [23] in the theory of inductive definitions. With the admissible ordinals playing the role of regular cardinals, analogues of Inaccessible, Mahlo and Indescribable cardinals were developed in this context.

To the aim of providing a general framework allowing an uniform treatment of these different analogues of such cardinals, Feferman proposed in [7], the *Operational Set Theory* (OST). The cardinal notions introduced there are for Inaccessible, Mahlo and Weakly Compact. A reflection principle entailing the existence of all these cardinals is also formulated in this context. The consistency strength of OST with this reflection principle adjoined, which we denote by $\mathsf{OST} + \mathsf{Rfn}^{\forall}_{\mathsf{op}}$, has not been established yet. A partial result in this direction has however been achieved: in Appendix A, it will be shown that the consistency of $\mathsf{OST} + \mathsf{Rfn}^{\forall}_{\mathsf{op}}$ is not provable in ZFC.

CHAPTER 1

ADMISSIBLE SET THEORY

Theories for admissible sets are generally based on Kripke-Platek set theory KP, a subsystem of Zermelo-Fraenkel set theory ZF, whose transitive standard models are the admissible sets.

One of the prominent extension, among several others, of KP is the theory KPu. KPu corresponds to Barwise's theory KPU^+ of admissible sets *above* natural numbers as urelements [2]. We have then two axiom schemata of induction, namely complete induction on the natural numbers and full $\in$-induction. The theory KPu is introduced and proof-theoretically analyzed by Jäger in [13], where it is shown that KPu proves the same arithmetical sentences as Feferman's system ID_1 of one positive, non-iterated inductive definition and its corresponding proof-theoretic ordinal is the Bachmann-Howard ordinal $\theta_{\varepsilon_{\Omega+1}}0$.

Our starting point is Jäger's theory KPu^r, described in [14]. KPu^r is obtained from KPu, by restricting each of the two axiom schemata of induction to sets (hence the superscript in KPu^r). It is also known from here that KPu^r is a conservative extension of Peano Arithmetic (PA) and its corresponding proof-theoretic ordinal is ε_0.

1.1 THE THEORY KPu^r

Let Peano Arithmetic, PA, be formulated in the first order language $\mathcal{L}$ with a constant for every natural number and countably many number variables $u, v, w, x, y, z, ...$ We assume that there are *no proper function symbols* in $\mathcal{L}$. Accordingly we have symbols for all primitive recursive relations and for the graphs of the primitive recursive functions. In particular, we let the binary relation symbol Sc denote the graph of the (primitive recursive) successor function. The *number terms* ($r, s, t, r_0, s_0, t_0, ...$ with or without numerical subscripts) of $\mathcal{L}$ are only the number constants and number variables. The *atomic formulae* of $\mathcal{L}$ are all expressions $R(s_1, ..., s_n)$ for R being a symbol for an n-ary primitive recursive relation. The *formulae* ($\varphi, \psi, \varphi_0, \psi_0, ...$ with or without numerical subscripts) of $\mathcal{L}$ form the smallest collection containing the atomic formulae of $\mathcal{L}$ closed under conjunction, negation and universal quantification.

The theory $\mathsf{KPu}^{\mathsf{r}}$ is formulated in the extended language $\mathcal{L}^* = \mathcal{L}(\in, \mathsf{N}, \mathsf{S})$ obtained from $\mathcal{L}$ by adjunction of the membership relation symbol $\in$, the set constant N for the set of natural numbers and the unary relation symbols S for sets. The *terms* $(a, b, c, a_0, b_0, c_0, ...)$ of $\mathcal{L}^*$ are the terms of $\mathcal{L}$ plus the set constant N. The *atomic formulae* of $\mathcal{L}^*$ are the atomic formulae of $\mathcal{L}$ plus all the expressions $a \in b$ and $\mathsf{S}(a)$ for any term a and b. The *formulae* $(\varphi, \psi, \varphi_0, \psi_0, ...)$ of $\mathcal{L}^*$ form the smallest collection containing all the atomic formulae of $\mathcal{L}^*$ closed under negation, conjunction and universal quantification. All the remaining logical operators are introduced as follows: The following abbreviations are introduced:

$$\begin{aligned} \varphi \vee \psi \ &:= \neg(\neg\varphi \wedge \neg\psi); \\ \varphi \rightarrow \psi \ &:= (\neg\varphi) \vee \psi; \\ \varphi \leftrightarrow \psi \ &:= (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi); \\ \exists x\varphi \ &:= \neg\forall x\neg\varphi. \end{aligned}$$

Classifications Δ_0, Σ, Π, Σ_n, Π_n of formulae of $\mathcal{L}^*$ correspond to the Lévy's standard hierarchy of formulae of ZF (Lévy [19]). The notation $\vec{a}$ is shorthand for a finite string $a_1, ..., a_n$ whose length will be clear from the context. Also, equality between objects is not a primitive symbol of the language, but it is taken to be defined by

$$(a = b) \ := \begin{cases} (a \in \mathsf{N} \wedge b \in \mathsf{N} \wedge (a =_{\mathsf{N}} b)) \\ \qquad\qquad\qquad \vee \\ (\mathsf{S}(a) \wedge \mathsf{S}(b) \wedge \forall x(x \in a \leftrightarrow x \in b)) \end{cases}$$

where $=_{\mathsf{N}}$ is the symbol for the primitive recursive equality on the natural numbers.

DEFINITION 1.1.1. For any term a and any formula φ of $\mathcal{L}^*$, the relativization of φ to a, denoted by $\varphi^{(a)}$, is the formula resulting from φ by binding all the unbounded quantifiers occurring in φ to a; that is, replacing

$$\begin{array}{lcl} \exists x(...) & \text{by} & \exists x[x \in a \wedge (...)], \\ \forall x(...) & \text{by} & \forall x[x \in a \rightarrow (...)]. \end{array}$$

Bounded quantification is abbreviated as usual:

$$\begin{array}{lcl} (\exists x \in a)\varphi & \text{for} & \exists x[x \in a \wedge \varphi], \\ (\forall x \in a)\varphi & \text{for} & \forall x[x \in a \rightarrow \varphi]. \end{array}$$

In addition, we freely make use of all standard set-theoretic notations and write, for example, $\mathsf{Tran}(a)$ and $\mathsf{On}(a)$ for the following Δ_0 formulae:

$$\begin{aligned} \mathsf{Tran}(a) \ &:= \mathsf{S}(a) \wedge \forall x(x \in a \rightarrow x \subseteq a) \\ \mathsf{On}(a) \ &:= \mathsf{Tran}(a) \wedge (\forall y \in a)(\mathsf{Tran}(y)). \end{aligned}$$

The logical axioms of $\mathsf{KPu^r}$ comprise the usual axioms of classical first order logic with equality. The non-logical axioms are divided into the following four groups.

I. Ontological axioms. We have for all terms a, b and $\vec{c}$ of $\mathcal{L}^*$, all relation symbols R of $\mathcal{L}$ and all axioms $\varphi(\vec{x})$ of group III whose free variables belong to the list $\vec{x}$:

- $a \in \mathsf{N} \leftrightarrow \neg \mathsf{S}(a)$,
- $R(\vec{c}) \rightarrow \vec{c} \in \mathsf{N}$,
- $a \in b \rightarrow \mathsf{S}(b)$.

II. Number-theoretic axioms. We have for all axioms $\varphi(\vec{x})$ of PA which are not instances of the schema of complete induction and whose free variables belong to the list $\vec{x}$:

- $(\forall \vec{x} \in \mathsf{N})\varphi^{(\mathsf{N})}(\vec{x})$.

III. Set-theoretic axioms. We have for all terms a, b and all Δ_0 formulae $\varphi(a)$ and $\psi(a, b)$ of $\mathcal{L}^*$:

- $\exists x(a \in x \wedge b \in x)$ (Pairing),
- $\exists x(b \subseteq x \wedge \mathsf{Tran}(x))$ (Transitive Hull),
- $\exists x(\mathsf{S}(x) \wedge \forall z(z \in x \leftrightarrow z \in a \wedge \varphi(z))$ (Δ_0-Sep),
- $(\forall x \in a)\exists y \psi(x, y) \rightarrow \exists z(\mathsf{S}(z) \wedge (\forall x \in a)(\exists y \in z)\psi(x, y))$ (Δ_0-Coll).

IV. Induction axioms. These consist of the following axioms of complete induction on the natural numbers for sets and of $\in$-induction respectively:

- $0 \in a \wedge (\forall x, y \in \mathsf{N})(x \in a \wedge \mathsf{Sc}(x, y) \rightarrow y \in a) \rightarrow \mathsf{N} \subseteq a$ (Δ_0-$\mathsf{I}_{\mathbb{N}}$),
- $\exists y(y \in a) \rightarrow \exists y(y \in a \wedge \forall z(z \in y \rightarrow z \notin a))$ (Δ_0-$\mathsf{I}_{\in}$).

Remark 1.1.2. It is worth mentioning that over the theory $\mathsf{KPu^r}$ the axioms of $\in$-induction and of complete induction on the natural numbers for sets are provably equivalent to the corresponding schemata restricted to the class of Δ_0-formulae of $\mathcal{L}^*$. Hence the notation Δ_0-$\mathsf{I}_{\in}$ and Δ_0-$\mathsf{I}_{\mathbb{N}}$.

Let us conclude this section with two observations which will be often invoked in the remaining part of our work.

Let the Union axiom be (i.e. the universal closure of):

$$\exists x[\mathsf{S}(x) \wedge \forall z(z \in x \leftrightarrow \exists v(z \in v \wedge v \in a))].$$

PROPOSITION 1.1.3. *The* UNION *axiom is derivable in* $\mathsf{KPu^r}$.

Proof. Let us argue informally within the theory $\mathsf{KPu^r}$. Consider the following instance of Δ_0-SEP:

$$\forall y \exists x[\mathsf{S}(x) \wedge \forall z(z \in x \leftrightarrow z \in y \wedge \exists v(z \in v \wedge v \in a))]. \tag{1}$$

Replacing the term b in the axiom of TRANSITIVE HULL by $\{a\}$, we obtain

$$\exists y(a \in y \wedge \mathsf{Tran}(y)).$$

And this, along with the following implication

$$\exists y(a \in y \wedge \mathsf{Tran}(y)) \rightarrow \exists y(a \subseteq y \wedge \mathsf{Tran}(y)),$$

logically entails, by MODUS PONENDO PONENS,

$$\exists y(a \subseteq y \wedge \mathsf{Tran}(y)).$$

Futher,

$$\exists y(a \subseteq y \wedge \mathsf{Tran}(y)) \rightarrow \exists y \forall z(\exists v(z \in v \wedge v \in a) \rightarrow z \in y).$$

These last two lines logically entail, by MODUS PONENDO PONENS, the following

$$\exists y \forall z(\exists v(z \in v \wedge v \in a) \rightarrow z \in y). \tag{2}$$

From (1) and (2) just using logic we therefore obtain:

$$\exists x[\mathsf{S}(x) \wedge \forall z(z \in x \leftrightarrow \exists v(z \in v \wedge v \in a))].$$

□

Let the PAIR axiom be (i.e. the universal closure of):

$$\exists y(\mathsf{S}(y) \wedge \forall z[z \in y \leftrightarrow (z = a \vee z = b)])$$

PROPOSITION 1.1.4. *The following are derivable in* $\mathsf{KPu^r}$:

(*a*) PAIR,

(*b*) Δ-SEP,

(*c*) Σ-COLL.

Proof. (*a*) follows from PAIRING and Δ_0-SEP. For a proof of (*b*) and (*c*) the reader is referred to Barwise [2], p.17, Theorem I.4.4 and Theorem I.4.5, respectively.

□

1.2 THE THEORY $\mathsf{sKPu}_2^{\mathsf{r}}\!\restriction$

The second-order language $\mathcal{L}_2^*$ of $\mathsf{sKPu}_2^{\mathsf{r}}\!\restriction$, is obtained from $\mathcal{L}^*$ by adjunction of an infinite stock of class (monadic predicate) variables $X, Y, Z, ...$, together with universal quantifiers binding them. Here class variables are our only class terms. The atomic formulae are then expanded to include $a \in X$ for any term a and class variable X. We are using the symbol "$\in$" ambiguously, to denote both a relation between sets and sets and a heterogenious relation between sets and classes, but no confusion will result. Formulae of $\mathcal{L}_2^*$ are built up from the atomic formulae of $\mathcal{L}_2^*$ by closing under the propositional operators "$\neg$", "$\wedge$" and universal quantification with respect both to set and class variables. The existential class quantifer is defined as follows:

$$\exists X\varphi \;:=\; \neg\forall X\neg\varphi.$$

The definition of classifications Δ_0^{C}, Σ^{C}, Π^{C}, Σ_n^{C} and Π_n^{C} of formulae of $\mathcal{L}_2^*$ is just as for the classifications Δ_0, Σ, Π, Σ_n, Π_n of $\mathcal{L}^*$, but with the understanding that formulae in the former classifications might contain class variables via the expanded class of atomic formulae; hence the superscript "C". A formula is said to be *predicative* if it contains no bound class variables. Hence predicative in the sense of not including a reference by a quantifier to the realm of classes. In line with the definition of the classifications Σ_n and Π_n for $\mathsf{KPu}^{\mathsf{r}}$, we define classes Σ_n^1 and Π_n^1 as follows: a formula φ of $\mathcal{L}_2^*$ is said to be in Σ_n^1 if it is given by prefixing n alternating class quantifiers to a predicative formula, the leading quantifier being existential, "$\exists$". The superscript in "Σ_n^1" tells us that we are measuring the second-order quantifier complexity of a formula φ. Dually, φ is said to be in Π_n^1 if it is given by prefixing n alternating class quantifiers to a predicative formula, the leading quantifier being universal, "$\forall$". Therefore, in particular, a Π_1^1 formula is a formula of the form $\forall X\varphi$ where φ is predicative. The definition of a strict Π_1^1 formula (S-Π_1^1) is just like the definition of Π_1^1 except that the formula φ is required to be Σ^{C}. Dually, a formula φ of $\mathcal{L}_2^*$ is said to be strict Σ_1^1 (S-Σ_1^1) if it is given by prefixing an existential class quantifier to a Π^{C} formula.

REMARK 1.2.1. Towards the definition of a S-Π_1^1 (S-Σ_1^1) formula, it is worth warning the reader that such a definition differs from the one given by Barwise in [2] (Definition VIII.2.1, on page 316). Barwise's class of S-Π_1^1 (S-Σ_1^1) formulae correspond to our class of *essentially strict* Π_1^1 (Σ_1^1) *formulae*, see Definition 1.4.1 on page 20.

In formulating the theory $\mathsf{KPu}^{\mathsf{r}}$ we chose to take equality as a defined notion, and accordingly we make the same choice here with respect to classes. Class equality is then only an expression for extensional equality:

$$X = Y \;:=\; \forall x(x \in X \leftrightarrow x \in Y).$$

A special axiom of extensionality for classes is therefore not needed. Neither do we need a special axiom expressing the substitutivity of equal classes. For, any instance of the schema

$$X = Y \rightarrow (\varphi(X) \leftrightarrow \varphi(Y))$$

is derivable from the previous definition of class equality, with the help of predicate claculus.

The class existence axiom in this initial part of our work is given by the following Comprehension schema restricted to the formulae of $\mathcal{L}_2^*$ of logical complexity Δ_1^{c}:

$$\forall x(\varphi(x) \leftrightarrow \neg\psi(x)) \rightarrow \exists Y \forall x(x \in Y \leftrightarrow \varphi(x)) \qquad (\Delta_1^{\mathrm{c}}\text{-CA}),$$

where φ and ψ are Σ_1^{c} and do not contain the class variable Y free but may contain free set and class parameters besides x.

REMARK 1.2.2. For any formula $\varphi_0(x)$ of $\mathcal{L}_2^*$ of logical complexity Δ_1^{c}, the corresponding instance of Δ_1^{c}-CA yields a class (depending on the other parameters occurring in $\varphi_0(x)$ other than x) consisting of just those sets x such that $\varphi_0(x)$. By class equality there is exactly one such a class.

Expressions of the form

$$\{\, x \mid \varphi_0(x) \,\}$$

are called *class abstracts*. Boldface upper case letters **A**,**B**,**C**,... are used as metamathematical symbols standing for class-abstracts. As examples of class abstracts we have,

$$\mathbf{ON} \;:=\; \{\, x \mid \mathsf{On}(x) \,\} \qquad \text{and} \qquad \mathbf{V} \;:=\; \{\, x \mid x = x \,\}.$$

Further, lower case Greek letters [1] α, β, γ,... are to be understood as "relativized" variables ranging over the class-abstact **ON**, that is

$$\begin{array}{lcl} \exists\alpha(...\alpha...) & \text{is} & \exists y[\mathsf{On}(y) \wedge (...y...)] \\ \forall\alpha(...\alpha...) & \text{is} & \forall y[\mathsf{On}(y) \rightarrow (...y...)]. \end{array}$$

PROPOSITION 1.2.3. *For any set term a, the following is a direct consequence of* Δ_1^{c}-CA:

$$\exists Y \forall x(x \in Y \leftrightarrow x \in a).$$

REMARK 1.2.4. It actually turns out that for any set a, Δ_1^{c}-CA yields a class consisting of exactly the same members as a. Thus there should be no distinction between the set a and the class $\{\, x \mid x \in a \,\}$.

This simple observation motivates our subsequent definition of equality between sets and classes:

$$X = y \;:=\; \forall x(x \in X \leftrightarrow x \in y).$$

Any instance of the schema of full substituvity of equality is now derivable from this definition of equality between sets and classes, with the help of predicate calculus.

---

[1] With the only exception of φ and ψ, with or without numerical subscripts, which will be always used to denote formulae.

PROPOSITION 1.2.5. *Any instance of the following schema is derivable*

$$X = y \to (\varphi(X) \leftrightarrow \varphi(y)).$$

Before stating the strict Π^1_1 reflection principle we need to extend the definition of relativization to second-order formulae of $\mathcal{L}^*_2$.

DEFINITION 1.2.6. For any term a and any formula φ of $\mathcal{L}^*_2$, we define $\varphi^{(a)}$, the relativization of φ to a, to be the formula obtained from φ by binding all the unbounded set quantifiers occurring in φ to a (as in Definition 1.1.1) and replacing

$$\begin{array}{lcl} \exists X(...) & \text{by} & \exists X[X \subseteq a \wedge (...)] \\ \forall X(...) & \text{by} & \forall X[X \subseteq a \to (...)] \end{array}$$

The reason for defining the relativization of the class quantifiers in this way will appear clear in Chapter 2.

The strict Π^1_1 reflection axiom schema reads as follows

$$\begin{aligned} &\varphi(v_0, ..., v_n, C_0, ..., C_m) \to \\ &\to \exists y[\mathsf{Tran}(y) \wedge v_0, ..., v_n \in y \wedge \varphi^{(y)}(v_0, ..., v_n, C_0, ..., C_m)] \qquad (\text{S-}\Pi^1_1\ \text{RFN}), \end{aligned}$$

for any S-Π^1_1 formulae φ of $\mathcal{L}^*_2$ in which y does not occur free and with no free variables besides the displayed ones and not necessarily all of them.

REMARK 1.2.7. Under our definition of relativization of $\varphi^{(a)}$ for $\mathcal{L}^*_2$ formulae φ if φ is S-Π^1_1 (S-Σ^1_1), then $\varphi^{(a)}$ is S-Π^1_1 (S-Σ^1_1) with free variables those of φ and the new variable a.

The underlying logic of $\mathsf{sKPu}_2^{\mathsf{r}}\restriction$ is the classical second-order with first-order equality. The non-logical axioms are divided into the following four groups.

I. ONTOLOGICAL AXIOMS. As in $\mathsf{KPu}^{\mathsf{r}}$.

II. NUMBER-THEORETIC AXIOMS. As in $\mathsf{KPu}^{\mathsf{r}}$.

III. CLASS/SET-THEORETIC AXIOMS.

- Δ_0-SEP,
- S-Π^1_1 RFN,
- Δ^{C}_1-CA

IV. INDUCTION AXIOMS. These consist of the following axioms for induction on the natural numbers and $\in$-induction respectively:

- $0 \in A \wedge (\forall x, y \in \mathsf{N})(x \in A \wedge \mathsf{Sc}(x, y) \to y \in A) \to \mathsf{N} \subseteq A \qquad (\mathsf{I}^2_{\mathbb{N}}),$

- $\exists y(y \in A) \to \exists y(y \in A \wedge \forall z(z \in y \to z \notin A))$ $(\mathsf{I}^2_\in)$.

It is worth stressing that,

Class parameters are not allowed in the defining formulae of Δ_0-SEP.

PROPOSITION 1.2.8. *For all* $\mathcal{L}^*_2$ *formulae* $\varphi(\vec{v}, \vec{C})$ *with no free variables besides the displayed ones and not necessarily all of them and for any set* b *wich does not occur free in the list* $\vec{v}$ *we have the following provable in* $\mathsf{sKPu}^{\mathsf{r}}_2\!\restriction$:

$$\vec{v} \in b \to \Big(\varphi^{(b)}(\vec{v}, \vec{C}) \leftrightarrow \varphi^{(b)}(\vec{v}, \vec{C} \cap b)\Big).$$

Proof. The proof proceeds by induction over the build-up of $\varphi(\vec{v}, \vec{C})$.

$\underline{\varphi(\vec{v}, \vec{C}) \equiv v \in C}$: Then we have the following derivable in $\mathsf{sKPu}^{\mathsf{r}}_2\!\restriction$:

$$v \in b \to \Big(v \in C \leftrightarrow v \in C \wedge v \in b\Big).$$

$\underline{\varphi(\vec{v}, \vec{C}) \equiv \neg\varphi_0(\vec{v}, \vec{C})}$: By I.H.

$$\vec{v} \in b \to \Big(\varphi_0^{(b)}(\vec{v}, \vec{C}) \leftrightarrow \varphi_0^{(b)}(\vec{v}, \vec{C} \cap b)\Big).$$

Whence by means of propositional calculus

$$\vec{v} \in b \to \Big(\neg\varphi_0^{(b)}(\vec{v}, \vec{C}) \leftrightarrow \neg\varphi_0^{(b)}(\vec{v}, \vec{C} \cap b)\Big).$$

That is

$$\vec{v} \in b \to \bigg(\Big(\neg\varphi_0(\vec{v}, \vec{C})\Big)^{(b)} \leftrightarrow \Big(\neg\varphi_0(\vec{v}, \vec{C} \cap b)\Big)^{(b)}\bigg).$$

$\underline{\varphi(\vec{v}, \vec{C}) \equiv \varphi_0(\vec{v}, \vec{C}) \wedge \varphi_1(\vec{v}, \vec{C})}$: By I.H.

$$\vec{v} \in b \to \Big(\varphi_0^{(b)}(\vec{v}, \vec{C}) \leftrightarrow \varphi_0^{(b)}(\vec{v}, \vec{C} \cap b)\Big)$$

and

$$\vec{v} \in b \to \Big(\varphi_1^{(b)}(\vec{v}, \vec{C}) \leftrightarrow \varphi_1^{(b)}(\vec{v}, \vec{C} \cap b)\Big).$$

Hence

$$\vec{v} \in b \to \Big(\varphi_0^{(b)}(\vec{v}, \vec{C}) \leftrightarrow \varphi_0^{(b)}(\vec{v}, \vec{C} \cap b)\Big) \wedge \Big(\varphi_1^{(b)}(\vec{v}, \vec{C}) \leftrightarrow \varphi_1^{(b)}(\vec{v}, \vec{C} \cap b)\Big).$$

Whence by means of propositional calculus

$$\vec{v} \in b \to \Big(\varphi_0^{(b)}(\vec{v}, \vec{C}) \wedge \varphi_1^{(b)}(\vec{v}, \vec{C})\Big) \leftrightarrow \Big(\varphi_0^{(b)}(\vec{v}, \vec{C} \cap b) \wedge \varphi_1^{(b)}(\vec{v}, \vec{C} \cap b)\Big).$$

That is

$$\vec{v} \in b \to \bigg(\Big(\varphi_0(\vec{v}, \vec{C}) \wedge \varphi_1(\vec{v}, \vec{C})\Big)^{(b)} \leftrightarrow \Big(\varphi_0(\vec{v}, \vec{C} \cap b) \wedge \varphi_1(\vec{v}, \vec{C} \cap b)\Big)^{(b)}\bigg).$$

$\underline{\varphi(\vec{v}, \vec{C}) \equiv \forall x \varphi_0(x, \vec{v}, \vec{C})}$: Fix an arbitary set term a such that a does not occur free anywhere else. By I.H.

$$\vec{v} \in b \wedge a \in b \to \Big(\varphi_0^{(b)}(a, \vec{v}, \vec{C}) \leftrightarrow \varphi_0^{(b)}(a, \vec{v}, \vec{C} \cap b)\Big).$$

Whence by means of propositional calculus

$$\vec{v} \in b \to \Big((a \in b \to \varphi_0^{(b)}(a, \vec{v}, \vec{C})) \leftrightarrow (a \in b \to \varphi_0^{(b)}(a, \vec{v}, \vec{C} \cap b))\Big).$$

By generalizing with respect to a, then

$$\vec{v} \in b \to \forall x\Big((x \in b \to \varphi_0^{(b)}(x, \vec{v}, \vec{C})) \leftrightarrow (x \in b \to \varphi_0^{(b)}(x, \vec{v}, \vec{C} \cap b))\Big).$$

From which we infer

$$\vec{v} \in b \to \Big(\forall x(x \in b \to \varphi_0^{(b)}(x, \vec{v}, \vec{C})) \leftrightarrow \forall x(x \in b \to \varphi_0^{(b)}(x, \vec{v}, \vec{C} \cap b))\Big).$$

That is

$$\vec{v} \in b \to \bigg(\Big(\forall x \varphi_0(x, \vec{v}, \vec{C})\Big)^{(b)} \leftrightarrow \Big(\forall x \varphi_0(x, \vec{v}, \vec{C} \cap b)\Big)^{(b)}\bigg).$$

$\underline{\varphi(\vec{v}, \vec{C}) \equiv \forall X \varphi_0(X, \vec{v}, \vec{C})}$: Fix an arbitary class variable A such that A does not occur free anywhere else. By I.H.

$$\vec{v} \in b \to \Big(\varphi_0^{(b)}(A, \vec{v}, \vec{C}) \leftrightarrow \varphi_0^{(b)}(A \cap b, \vec{v}, \vec{C} \cap b)\Big).$$

From which we infer

$$\vec{v} \in b \wedge A \subseteq b \to \Big(\varphi_0^{(b)}(A, \vec{v}, \vec{C}) \leftrightarrow \varphi_0^{(b)}(A \cap b, \vec{v}, \vec{C} \cap b)\Big).$$

Note that the upon the assumption that $A \subseteq b$, then $A \cap b = A$. Hence

$$\vec{v} \in b \wedge A \subseteq b \to \Big(\varphi_0^{(b)}(A, \vec{v}, \vec{C}) \leftrightarrow \varphi_0^{(b)}(A, \vec{v}, \vec{C} \cap b)\Big).$$

Whence

$$\vec{v} \in b \rightarrow \Big((A \subseteq b \rightarrow \varphi_0^{(b)}(A, \vec{v}, \vec{C})) \leftrightarrow (A \subseteq b \rightarrow \varphi_0^{(b)}(A, \vec{v}, \vec{C} \cap b)) \Big).$$

By generalizing with respect to A, then

$$\vec{v} \in b \rightarrow \forall X \Big((X \subseteq b \rightarrow \varphi_0^{(b)}(X, \vec{v}, \vec{C})) \leftrightarrow (X \subseteq b \rightarrow \varphi_0^{(b)}(X, \vec{v}, \vec{C} \cap b)) \Big).$$

And from this

$$\vec{v} \in b \rightarrow \Big(\forall X (X \subseteq b \rightarrow \varphi_0^{(b)}(X, \vec{v}, \vec{C})) \leftrightarrow \forall X (X \subseteq b \rightarrow \varphi_0^{(b)}(X, \vec{v}, \vec{C} \cap b)) \Big).$$

That is

$$\vec{v} \in b \rightarrow \Bigg(\Big(\forall X \varphi_0(X, \vec{v}, \vec{C}) \Big)^{(b)} \leftrightarrow \Big(\forall X \varphi_0(X, \vec{v}, \vec{C} \cap b) \Big)^{(b)} \Bigg).$$

□

PROPOSITION 1.2.9. *For any* s-Π^1_1 *formula* $\varphi(\vec{v}, \vec{C})$ *in which* y *does not occur free and with no free variables besides the displayed ones and not necessarily all of them, the following are shown to be provably equivalent in* $\mathsf{sKPu}_2^{\mathsf{r}}\!\restriction$:

(*a*) $\varphi(\vec{v}, \vec{C}) \rightarrow \exists y[\mathsf{Tran}(y) \wedge \vec{v} \in y \wedge \varphi^{(y)}(\vec{v}, \vec{C})]$,

(*b*) $\varphi(\vec{v}, \vec{C}) \rightarrow \exists y[\mathsf{Tran}(y) \wedge \vec{v} \in y \wedge \varphi^{(y)}(\vec{v}, \vec{C} \cap y)]$,

(*c*) $\varphi(\vec{v}, \vec{C}) \rightarrow \exists y[\mathsf{Tran}(y) \wedge \varphi^{(y)}(\vec{v}, \vec{C})]$,

(*d*) $\varphi(\vec{v}, \vec{C}) \rightarrow \exists y[\mathsf{Tran}(y) \wedge \varphi^{(y)}(\vec{v}, \vec{C} \cap y)]$.

Proof. $(a) \leftrightarrow (b)$ immediately follows from Proposition 1.2.8, after noticing that

$$\vec{v} \in b \rightarrow \Big(\varphi^{(b)}(\vec{v}, \vec{C}) \leftrightarrow \varphi^{(b)}(\vec{v}, \vec{C} \cap b) \Big)$$

is logically equivalent to

$$\vec{v} \in b \wedge \varphi^{(b)}(\vec{v}, \vec{C}) \leftrightarrow \vec{v} \in b \wedge \varphi^{(b)}(\vec{v}, \vec{C} \cap b).$$

$(a) \rightarrow (c)$ and $(b) \rightarrow (d)$ are trivial. We are left with showing that $(c) \rightarrow (a)$ and $(d) \rightarrow (b)$. Let us take the former first. For any s-Π^1_1 formula $\varphi(\vec{v}, \vec{C}) \equiv \varphi(v_0, ..., v_n, C_0, ..., C_m)$ by means of the equality axioms we have the following derivable in $\mathsf{sKPu}_2^{\mathsf{r}}\!\restriction$:

$$\varphi(v_0, ..., v_n, C_0, ..., C_m) \leftrightarrow \bigwedge_{0 \leq i \leq n} \exists z (z = v_i) \wedge \varphi(v_0, ..., v_n, C_0, ..., C_m).$$

By the definition of a s-Π^1_1 formula we know that

$$\varphi(v_0,...,v_n,C_0,...,C_m)\equiv\forall X\psi(v_0,...,v_n,C_0,...,C_m,X)$$

where ψ has logical complexity Σ. Hence we have

$$\varphi(v_0,...,v_n,C_0,...,C_m)\leftrightarrow\bigwedge_{0\leq i\leq n}\exists z(z=v_i)\wedge\forall X\psi(v_0,...,v_n,C_0,...,C_m,X).$$

That is

$$\varphi(v_0,...,v_n,C_0,...,C_m)\leftrightarrow\underbrace{\forall X\left(\bigwedge_{0\leq i\leq n}\exists z(z=v_i)\wedge\psi(v_0,...,v_n,C_0,...,C_m,X)\right)}_{\text{s-}\Pi^1_1}.$$

Hence from (c) we obtain

$$\begin{aligned}&\varphi(v_0,...,v_n,C_0,...,C_m)\rightarrow\\&\rightarrow\exists y\left[\mathsf{Tran}(y)\wedge\left(\forall X\left(\bigwedge_{0\leq i\leq n}\exists z(z=v_i)\wedge\psi(v_0,...,v_n,C_0,...,C_m,X)\right)\right)^{(y)}\right].\end{aligned}$$

Whence

$$\begin{aligned}&\varphi(v_0,...,v_n,C_0,...,C_m)\rightarrow\\&\rightarrow\exists y\left[\mathsf{Tran}(y)\wedge(\forall X\subseteq y)\left(\bigwedge_{0\leq i\leq n}\exists z(z=v_i)\wedge\psi(v_0,...,v_n,C_0,...,C_m,X)\right)^{(y)}\right].\end{aligned}$$

That is

$$\begin{aligned}&\varphi(v_0,...,v_n,C_0,...,C_m)\rightarrow\\&\rightarrow\exists y\left[\mathsf{Tran}(y)\wedge(\forall X\subseteq y)\left(\left(\bigwedge_{0\leq i\leq n}\exists z(z=v_i)\right)^{(y)}\wedge\psi^{(y)}(v_0,...,v_n,C_0,...,C_m,X)\right)\right].\end{aligned}$$

This last implication, along with "$\forall y(\emptyset\subseteq y)$", Proposition 1.2.3 and Proposition 1.2.5, entails the following:

$$\begin{aligned}&\varphi(v_0,...,v_n,C_0,...,C_m)\rightarrow\\&\rightarrow\exists y\left[\mathsf{Tran}(y)\wedge\left(\bigwedge_{0\leq i\leq n}\exists z(z=v_i)\right)^{(y)}\wedge(\forall X\subseteq y)(\psi^{(y)}(v_0,...,v_n,C_0,...,C_m,X))\right].\end{aligned}$$

That is

$$\begin{aligned}&\varphi(v_0,...,v_n,C_0,...,C_m)\rightarrow\\&\rightarrow\exists y\left[\mathsf{Tran}(y)\wedge\left(\bigwedge_{0\leq i\leq n}\exists z(z=v_i)\right)^{(y)}\wedge\varphi^{(y)}(v_0,...,v_n,C_0,...,C_m)\right].\end{aligned}$$

By resolving the relativization to y,

$$\begin{aligned}&\varphi(v_0, ..., v_n, C_0, ..., C_m) \to \\ &\to \exists y \left[\mathsf{Tran}(y) \wedge \bigwedge_{0 \le i \le n} \exists z(z \in y \wedge z = v_i) \wedge \varphi^{(y)}(v_0, ..., v_n, C_0, ..., C_m) \right].\end{aligned}$$

And from this

$$\begin{aligned}&\varphi(v_0, ..., v_n, C_0, ..., C_m) \to \\ &\to \exists y \left[\mathsf{Tran}(y) \wedge \bigwedge_{0 \le i \le n} v_i \in y \wedge \varphi^{(y)}(v_0, ..., v_n, C_0, ..., C_m) \right].\end{aligned}$$

That is

$$\varphi(\vec{v}, \vec{C}) \to \exists y[\mathsf{Tran}(y) \wedge \vec{v} \in y \wedge \varphi^{(y)}(\vec{v}, \vec{C})].$$

Analogously for $(d) \to (b)$. □

PROPOSITION 1.2.10. *The following two schemata are provably equivalent in* $\mathsf{sKPu}_2^{\mathsf{r}}\!\restriction$:

(a) $\Pi^{\rm C}$ RFN,

(b) S-Σ^1_1 RFN.

Proof. In the substantive direction, let $\exists X \psi(X, \vec{v}, \vec{C})$ be a S-Σ^1_1 formula where ψ is $\Pi^{\rm C}$. Assume $\exists X \psi(X, \vec{v}, \vec{C})$. So there is a class C_0 such that $\psi(C_0, \vec{v}, \vec{C})$. By applying $\Pi^{\rm C}$ RFN to this formula then we get

$$\exists y[\mathsf{Tran}(y) \wedge \vec{v} \in y \wedge \psi^{(y)}(C_0, \vec{v}, \vec{C})],$$

and from this in virtue of Proposition 1.2.8 we obtain

$$\exists y[\mathsf{Tran}(y) \wedge \vec{v} \in y \wedge \psi^{(y)}(C_0 \cap y, \vec{v}, \vec{C} \cap y)].$$

Therefore

$$\exists y[\mathsf{Tran}(y) \wedge \vec{v} \in y \wedge \exists X(X \subseteq y \wedge \psi^{(y)}(X, \vec{v}, \vec{C} \cap y)].$$

And again by Proposition 1.2.8, we get

$$\exists y[\mathsf{Tran}(y) \wedge \vec{v} \in y \wedge (\exists X \psi(X, \vec{v}, \vec{C}))^{(y)}].$$

□

1.3 $\mathsf{KPu^r}$ SUBSYSTEM OF $\mathsf{sKPu^r_2}{\upharpoonright}$

We are concerned with showing that any theorem of $\mathsf{KPu^r}$ is also a theorem of $\mathsf{sKPu^r_2}{\upharpoonright}$. We will show, in fact, that all the single axioms and the axiom schemata of $\mathsf{KPu^r}$, that do not already appear among the axioms of $\mathsf{sKPu^r_2}{\upharpoonright}$, are derivable within the theory $\mathsf{sKPu^r_2}{\upharpoonright}$. This in turn reduces down to prove the following propositions.

PROPOSITION 1.3.1. *Any instance of* Δ_0-COLL *is derivable in* $\mathsf{sKPu^r_2}{\upharpoonright}$.

Proof. Any instance of Δ_0-COLL is also an instance of S-Π^1_1 RFN. $\square$

PROPOSITION 1.3.2. PAIRING *is derivable in* $\mathsf{sKPu^r_2}{\upharpoonright}$.

Proof. PAIRING is simply obtained once we apply S-Π^1_1 RFN to the formula

$$\forall x(x \in a \leftrightarrow x \in a) \wedge \forall x(x \in b \leftrightarrow x \in b)$$

which is derivable from $a = a$. Denoting this last formula by "$\varphi(a, b)$" we then get, by MODUS PONENDO PONENS,

$$\exists y[\mathsf{Tran}(y) \wedge a \in y \wedge b \in y \wedge \varphi^{(y)}(a, b)],$$

yielding in particular

$$\exists y[a \in y \wedge b \in y].$$

$\square$

PROPOSITION 1.3.3. TRANSITIVE HULL *is derivable in* $\mathsf{sKPu^r_2}{\upharpoonright}$.

Proof. In order to derive the axiom of TRANSITIVE HULL, we argue as follows. If $\varphi(a)$ is a provable S-Π^1_1 formula, then we obtain from S-Π^1_1 RFN, provided that the variable y does not occurr free in $\varphi(a)$,

$$\varphi(a) \to \exists y[\mathsf{Tran}(y) \wedge a \in y \wedge \varphi^{(y)}(a)],$$

yielding, by MODUS PONENDO PONENS,

$$\exists y[\mathsf{Tran}(y) \wedge a \in y \wedge \varphi^{(y)}(a)].$$

From this we infer in particular,

$$\exists y[\mathsf{Tran}(y) \wedge a \in y].$$

And this, along with the following implication,

$$\exists y[\mathsf{Tran}(y) \wedge a \in y] \to \exists y[\mathsf{Tran}(y) \wedge a \subseteq y],$$

logically entails, by MODUS PONENDO PONENS,

$$\exists y[\mathsf{Tran}(y) \wedge a \subseteq y].$$

$\square$

PROPOSITION 1.3.4. Δ_0-$\mathsf{I}_\in$ *is derivable in* $\mathsf{sKPu}^{\mathsf{r}}_2\!\upharpoonright$.

Proof. Propostion 1.2.3 and Proposition 1.2.5 along with $\mathsf{I}^2_\in$ logically entail Δ_0-$\mathsf{I}_\in$. ☐

PROPOSITION 1.3.5. Δ_0-$\mathsf{I}_\mathbb{N}$ *is derivable in* $\mathsf{sKPu}^{\mathsf{r}}_2\!\upharpoonright$.

Proof. Propostion 1.2.3 and Proposition 1.2.5 along with $\mathsf{I}^2_\mathbb{N}$ logically entail Δ_0-$\mathsf{I}_\mathbb{N}$. ☐

COROLLARY 1.3.6. *Every theorem* φ *of* $\mathsf{KPu}^{\mathsf{r}}$ *is also a theorem of* $\mathsf{sKPu}^{\mathsf{r}}_2\!\upharpoonright$,

$$\mathsf{KPu}^{\mathsf{r}} \vdash \varphi \quad \Longrightarrow \quad \mathsf{sKPu}^{\mathsf{r}}_2\!\upharpoonright \vdash \varphi.$$

1.4 $\mathsf{sKPu}^{\mathsf{r}}_2\!\upharpoonright$ CONSERVATIVE EXTENSION OF $\mathsf{KPu}^{\mathsf{r}}$

So far we have seen that that any theorem of $\mathsf{KPu}^{\mathsf{r}}$ is also a theorem of $\mathsf{sKPu}^{\mathsf{r}}_2\!\upharpoonright$. The next step we are concerned with is to prove that the theory $\mathsf{sKPu}^{\mathsf{r}}_2\!\upharpoonright$ is conservative over $\mathsf{KPu}^{\mathsf{r}}$ for a certain class of formulae. In other words, we will show that as far as the derivability of a particular class of formulae is concerned, one can prove in $\mathsf{sKPu}^{\mathsf{r}}_2\!\upharpoonright$ nothing more than one can prove already in $\mathsf{KPu}^{\mathsf{r}}$. The result will be established through an adaptation of the tecnique employed by Cantini [5] to the current context. The main modifications are worked out.

The reduction proceeds into two steps. First, we sketch a Tait-style reformulation of $\mathsf{sKPu}^{\mathsf{r}}_2\!\upharpoonright$ allowing us to establish a partial cut-elimination theorem, yielding quasi-normal derivations. In a second step quasi-normal derivations of such a Tait-style reformulation of $\mathsf{sKPu}^{\mathsf{r}}_2\!\upharpoonright$ are then reduced to $\mathsf{KPu}^{\mathsf{r}}$ by means of an asymmetric interpretation. We take up the first step.

DEFINITION 1.4.1. The essentially strict Π^1_1 formulae ($[\text{s-}\Pi^1_1]^{\text{E}}$) form the smallest class containing the Δ^{c}_0 formulae and closed under $\wedge, \vee, \forall x \in t, \exists x \in t, \exists x$ and the clause $\forall X$.
The essentially strict Σ^1_1 formulae $[\text{s-}\Sigma^1_1]^{\text{E}}$ form the dual class: that is, the smallest class containing the Δ^{c}_0 formulae and closed under $\wedge, \vee, \forall x \in t, \exists x \in t, \forall x$ and the clause $\exists X$.

REMARK 1.4.2. It is worth mentioning that one of the basic features of the essentially strict Π^1_1 formulae is that each of them is equivalent to one of the form:

$$\forall X \exists y \varphi(X, y, ...).$$

where $\varphi(X, y, ..)$ is Δ^{c}_0. For a proof, the reader is referred to Barwise [2], Lemma VIII.2.5, p. 318. This is done by simple quantifier-pushing manipulations. Unfortunately, this is no longer the case in our logico-axiomatic framework. In order to advance a set quantifier over a class quantifier in a suitable way, it seems necessary to assume some kind of axiom of choice. Consider for example, the following $[\text{s-}\Pi^1_1]^{\text{E}}$ formula:

$$\forall X \exists y \forall Z \exists x \varphi(X, y, Z, x, ...).$$

In order to show that

$$\forall X \exists y \forall Z \exists x \varphi(X, y, Z, x, ...) \leftrightarrow \forall X \forall Z \exists y \exists x \varphi(X, y, Z, x, ...)$$

we need to switch the universal class quantifier " $\forall Z$" with the existential set quantifier "$\exists y$". But in our framework this manipulation is only possible in presence of Σ^1_1-AC,

$$\forall x \exists Y \psi(x, Y, ...) \rightarrow \exists Y \forall x \psi(x, Y_x, ...)$$

for ψ being Σ^1_1 and $Y_x := \{v : \langle x, v\rangle \in Y\}$ being the standard coding for sequences of classes. The result is then simply obtained by contracting both universal class and existential set quantifiers.

A Tait-style reformulation of $\mathsf{sKPu}^{\mathsf{r}}_{2}\!\restriction$ can be regarded as the one-sided counterpart of Gentzen systems for $\mathsf{sKPu}^{\mathsf{r}}_{2}\!\restriction$ or as "Gentzen-symmetric", since symmetries of classical logic given by the De Morgan duality are built in. We need then a different treatment of negation. We assume that formulae are constructed from *positive* and *negative atomic formulae* [2] by closing against conjunction and disjunction as well as existential and universal quantification in both sorts. Negation $\neg$ satisfies $\neg\neg\varphi \equiv \varphi$ for atomic formulae φ, and is defined for compound formulae by De Morgan duality. In the sequel we identify formulae of $\mathcal{L}^*_2$ and their translations in the Tait-style language corresponding to $\mathcal{L}^*_2$. It is worth noticing that for the proof-theoretic analysis of $\mathsf{sKPu}^{\mathsf{r}}_{2}\!\restriction$ we aim at, it is not required to analyze the structure of formulae of complexity $[\text{s-}\Pi^1_1]^{\text{E}}/[\text{s-}\Sigma^1_1]^{\text{E}}$. This fact also motivates the subsequent definition of *rank* of a formula.

The rank of a formula φ, $\mathsf{rk}(\varphi)$, is recursively defined as follows:

- $\mathsf{rk}(\varphi) = 0$ if φ is $[\text{s-}\Pi^1_1]^{\text{E}}$ or $[\text{s-}\Sigma^1_1]^{\text{E}}$,

  otherwise,

- $\mathsf{rk}(\varphi \circ \psi) = \mathsf{max}(\mathsf{rk}(\varphi), \mathsf{rk}(\psi)) + 1$,

- $\mathsf{rk}(\mathcal{Q}x \in y.\varphi) = \mathsf{rk}(\varphi) + 2$,

- $\mathsf{rk}(\mathcal{Q}x.\varphi) = \mathsf{rk}(\mathcal{Q}X.\varphi) = \mathsf{rk}(\varphi) + 1$.

where $\mathcal{Q} = \forall, \exists$ and $\circ = \wedge, \vee$.

Let T_1 denote a Tait-style reformulation of $\mathsf{sKPu}^{\mathsf{r}}_{2}\!\restriction$. Axioms and inference rules of T_1 are stated for finite set $\Gamma, \Delta, ...$ of formulae which have to be interpreted disjunctively. We write, for example, $\Gamma, \Delta, \varphi, \psi$ for $\Gamma, \Delta \cup \{\varphi, \psi\}$. We distinguish between free and bound occurrences of variables. For a set Γ of $\mathcal{L}^*_2$ formulae we let $\text{FV}(\Gamma)$ denote the set of parameters (free variables) occurring in the formulae of Γ. If Γ is the singleton $\{\varphi\}$, we omit the curly brackets "$\{\}$". Let $\Gamma = \{\varphi_0, ..., \varphi_{n-1}\}$, we shall use the following notations:

$$\bigvee \Gamma := \varphi_0 \vee ... \vee \varphi_{n-1},$$

---

[2] Both types of atomic formulae are treated as *primitives*.

$$\neg\Gamma := \{\neg\varphi_0, ..., \neg\varphi_{n-1}\}$$

Let $\mathfrak{a}$ denote either a set or class variable and let $\mathfrak{t}$ denote either a set or class term. By $\mathcal{E}[\mathfrak{a}/\mathfrak{t}]$ we denote the result of substituting the term $\mathfrak{t}$ for the variable $\mathfrak{a}$ in the expression $\mathcal{E}$. Similarly, $\mathcal{E}[\vec{\mathfrak{a}}/\vec{\mathfrak{t}}]$

denotes the result of simultaneously substituting the terms $\vec{\mathfrak{t}} \equiv \mathfrak{t}_1, ..., \mathfrak{t}_n$ for the variables $\vec{\mathfrak{a}} \equiv \mathfrak{a}_1, ..., \mathfrak{a}_n$.

The logical axioms and inference rules of $\mathsf{T_1}$ are as follows.

Logical Axioms:

$$\Gamma, \neg\varphi, \varphi$$

for all atomic formulae φ.

Inference Rules:

$$\frac{\Gamma, \varphi \quad \Gamma, \psi}{\Gamma, \varphi \wedge \psi}\ (\wedge) \qquad \frac{\Gamma, \varphi, \psi}{\Gamma, \varphi \vee \psi}\ (\vee)$$

$$\frac{\Gamma, \varphi[x/y]}{\Gamma, \forall x\varphi}\ (\forall) \qquad \frac{\Gamma, \varphi[x/t]}{\Gamma, \exists x\varphi}\ (\exists)$$

$$\frac{\Gamma, \varphi[X/Y]}{\Gamma, \forall X\varphi}\ (\forall^2) \qquad \frac{\Gamma, \varphi[X/Y]}{\Gamma, \exists X\varphi}\ (\exists^2)$$

where for each of the two universal rules the variables y and Y do not occur free within the conclusion.

We further introduce two derived rules, (b∀) and (b∃), which are in fact just particular instances of (∀) and (∃) respectively,

$$\frac{\Gamma, y \in a \rightarrow \varphi[x/y]}{\Gamma, (\forall x \in a)\varphi}\ (\mathrm{b}\forall) \qquad \frac{\Gamma, t \in a \wedge \varphi[x/t]}{\Gamma, (\exists x \in a)\varphi}\ (\mathrm{b}\exists)$$

where (b∀) is under the same restrictive conditions as above.

Cut rule:

$$\frac{\Gamma, \varphi \quad \Gamma, \neg\varphi}{\Gamma}\ (\mathrm{Cut})$$

The *rank of a cut* is defined to be the rank $\mathsf{rk}(\varphi) = \mathsf{rk}(\neg\varphi)$ of its cut formulae.

As far as the non-logical axioms are concerned, we notice that all axioms of $\mathsf{sKPu}_2^{\mathsf{r}}\!\restriction$, except $\text{S-}\Pi^1_1$ RFN and Δ^{c}_1-CA, can easily be written in a Tait-style manner so that the principal formulae are at most $[\text{S-}\Pi^1_1]^{\mathrm{E}}/[\text{S-}\Sigma^1_1]^{\mathrm{E}}$. For example, $\mathsf{I}^2_{\in}$ and $\mathsf{I}^2_{\mathbb{N}}$ are reformulated respectively as:

$$\Gamma, \forall y(y \notin A), \exists y(y \in A \wedge \forall z(z \in y \rightarrow z \notin A)) \quad (\mathsf{I}^2_{\in});$$

$$\Gamma, 0 \notin A, (\exists x, y \in \mathsf{N})(x \in A \wedge \mathsf{Sc}(x, y) \wedge y \notin A), \mathsf{N} \subseteq A \quad (\mathsf{I}^2_{\mathbb{N}}).$$

In order to allow partial cut-elimination up to $[\text{S-}\Pi^1_1]^{\mathrm{E}}/[\text{S-}\Sigma^1_1]^{\mathrm{E}}$ formulae, $\text{S-}\Pi^1_1$ RFN and Δ^{c}_1-CA are replaced by the following two non-logical inference rules, where the principal formulae are $[\text{S-}\Pi^1_1]^{\mathrm{E}}$ and $[\text{S-}\Sigma^1_1]^{\mathrm{E}}$ respectively:

$$\frac{\Gamma, \varphi(\vec{a}, \vec{C})}{\Gamma, \underbrace{\exists w[\mathsf{Tran}(w) \wedge \vec{a} \in w \wedge \varphi^{(w)}(\vec{a}, \vec{C})]}_{[\text{S-}\Pi^1_1]^{\mathrm{E}}}} \ (\text{S-}\Pi^1_1 \ \text{RFN})$$

for φ being $\text{S-}\Pi^1_1$ and $w \notin \mathrm{FV}(\varphi)$.

$$\frac{\Gamma, \forall x(\varphi(x) \rightarrow \neg\psi(x)) \quad \Gamma, \forall x(\neg\psi(x) \rightarrow \varphi(x))}{\Gamma, \underbrace{\exists Y[\forall x(x \in Y \rightarrow \neg\psi(x)) \wedge \forall x(\varphi(x) \rightarrow x \in Y)]}_{[\text{S-}\Sigma^1_1]^{\mathrm{E}}}} \ (\Delta^{\mathrm{c}}_1\text{-CA})$$

for φ and ψ being Σ^{c}_1 and $Y \notin \mathrm{FV}(\{\varphi, \psi\})$.

Since any derivation is a finite syntactic object, this implies that only a finite number of instances of the schema of Δ_0-SEP are involved in any such derivation. Collect together all the Δ_0 formulae of such instances and let $\mathcal{C}_{\Delta_0}$ be such a finite collection of Δ_0 formulae of $\mathcal{L}^*$ (*not* containing class parameters). By $\mathsf{T}_1{\restriction}_{\mathcal{C}_{\Delta_0}}$ we then denote the subsystem of T_1 where the schema of Δ_0-SEP is restricted to the formulae of $\mathcal{C}_{\Delta_0}$.

$\mathsf{T}_1{\restriction}_{\mathcal{C}_{\Delta_0}} \vdash^n_k \Gamma$ expresses that there is a derivation in $\mathsf{T}_1{\restriction}_{\mathcal{C}_{\Delta_0}}$ of depth $\leqslant n$ ending with the finite set Γ of $\mathcal{L}^*_2$ formulae, where all cuts in the derivation have rank $< k$.

EMBEDDING OF $\mathsf{sKPu}^{\mathsf{r}}_{2}{\restriction}$ INTO $\mathsf{T}_1{\restriction}_{\mathcal{C}_{\Delta_0}}$. *Let φ be a $\mathcal{L}^*_2$ formula such that*

$$\mathsf{sKPu}^{\mathsf{r}}_{2}{\restriction} \ \vdash \varphi.$$

Then there are two natural numbers n and k and a finite collection $\mathcal{C}_{\Delta_0}$ of Δ_0 formulae of $\mathcal{L}^*$ such that

$$\mathsf{T}_1{\restriction}_{\mathcal{C}_{\Delta_0}} \vdash^n_k \varphi$$

Standard cut elimination techniques are then applied in order to show that all cuts of rank greater than zero can be eliminated. The depth of the so-obtained quasi-normal derivations is measured as usual by $2_k(n)$ where we set

$$2_0(n) = n,$$

$$2_{k+1}(n) = 2^{2_k(n)}$$

The above-mentioned considerations are synthetized in the following partial cut elimination theorem.

PARTIAL CUT ELIMINATION FOR $\mathsf{T}_1{\restriction}_{\mathcal{C}_{\Delta_0}}$. *For all finite sets Γ of $\mathcal{L}_2^*$ formulae and all natural numbers n and k,*

$$\mathsf{T}_1{\restriction}_{\mathcal{C}_{\Delta_0}} \vdash^n_{k+1} \Gamma \quad \Longrightarrow \quad \mathsf{T}_1{\restriction}_{\mathcal{C}_{\Delta_0}} \vdash^{2_k(n)}_1 \Gamma.$$

Proof. Observe that the principal formulae of the axioms and of each of the two non-logical rule of inference are all $[\text{s-}\Pi^1_1]^{\mathrm{E}}$ or $[\text{s-}\Sigma^1_1]^{\mathrm{E}}$. Then the result is obtained by the same proof as, for example, in Schwichtenberg [24]. □

COROLLARY 1.4.3. *Let φ be a $\mathcal{L}_2^*$ formula such that*

$$\mathsf{sKPu}^{\mathsf{r}}_2{\restriction} \vdash \varphi.$$

Then there is a natural number and a finite collection $\mathcal{C}_{\Delta_0}$ of Δ_0 formulae of $\mathcal{L}^*$ such that

$$\mathsf{T}_1{\restriction}_{\mathcal{C}_{\Delta_0}} \vdash^n_1 \varphi$$

The next step of reducing $\mathsf{sKPu}^{\mathsf{r}}_2{\restriction}$ to $\mathsf{KPu}^{\mathsf{r}}$ consists in setting up a partial model for $\mathsf{sKPu}^{\mathsf{r}}_2{\restriction}$ (e.g. a model for the set-theoretic Π_2 sentences of $\mathsf{sKPu}^{\mathsf{r}}_2{\restriction}$), which will subsequently be used in order to prove an asymmetric interpretation theorem for quasi-normal $\mathsf{T}_1{\restriction}_{\mathcal{C}_{\Delta_0}}$ derivations. It is argued that the whole procedure can be formalized in $\mathsf{KPu}^{\mathsf{r}}$. In particular, the partial models needed for an interpretation of $\mathsf{sKPu}^{\mathsf{r}}_2{\restriction}$ are available in $\mathsf{KPu}^{\mathsf{r}}$.

For any set a, let

$$\bigcup a \;:=\; \{\, z \,|\, (\exists v \in a)(z \in v) \,\}.$$

This is a set by Proposition 1.1.3.

For each formula φ in $\mathcal{C}_{\Delta_0}$, we define a Σ-function symbol $\mathsf{F}_\varphi(a, \vec{b})$ such that:

$$\mathsf{F}_\varphi(a, \vec{b}) = \{\, x \in a \,|\, \varphi(x, \vec{b}) \,\}.$$

Given $\mathcal{C}_{\Delta_0}$ and an arbitrary set term h we define by recursion on n a finite hierarchy $\langle L_n(h)\rangle_{n\in\mathbb{N}}$ of set terms $L_n(h)$ depending on $\mathcal{C}_{\Delta_0}$:

$$\begin{aligned} L_0(h) &:= h, \\ L_{n+1}(h) &:= L_n(h) \cup \\ &\quad\; \{L_n(h)\} \cup \\ &\quad\; \{\, \mathsf{F}_\varphi(a, \vec{b}) \,|\, a, \vec{b} \in L_n(h) \;\&\; \varphi \in \mathcal{C}_{\Delta_0} \,\}. \end{aligned}$$

LEMMA 1.4.4. *For any natural number $n \in \mathbb{N}$,*

$$\mathsf{KPu}^{\mathsf{r}} \vdash \forall h \Big(\mathsf{Tran}(h) \rightarrow \mathsf{Tran}(L_n(h)) \Big).$$

Proof. The proof runs by induction on n. We work informally within the theory $\mathsf{KPu}^{\mathsf{r}}$. Fix an arbitrary h and assume $\mathsf{Tran}(h)$.

$\underline{n = 0}$ Trivial.

$\underline{n \mapsto n+1}$ By I. H. we have provable in $\mathsf{KPu}^{\mathsf{r}}$: $L_n(h)$ is a transitive set. We have to prove that $L_{n+1}(h)$ is a transitive set in $\mathsf{KPu}^{\mathsf{r}}$. We first show that $\mathsf{KPu}^{\mathsf{r}}$ proves that $L_{n+1}(h)$ is transitive. Assuming $\mathsf{Tran}(L_n(h))$ we have to show that each operation for generating $L_{n+1}(h)$ preserves transitivity. The induction-step breaks up into three subcases; we restrict ourselves to the separation case. Assume $d \in L_{n+1}(h)$ and $c \in d$. Then

$$d = \{\, x \in a \mid \varphi(x, \vec{b}) \,\}$$

with $a, \vec{b} \in L_n(h)$. From $c \in d$ we then infer $c \in a \in L_n(h)$ and by I.H. $\mathsf{Tran}(L_n(h))$. Therefore $c \in L_n(h)$ and $c \in L_{n+1}(h)$. The desired result is then obtained by summing-up with respect to the remaining transitive members of $L_{n+1}(h)$. It remains to show that $L_{n+1}(h)$ is a set. The only operation for which $L_{n+1}(h)$ could fail to be a set is separation. Thus proving the result reduces to showing that

$$\{\, \mathsf{F}_{\varphi}(a, \vec{b}) \mid a, \vec{b} \in L_n(h) \;\&\; \varphi \in \mathcal{C}_{\Delta_0} \,\}, \tag{1}$$

is a set in $\mathsf{KPu}^{\mathsf{r}}$. Once we have this, then the result is obtained again by summing-up with respect to the remaining members of $L_{n+1}(h)$. Note that (1) corresponds to

$$\bigcup_{\varphi \in \mathcal{C}_{\Delta_0}} \{\, \mathsf{F}_{\varphi}(a, \vec{b}) \mid a, \vec{b} \in L_n(h) \,\}.$$

And since $\mathcal{C}_{\Delta_0}$ is finite it is enough to prove for an arbitrary $\varphi \in \mathcal{C}_{\Delta_0}$ that

$$\{\, \mathsf{F}_{\varphi}(a, \vec{b}) \mid a, \vec{b} \in L_n(h) \,\}$$

is a set in $\mathsf{KPu}^{\mathsf{r}}$. Thus, given $\varphi \in \mathcal{C}_{\Delta_0}$ we know

$$\forall a, \vec{b}\, (a, \vec{b} \in L_n(h) \rightarrow \exists y (\mathsf{S}(y) \wedge \mathsf{F}_{\varphi}(a, \vec{b}) = y)).$$

Since $L_n(h)$ is a set by I.H., then by Σ-COLL there exists a set v such that

$$\forall a, \vec{b}\, (a, \vec{b} \in L_n(h) \rightarrow \exists y (y \in v \wedge \mathsf{S}(y) \wedge \mathsf{F}_{\varphi}(a, \vec{b}) = y)).$$

Through Δ-SEP we then isolate from the set v a set v_0 consisting of all the y's such that $y = \mathsf{F}_{\varphi}(a, \vec{b})$, that is

$$v_0 = \{\, F_{\varphi}(a, \vec{b}) \mid a, \vec{b} \in L_n(h) \,\}.$$

□

Sets and classes are interpreted, respectively, as elements and subsets of

$$\bigcup_{n \in \mathbb{N}} L_n(h).$$

We adopt the following convention. Let $\varphi(\vec{s}, \vec{C})$ be any formula of $\mathcal{L}_2^*$, whose all set and class parameters came from the lists $\vec{s}, \vec{C}$ respectively. We write $\varphi^{(L_n(h))}(\vec{s}, \vec{c})$ to denote the result of replacing in $\varphi(\vec{s}, \vec{C})$

- every unbounded set quantifier $\mathcal{Q}x$ by $\mathcal{Q}x \in L_n(h)$,
- every class quantifier $\mathcal{Q}Y$ by $\mathcal{Q}y \subseteq L_n(h)$,
- every class variable C by a set variable c.

We avoid conflict of variables.

Lemma 1.4.5. *For any formula $\varphi(\vec{s}, \vec{C}, \vec{D})$ of $\mathcal{L}_2^*$, with no free variables besides the displayed ones and not necessarily all of them and for any set b wich does not occur free in the list $\vec{s}$ we have the following provable in* $\mathsf{KPu^r}$:

$$\vec{s} \in b \to \Big(\varphi^{(b)}(\vec{s}, \vec{c}, \vec{d}) \leftrightarrow \varphi^{(b)}(\vec{s}, \vec{c} \cap b, \vec{d}) \Big).$$

Proof. The proof, adapted in the obvious way, is as for Proposition 1.2.8. □

Before providing an asymmetric interpretation of $\mathsf{T}_1{\restriction}_{\mathcal{C}_{\Delta_0}}$ into $\mathsf{KPu^r}$, let us state essential persistence properties of $[\text{S-}\Pi^1_1]^{\mathrm{E}}$ and $[\text{S-}\Sigma^1_1]^{\mathrm{E}}$ formulae with respect to the hierarchy $\langle L_n(h) \rangle_{n \in \mathbb{N}}$.

Persistency. *For all $[\text{S-}\Pi^1_1]^{\mathrm{E}}$ formulae $\varphi(\vec{s}, \vec{C})$ and $[\text{S-}\Sigma^1_1]^{\mathrm{E}}$ formulae $\psi(\vec{s}, \vec{C})$ of $\mathcal{L}_2^*$, we have:*

$$\begin{aligned} \mathsf{KPu^r} \vdash \forall h \, \forall q \, \forall m \, \forall \vec{s} \forall \vec{c} \bigg(\Big(& \mathsf{Tran}(h) \wedge q > m \wedge m > 0 \, \wedge \\ & \wedge \vec{s} \in L_m(h) \wedge \vec{c} \subseteq L_q(h) \wedge \varphi^{(L_m(h))}(\vec{s}, \vec{c}) \Big) \to \\ & \to \varphi^{(L_q(h))}(\vec{s}, \vec{c}) \bigg) \qquad (\text{Upward Persistency}); \end{aligned}$$

$$\begin{aligned} \mathsf{KPu^r} \vdash \forall h \, \forall q \, \forall m \, \forall \vec{s} \forall \vec{c} \bigg(\Big(& \mathsf{Tran}(h) \wedge q > m \wedge m > 0 \, \wedge \\ & \wedge \vec{s} \in L_m(h) \wedge \vec{c} \subseteq L_q(h) \wedge \psi^{(L_q(h))}(\vec{s}, \vec{c}) \Big) \to \\ & \to \psi^{(L_m(h))}(\vec{s}, \vec{c}) \bigg) \qquad (\text{Downward Persistency}). \end{aligned}$$

Proof. The proof proceeds by induction over build-up of formulae. We content ourselves to showing Upward Persistency for $[\text{S-}\Pi^1_1]^{\mathrm{E}}$ formulae.

$\underline{\Delta_0^{\mathrm{C}}}$: Immediate by absoluteness of Δ_0-formulae for transitive sets.

$\underline{\varphi(\vec{s},\vec{C})\equiv\varphi_0(\vec{s},\vec{C})\wedge\varphi_1(\vec{s},\vec{C})}$. By I.H.,

$$\begin{aligned}\mathsf{KPu}^{\mathsf{r}}\vdash\forall h\,\forall q\,\forall m\,\forall\vec{s}\forall\vec{c}\Bigg(\Bigg(&\mathsf{Tran}(h)\wedge q>m\wedge m>0\wedge\\ &\wedge\vec{s}\in L_m(h)\wedge\vec{c}\subseteq L_q(h)\wedge\varphi_0^{(L_m(h))}(\vec{s},\vec{c})\Bigg)\rightarrow\\ &\rightarrow\varphi_0^{(L_q(h))}(\vec{s},\vec{c})\Bigg)\end{aligned}\tag{1}$$

and

$$\begin{aligned}\mathsf{KPu}^{\mathsf{r}}\vdash\forall h\,\forall q\,\forall m\,\forall\vec{s}\forall\vec{c}\Bigg(\Bigg(&\mathsf{Tran}(h)\wedge q>m\wedge m>0\wedge\\ &\wedge\vec{s}\in L_m(h)\wedge\vec{c}\subseteq L_q(h)\wedge\varphi_1^{(L_m(h))}(\vec{s},\vec{c})\Bigg)\rightarrow\\ &\rightarrow\varphi_1^{(L_q(h))}(\vec{s},\vec{c})\Bigg).\end{aligned}\tag{2}$$

From (1) and (2), we infer respectively

$$\begin{aligned}\mathsf{KPu}^{\mathsf{r}}\vdash\forall h\,\forall q\,\forall m\,\forall\vec{s}\forall\vec{c}\Bigg(\Bigg(&\mathsf{Tran}(h)\wedge q>m\wedge m>0\wedge\\ &\wedge\vec{s}\in L_m(h)\wedge\vec{c}\subseteq L_q(h)\wedge\varphi_0^{(L_m(h))}(\vec{s},\vec{c})\wedge\\ &\wedge\varphi_1^{(L_m(h))}(\vec{s},\vec{c})\Bigg)\rightarrow\varphi_0^{(L_q(h))}(\vec{s},\vec{c})\Bigg)\end{aligned}\tag{3}$$

and

$$\begin{aligned}\mathsf{KPu}^{\mathsf{r}}\vdash\forall h\,\forall q\,\forall m\,\forall\vec{s}\forall\vec{c}\Bigg(\Bigg(&\mathsf{Tran}(h)\wedge q>m\wedge m>0\wedge\\ &\wedge\vec{s}\in L_m(h)\wedge\vec{c}\subseteq L_q(h)\wedge\varphi_1^{(L_m(h))}(\vec{s},\vec{c})\wedge\\ &\wedge\varphi_0^{(L_m(h))}(\vec{s},\vec{c})\Bigg)\rightarrow\varphi_1^{(L_q(h))}(\vec{s},\vec{c})\Bigg).\end{aligned}\tag{4}$$

Hence from (3) and (4) we obtain

$$\begin{aligned}\mathsf{KPu^r} \vdash \forall h\,\forall q\,\forall m\,\forall \vec{s}\forall \vec{c}\biggl(\Bigl(&\mathsf{Tran}(h) \wedge q > m \wedge m > 0\,\wedge\\ &\wedge \vec{s} \in L_m(h) \wedge \vec{c} \subseteq L_q(h)\,\wedge\\ &\wedge \varphi_1^{(L_m(h))}(\vec{s},\vec{c}) \wedge \varphi_0^{(L_m(h))}(\vec{s},\vec{c})\Bigr)\rightarrow\\ &\rightarrow \varphi_1^{(L_q(h))}(\vec{s},\vec{c}) \wedge \varphi_0^{(L_q(h))}(\vec{s},\vec{c})\biggr).\end{aligned}$$

Similarly for disjunction and bounded set quantifiers.

$\underline{\varphi(\vec{s},\vec{C}) \equiv \exists x \varphi_0(x,\vec{s},\vec{C})}$. Fix an a such that a does not occur free anywhere else.

By I.H.,

$$\begin{aligned}\mathsf{KPu^r} \vdash \forall h\,\forall q\,\forall m\,\forall \vec{s}\forall \vec{c}\biggl(\Bigl(&\mathsf{Tran}(h) \wedge q > m \wedge m > 0\,\wedge\\ &\wedge \vec{s} \in L_m(h) \wedge a \in L_m(h) \wedge \vec{c} \subseteq L_q(h)\,\wedge\\ &\wedge \varphi_0^{(L_m(h))}(a,\vec{s},\vec{c})\Bigr)\rightarrow \varphi_0^{(L_q(h))}(a,\vec{s},\vec{c})\biggr).\end{aligned}\tag{5}$$

By construction of $\langle L_n(h)\rangle_{n\in\mathbb{N}}$ we have

$$\mathsf{KPu^r} \vdash \forall h\,\forall q\,\forall m\,\Bigl(q > m \wedge m > 0 \wedge a \in L_m(h) \rightarrow a \in L_q(h)\Bigr).$$

From this last line we infer

$$\begin{aligned}\mathsf{KPu^r} \vdash \forall h\,\forall q\,\forall m\,\forall \vec{s}\forall \vec{c}\biggl(\Bigl(&\mathsf{Tran}(h) \wedge q > m \wedge m > 0\,\wedge\\ &\wedge \vec{s} \in L_m(h) \wedge a \in L_m(h) \wedge \vec{c} \subseteq L_q(h)\,\wedge\\ &\wedge \varphi_0^{(L_m(h))}(a,\vec{s},\vec{c})\Bigr)\rightarrow a \in L_q(h)\biggr).\end{aligned}\tag{6}$$

Hence from (5) and (6) we obtain

$$\begin{aligned}\mathsf{KPu^r} \vdash \forall h\,\forall q\,\forall m\,\forall \vec{s}\forall \vec{c}\biggl(\Bigl(&\mathsf{Tran}(h) \wedge q > m \wedge m > 0\,\wedge\\ &\wedge \vec{s} \in L_m(h) \wedge a \in L_m(h) \wedge \vec{c} \subseteq L_q(h)\,\wedge\\ &\wedge \varphi_0^{(L_m(h))}(a,\vec{s},\vec{c})\Bigr)\rightarrow a \in L_q(h) \wedge \varphi_0^{(L_q(h))}(a,\vec{s},\vec{c})\biggr).\end{aligned}$$

Thus

$$\begin{aligned}\mathsf{KPu}^{\mathsf{r}} \vdash \forall h\,\forall q\,\forall m\,\forall\vec{s}\forall\vec{c}\Bigg(\Big(&\mathsf{Tran}(h)\wedge q>m\wedge m>0\wedge\\ &\wedge\vec{s}\in L_m(h)\wedge a\in L_m(h)\wedge\vec{c}\subseteq L_q(h)\wedge\\ &\wedge\varphi_0^{(L_m(h))}(a,\vec{s},\vec{c})\Big)\rightarrow\\ &\rightarrow\exists x(x\in L_q(h)\wedge\varphi_0^{(L_q(h))}(x,\vec{s},\vec{c}))\Bigg).\end{aligned}$$

And from this we obtain

$$\begin{aligned}\mathsf{KPu}^{\mathsf{r}} \vdash \forall h\,\forall q\,\forall m\,\forall\vec{s}\forall\vec{c}\Bigg(\Big(&\mathsf{Tran}(h)\wedge q>m\wedge m>0\wedge\\ &\wedge\vec{s}\in L_m(h)\wedge\vec{c}\subseteq L_q(h)\wedge\\ &\wedge\exists x(x\in L_m(h)\wedge\varphi_0^{(L_m(h))}(x,\vec{s},\vec{c}))\Big)\rightarrow\\ &\rightarrow\exists x(x\in L_q(h)\wedge\varphi_0^{(L_q(h))}(x,\vec{s},\vec{c}))\Bigg).\end{aligned}$$

$\underline{\varphi(\vec{s},\vec{C})\equiv\forall X\varphi_0(X,\vec{s},\vec{C})}$. Fix an a such that a does not occur free anywhere else.
By I.H.

$$\begin{aligned}\mathsf{KPu}^{\mathsf{r}} \vdash \forall h\,\forall q\,\forall m\,\forall\vec{s}\forall\vec{c}\Bigg(\Big(&\mathsf{Tran}(h)\wedge q>m\wedge m>0\wedge\\ &\wedge\vec{s}\in L_m(h)\wedge a\subseteq L_q(h)\wedge\vec{c}\subseteq L_q(h)\wedge\\ &\wedge\varphi_0^{(L_m(h))}(a,\vec{s},\vec{c})\Big)\rightarrow\varphi_0^{(L_q(h))}(a,\vec{s},\vec{c})\Bigg).\end{aligned}$$

From which we infer

$$\begin{aligned}\mathsf{KPu}^{\mathsf{r}} \vdash \forall h\,\forall q\,\forall m\,\forall\vec{s}\forall\vec{c}\Bigg(\Big(&\mathsf{Tran}(h)\wedge q>m\wedge m>0\wedge\\ &\wedge\vec{s}\in L_m(h)\wedge a\subseteq L_q(h)\wedge\vec{c}\subseteq L_q(h)\wedge\\ &\wedge\varphi_0^{(L_m(h))}(a\cap L_m(h),\vec{s},\vec{c})\Big)\rightarrow\\ &\rightarrow\Big(\varphi_0^{(L_m(h))}(a,\vec{s},\vec{c})\rightarrow\varphi_0^{(L_q(h))}(a,\vec{s},\vec{c})\Big)\Bigg).\end{aligned}\tag{7}$$

By Lemma 1.4.5,

$$\mathsf{KPu^r} \vdash \forall h\, \forall m\, \forall \vec{s} \forall \vec{c} \left(\vec{s} \in L_m(h) \wedge \varphi_0^{L_m(h)}(\vec{s}, a \cap L_m(h), \vec{c}) \to \varphi_0^{L_m(h)}(\vec{s}, a, \vec{c}) \right).$$

From this last line we infer

$$\begin{aligned} \mathsf{KPu^r} \vdash \forall h\, \forall q\, \forall m\, \forall \vec{s} \forall \vec{c} \Bigg(\bigg(& \mathsf{Tran}(h) \wedge q > m \wedge m > 0 \wedge \\ & \wedge \vec{s} \in L_m(h) \wedge a \subseteq L_q(h) \wedge \vec{c} \subseteq L_q(h) \wedge \\ & \wedge \varphi_0^{(L_m(h))}(a \cap L_m(h), \vec{s}, \vec{c}) \bigg) \to \\ & \to \varphi_0^{(L_m(h))}(a, \vec{s}, \vec{c}) \Bigg). \end{aligned} \tag{8}$$

From (7) and (8) we then get

$$\begin{aligned} \mathsf{KPu^r} \vdash \forall h\, \forall q\, \forall m\, \forall \vec{s} \forall \vec{c} \Bigg(\bigg(& \mathsf{Tran}(h) \wedge q > m \wedge m > 0 \wedge \\ & \wedge \vec{s} \in L_m(h) \wedge a \subseteq L_q(h) \wedge \vec{c} \subseteq L_q(h) \wedge \\ & \wedge \varphi_0^{(L_m(h))}(a \cap L_m(h), \vec{s}, \vec{c}) \bigg) \to \\ & \to \varphi_0^{(L_q(h))}(a, \vec{s}, \vec{c}) \Bigg). \end{aligned} \tag{9}$$

Obviously

$$\mathsf{KPu^r} \vdash \forall h\, \forall m\, (a \cap L_m(h) \subseteq L_m(h)). \tag{10}$$

(9) along with (10) logically entails the following

$$\begin{aligned} \mathsf{KPu^r} \vdash \forall h\, \forall q\, \forall m\, \forall \vec{s} \forall \vec{c} \Bigg(\bigg(& \mathsf{Tran}(h) \wedge q > m \wedge m > 0 \wedge \\ & \wedge \vec{s} \in L_m(h) \wedge \vec{c} \subseteq L_q(h) \wedge \\ & \wedge \left(a \cap L_m(h) \subseteq L_m(h) \to \varphi_0^{(L_m(h))}(a \cap L_m(h), \vec{s}, \vec{c}) \right) \bigg) \to \\ & \to \left(a \subseteq L_q(h) \to \varphi_0^{(L_q(h))}(a, \vec{s}, \vec{c}) \right) \Bigg). \end{aligned}$$

Therefore

$$\begin{aligned}\mathsf{KPu}^{\mathsf{r}} \vdash \forall h \forall q \forall m \forall \vec{s} \forall \vec{c} \Bigg(\Bigg(& \mathsf{Tran}(h) \wedge q > m \wedge m > 0 \wedge \\ & \wedge \vec{s} \in L_m(h) \wedge \vec{c} \subseteq L_q(h) \wedge \\ & \wedge \forall x (x \subseteq L_m(h) \to \varphi_0^{(L_m(h))}(x, \vec{s}, \vec{c})) \Bigg) \to \\ & \to \left(a \subseteq L_q(h) \to \varphi_0^{(L_q(h))}(a, \vec{s}, \vec{c}) \right) \Bigg).\end{aligned}$$

Finally,

$$\begin{aligned}\mathsf{KPu}^{\mathsf{r}} \vdash \forall h \forall q \forall m \forall \vec{s} \forall \vec{c} \Bigg(\Bigg(& \mathsf{Tran}(h) \wedge q > m \wedge m > 0 \wedge \\ & \wedge \vec{s} \in L_m(h) \wedge \vec{c} \subseteq L_q(h) \wedge \\ & \wedge \forall x (x \subseteq L_m(h) \to \varphi_0^{(L_m(h))}(x, \vec{s}, \vec{c})) \Bigg) \to \\ & \to \forall x (x \subseteq L_q(h) \to \varphi_0^{(L_q(h))}(x, \vec{s}, \vec{c})) \Bigg).\end{aligned}$$

Downward Persistency for $[\text{s-}\Sigma^1_1]^{\mathrm{E}}$ formulae is proved following the same pattern. $\square$

Let $\Gamma_{\vec{s},\vec{C}}$ be a finite set of $[\text{s-}\Pi^1_1]^{\mathrm{E}}$ and $[\text{s-}\Sigma^1_1]^{\mathrm{E}}$ formulae of $\mathcal{L}^*_2$ whose all set and class parameters come from the lists $\vec{s}, \vec{C}$ respectively and let $q > m > 0$. We write $\Gamma_{\vec{s},\vec{c}}\left[m, q\right]$ to denote the result of replacing in $\Gamma_{\vec{s},\vec{C}}$

- every $[\text{s-}\Sigma^1_1]^{\mathrm{E}}$ formula $\psi(\vec{s}, \vec{C})$ by $\psi^{(L_m(h))}(\vec{s}, \vec{c})$,
- every $[\text{s-}\Pi^1_1]^{\mathrm{E}}$ formula $\varphi(\vec{s}, \vec{C})$ by $\varphi^{(L_q(h))}(\vec{s}, \vec{c})$.

Note that upon the assumption that $\vec{s} \in L_m(h)$ then, by Lemma 1.4.5 and the construction of $\langle L_n(h) \rangle_{n \in \mathbb{N}}$, $\Gamma_{\vec{s},\vec{c}}\left[m, q\right]$ equivals to the result of replacing in $\Gamma_{\vec{s},\vec{C}}$

- every $[\text{s-}\Sigma^1_1]^{\mathrm{E}}$ formula $\psi(\vec{s}, \vec{C})$ by $\psi^{(L_m(h))}(\vec{s}, \vec{c} \cap L_m(h))$,
- every $[\text{s-}\Pi^1_1]^{\mathrm{E}}$ formula $\varphi(\vec{s}, \vec{C})$ by $\varphi^{(L_q(h))}(\vec{s}, \vec{c} \cap L_q(h))$.

Corollary 1.4.6. *For all* $[\text{s-}\Pi^1_1]^{\mathrm{E}}$ *formulae* $\varphi(\vec{s}, \vec{C})$ *and* $[\text{s-}\Sigma^1_1]^{\mathrm{E}}$ *formulae* $\psi(\vec{s}, \vec{C})$ *of* $\mathcal{L}^*_2$ *and for any finite set* $\Gamma_{\vec{s},\vec{C}}$ *of* $[\text{s-}\Pi^1_1]^{\mathrm{E}}$ *and* $[\text{s-}\Sigma^1_1]^{\mathrm{E}}$ *formulae of*

$\mathcal{L}_2^*$, *we have:*

(i)
$$\begin{aligned}\mathsf{KPu^r} \vdash \forall h \forall q \forall m \forall \vec{s} \forall \vec{c}\Bigg(\Bigg(&\mathsf{Tran}(h) \wedge q > m \wedge m > 0 \wedge \\ &\wedge \vec{s} \in L_m(h) \wedge \vec{c} \subseteq L_q(h) \wedge \\ &\wedge \varphi^{(L_m(h))}(\vec{s}, \vec{c} \cap L_m(h))\Bigg) \to \\ &\to \varphi^{(L_q(h))}(\vec{s}, \vec{c})\Bigg);\end{aligned}$$

(ii)
$$\begin{aligned}\mathsf{KPu^r} \vdash \forall h \forall q \forall m \forall \vec{s} \forall \vec{c}\Bigg(\Bigg(&\mathsf{Tran}(h) \wedge q > m \wedge m > 0 \wedge \\ &\wedge \vec{s} \in L_m(h) \wedge \vec{c} \subseteq L_q(h) \wedge \\ &\wedge \varphi^{(L_q(h))}(\vec{s}, \vec{c})\Bigg) \to \\ &\to \varphi^{(L_m(h))}(\vec{s}, \vec{c} \cap L_m(h))\Bigg);\end{aligned}$$

(iii)
$$\begin{aligned}\mathsf{KPu^r} \vdash \forall h \forall q \forall r \forall p \forall m \forall \vec{s} \forall \vec{c}\Bigg(\Bigg(&\mathsf{Tran}(h) \wedge q > r \wedge r > p \wedge \\ &\wedge p > m \wedge m > 0 \wedge \\ &\wedge \vec{s} \in L_m(h) \wedge \vec{c} \subseteq L_q(h) \wedge \\ &\wedge \Big[\bigvee \Gamma_{\vec{s}, \vec{c} \cap L_r(h)}\Big[p, r\Big] \vee \bigvee \Delta\Big]\Bigg) \to \\ &\to \Big[\bigvee \Gamma_{\vec{s}, \vec{c}}\Big[m, q\Big] \vee \bigvee \Delta\Big]\Bigg).\end{aligned}$$

Proof. (i) and (ii) immediately follow from Lemma 1.4.5 and the Persistency result. (iii) is immediate by the definition of $\Gamma_{\vec{s}, \vec{c}}\Big[m, p\Big]$, (i) and (ii). □

ASYMMETRIC INTERPRETATION OF $\mathsf{T}_1{\restriction}_{\mathcal{C}_{\Delta_0}}$ INTO $\mathsf{KPu^r}$. *Assume that $\Gamma_{\vec{s}, \vec{C}}$ is a finite set of $[\text{S-}\Pi^1_1]^{\text{E}}$ and $[\text{S-}\Sigma^1_1]^{\text{E}}$ formulae of $\mathcal{L}_2^*$ so that*

$$\mathsf{T}_1{\restriction}_{\mathcal{C}_{\Delta_0}} \vdash^n_1 \Gamma_{\vec{s}, \vec{C}}$$

for some natural number n. Then for all natural numbers $m > 0$ we have

$$\begin{aligned}\mathsf{KPu^r} \vdash \forall h \forall \vec{s} \forall \vec{c}\Big(&\mathsf{Tran}(h) \wedge \mathsf{N} \in h \wedge \vec{s} \in L_m(h) \wedge \vec{c} \subseteq L_{m+2^n}(h) \to \\ &\to \bigvee \Gamma_{\vec{s}, \vec{c}}\Big[m, m + 2^n\Big]\Big).\end{aligned}$$

Proof. By induction on n. This is essentially the same treatment carried out by Cantini in [5]. We just show how the current asymmetric interpretation verifies S-Π^1_1 RFN.

<u>S-Π^1_1 RFN</u> Suppose that $\Gamma_{\vec{s},\vec{C}}$ is the conclusion of the non-logical rule of inference for S-Π^1_1 RFN. Then there exists a S-Π^1_1 formula $\varphi(\vec{s},\vec{C})$ and a natural number $n_0 < n$ such that

$$\mathsf{T}_1\!\restriction_{\mathcal{C}_{\Delta_0}} \vdash^{n_0}_1 \Gamma_{\vec{s},\vec{C}}, \varphi(\vec{s},\vec{C}). \tag{1}$$

The I.H. applied to (1) yields for all natural numbers $m > 0$

$$\mathsf{KPu}^{\mathsf{r}} \vdash \forall h \forall \vec{s} \forall \vec{c} \Big(\mathsf{Tran}(h) \wedge \mathsf{N} \in h \wedge \vec{s} \in L_m(h) \wedge \vec{c} \subseteq L_{m+2^{n_0}}(h) \rightarrow \\ \rightarrow \Big[\bigvee \Gamma_{\vec{s},\vec{c}} \Big[m, m+2^{n_0} \Big] \vee \varphi^{(L_{m+2^{n_0}}(h))}(\vec{s},\vec{c}) \Big] \Big).$$

From this by instaciating $\vec{c}$ by $\vec{c} \cap L_{m+2^{n_0}}(h)$ we obtain

$$\mathsf{KPu}^{\mathsf{r}} \vdash \forall h \forall \vec{s} \Big(\mathsf{Tran}(h) \wedge \mathsf{N} \in h \wedge \vec{s} \in L_m(h) \wedge \vec{c} \cap L_{m+2^{n_0}}(h) \subseteq L_{m+2^{n_0}}(h) \rightarrow \\ \rightarrow \Big[\bigvee \Gamma_{\vec{s},\vec{c} \cap L_{m+2^{n_0}}(h)} \Big[m, m+2^{n_0} \Big] \vee \\ \vee \varphi^{(L_{m+2^{n_0}}(h))}(\vec{s}, \vec{c} \cap L_{m+2^{n_0}}(h)) \Big] \Big). \tag{2}$$

By construction of $\langle L_n(h) \rangle_{n \in \mathbb{N}}$ we have

$$\mathsf{KPu}^{\mathsf{r}} \vdash \forall h \, (\vec{s} \in L_m(h) \rightarrow \vec{s} \in L_{m+2^{n_0}}(h)). \tag{3}$$

From (2) and (3), just using logic we obtain

$$\mathsf{KPu}^{\mathsf{r}} \vdash \forall h \forall \vec{s} \Big(\mathsf{Tran}(h) \wedge \mathsf{N} \in h \wedge \vec{s} \in L_m(h) \wedge \vec{c} \cap L_{m+2^{n_0}}(h) \subseteq L_{m+2^{n_0}}(h) \rightarrow \\ \rightarrow \Big[\bigvee \Gamma_{\vec{s},\vec{c} \cap L_{m+2^{n_0}}(h)} \Big[m, m+2^{n_0} \Big] \vee \\ \vee \Big(\vec{s} \in L_{m+2^{n_0}}(h) \wedge \varphi^{(L_{m+2^{n_0}}(h))}(\vec{s}, \vec{c} \cap L_{m+2^{n_0}}(h)) \Big) \Big] \Big). \tag{4}$$

Lemma 1.4.4 trivially entails that

$$\mathsf{KPu}^{\mathsf{r}} \vdash \forall h (\mathsf{Tran}(h) \rightarrow \mathsf{Tran}(L_{m+2^{n_0}}(h))). \tag{5}$$

Therefore from (4) and (5) we infer

$$\begin{aligned}\mathsf{KPu}^{\mathsf{r}} \vdash \forall h \, \forall \vec{s} \Big(& \mathsf{Tran}(h) \wedge \mathsf{N} \in h \wedge \vec{s} \in L_m(h) \wedge \vec{c} \cap L_{m+2^{n_0}}(h) \subseteq L_{m+2^{n_0}}(h) \rightarrow \\ & \rightarrow \Big[\bigvee \Gamma_{\vec{s}, \vec{c} \cap L_{m+2^{n_0}}(h)} \Big[m, m + 2^{n_0} \Big] \vee \Big(\mathsf{Tran}(L_{m+2^{n_0}}(h)) \wedge \\ & \wedge \vec{s} \in L_{m+2^{n_0}}(h) \wedge \varphi^{(L_{m+2^{n_0}}(h))}(\vec{s}, \vec{c} \cap L_{m+2^{n_0}}(h)) \Big) \Big] \Big).\end{aligned} \tag{6}$$

By construction of $\langle L_n(h) \rangle_{n \in \mathbb{N}}$ we have

$$\mathsf{KPu}^{\mathsf{r}} \vdash \forall h \, (L_{m+2^{n_0}}(h) \in L_{m+2^n}(h)). \tag{7}$$

Therefore from (6) and (7) we infer

$$\begin{aligned}\mathsf{KPu}^{\mathsf{r}} \vdash \forall h \, \forall \vec{s} \Big(& \mathsf{Tran}(h) \wedge \mathsf{N} \in h \wedge \vec{s} \in L_m(h) \wedge \vec{c} \cap L_{m+2^{n_0}}(h) \subseteq L_{m+2^{n_0}}(h) \rightarrow \\ & \rightarrow \Big[\bigvee \Gamma_{\vec{s}, \vec{c} \cap L_{m+2^{n_0}}(h)} \Big[m, m + 2^{n_0} \Big] \vee \Big(L_{m+2^{n_0}}(h) \in L_{m+2^n}(h) \wedge \\ & \wedge \mathsf{Tran}(L_{m+2^{n_0}}(h)) \wedge \vec{s} \in L_{m+2^{n_0}}(h) \wedge \\ & \wedge \varphi^{(L_{m+2^{n_0}}(h))}(\vec{s}, \vec{c} \cap L_{m+2^{n_0}}(h)) \Big) \Big] \Big).\end{aligned}$$

From this last expression we then infer,

$$\begin{aligned}\mathsf{KPu}^{\mathsf{r}} \vdash \forall h \, \forall \vec{s} \Big(& \mathsf{Tran}(h) \wedge \mathsf{N} \in h \wedge \vec{s} \in L_m(h) \wedge \vec{c} \cap L_{m+2^{n_0}}(h) \subseteq L_{m+2^{n_0}}(h) \rightarrow \\ & \rightarrow \Big[\bigvee \Gamma_{\vec{s}, \vec{c} \cap L_{m+2^{n_0}}(h)} \Big[m, m + 2^{n_0} \Big] \vee \\ & \vee \exists w [w \in L_{m+2^n}(h) \wedge \mathsf{Tran}(w) \wedge \vec{s} \in w \wedge \varphi^{(w)}(\vec{s}, \vec{c} \cap w)] \Big] \Big).\end{aligned} \tag{8}$$

(8) along with Lemma 1.4.5 trivially entails

$$\begin{aligned}\mathsf{KPu}^{\mathsf{r}} \vdash \forall h \, \forall \vec{s} \Big(& \mathsf{Tran}(h) \wedge \mathsf{N} \in h \wedge \vec{s} \in L_m(h) \wedge \vec{c} \cap L_{m+2^{n_0}}(h) \subseteq L_{m+2^{n_0}}(h) \rightarrow \\ & \rightarrow \Big[\bigvee \Gamma_{\vec{s}, \vec{c} \cap L_{m+2^{n_0}}(h)} \Big[m, m + 2^{n_0} \Big] \vee \\ & \vee \exists w [w \in L_{m+2^n}(h) \wedge \mathsf{Tran}(w) \wedge \vec{s} \in w \wedge \varphi^{(w)}(\vec{s}, \vec{c})] \Big] \Big).\end{aligned} \tag{9}$$

Obviously,

$$\mathsf{KPu}^{\mathsf{r}} \vdash \forall h \, (\vec{c} \cap L_{m+2^{n_0}}(h) \subseteq L_{m+2^{n_0}}(h)). \tag{10}$$

Therefore from (9) and (10) we obtain

$$\begin{aligned}\mathsf{KPu}^{\mathsf{r}} \vdash \forall h \forall \vec{s} \Big(& \mathsf{Tran}(h) \wedge \mathsf{N} \in h \wedge \vec{s} \in L_m(h) \rightarrow \\ & \rightarrow \Big[\bigvee \Gamma_{\vec{s},\vec{c}\cap L_{m+2^{n_0}}(h)} \Big[m, m+2^{n_0} \Big] \vee \\ & \vee \exists w [w \in L_{m+2^n}(h) \wedge \mathsf{Tran}(w) \wedge \vec{s} \in w \wedge \varphi^{(w)}(\vec{s},\vec{c})] \Big] \Big).\end{aligned} \tag{11}$$

From (11) through Corollary 1.4.6.(iii), we obtain

$$\begin{aligned}\mathsf{KPu}^{\mathsf{r}} \vdash \forall h \forall \vec{s} \forall \vec{c} \Big(& \mathsf{Tran}(h) \wedge \mathsf{N} \in h \wedge \vec{s} \in L_m(h) \wedge \vec{c} \subseteq L_{m+2^n}(h) \rightarrow \\ & \rightarrow \Big[\bigvee \Gamma_{\vec{s},\vec{c}} \Big[m, m+2^n \Big] \vee \\ & \vee \exists w [w \in L_{m+2^n}(h) \wedge \mathsf{Tran}(w) \wedge \vec{s} \in w \wedge \varphi^{(w)}(\vec{s},\vec{c})] \Big] \Big).\end{aligned}$$

Since the formula $\exists w [w \in L_{m+2^n}(h) \wedge \mathsf{Tran}(w) \wedge \vec{s} \in w \wedge \varphi^{(w)}(\vec{s},\vec{c})]$ is contained in $\Gamma_{\vec{s},\vec{c}} \Big[m, m+2^n \Big]$, the asymmetric treatment of the non-logical inference rule of S-Π^1_1 RFN is complete. ☐

Π_2-CONSERVATIVITY. $\mathsf{sKPu}^{\mathsf{r}}_2\restriction$ *conservatively extends* $\mathsf{KPu}^{\mathsf{r}}$ *for set-theoretic* Π_2 *sentences.*

Proof. Suppose that φ is a set-theoretic Π_2 sentence derivable in $\mathsf{sKPu}^{\mathsf{r}}_2\restriction$. Writing φ as $\forall a \exists y \psi(a,y)$ where ψ is Δ_0, then

$$\mathsf{sKPu}^{\mathsf{r}}_2\restriction \ \vdash \forall a \exists y \psi(a,y).$$

From which we infer, by Inversion, for an arbitrary a,

$$\mathsf{sKPu}^{\mathsf{r}}_2\restriction \ \vdash \exists y \psi(a,y).$$

By Corollary 1.4.3, there is a natural number n and a finite collection of $\mathcal{C}_{\Delta_0}$ of Δ_0 formulae of $\mathcal{L}^*$ such that

$$\mathsf{T}_1\restriction_{\mathcal{C}_{\Delta_0}} \vdash^n_1 \exists y \psi(a,y).$$

By the asymmetric interpretation of $\mathsf{T}_1\restriction_{\mathcal{C}_{\Delta_0}}$ into $\mathsf{KPu}^{\mathsf{r}}$ then for any natural numbers $m > 0$ we have that

$$\begin{aligned}\mathsf{KPu}^{\mathsf{r}} \vdash \forall h \forall a \Big(& \mathsf{Tran}(h) \wedge \mathsf{N} \in h \wedge a \in L_m(h) \rightarrow \\ & \rightarrow \exists y \, (\, y \in L_{m+2^n}(h) \wedge \psi(a,y) \,) \Big).\end{aligned} \tag{1}$$

Obviously

$$\mathsf{KPu^r} \vdash \forall h \forall a \Big(\exists y\,(\, y \in L_{m+2^n}(h) \wedge \psi(a,y)\,) \to \exists y \psi(a,y) \Big). \tag{2}$$

Hence from (1) and (2) we obtain

$$\mathsf{KPu^r} \vdash \forall h \forall a \Big(\mathsf{Tran}(h) \wedge \mathsf{N} \in h \wedge a \in L_m(h) \to \exists y \psi(a,y) \Big). \tag{3}$$

Since $h \notin \mathrm{FV}(\psi)$, then (3) logically entails

$$\mathsf{KPu^r} \vdash \forall a \exists h \Big(\mathsf{Tran}(h) \wedge \mathsf{N} \in h \wedge a \in L_m(h) \Big) \to \forall a \exists y \psi(a,y). \tag{4}$$

By construction of $\langle L_n(h) \rangle_{n \in \mathbb{N}}$ we have

$$\mathsf{KPu^r} \vdash \forall a \forall h \Big(a \in h \to a \in L_m(h) \Big). \tag{5}$$

Therefore from (4) and (5) we infer

$$\mathsf{KPu^r} \vdash \forall a \exists h \Big(\mathsf{Tran}(h) \wedge \mathsf{N} \in h \wedge a \in h \Big) \to \forall a \exists y \psi(a,y). \tag{6}$$

By PAIRING,

$$\mathsf{KPu^r} \vdash \forall a \exists y (a \in y \wedge \mathsf{N} \in y).$$

By TRANSITIVE HULL,

$$\mathsf{KPu^r} \vdash \forall y \exists h \Big(\mathsf{Tran}(h) \wedge y \subseteq h \Big).$$

From these last two expressions, just using logic, we infer

$$\mathsf{KPu^r} \vdash \forall a \exists h \Big(\mathsf{Tran}(h) \wedge \mathsf{N} \in h \wedge a \in h \Big). \tag{7}$$

Finally from (6) and (7), by MODUS PONENDO PONENS, we infer

$$\mathsf{KPu^r} \vdash \forall a \exists y \psi(a,y).$$

□

THEOREM 1.4.7. $\mathsf{KPu^r}$ *is a conservative extension of* PA.

For a proof of this result the reader is referred to Jäger [14].

THEOREM 1.4.8. $\mathsf{sKPu^r_2}{\restriction}$ *is a conservative extension of* PA.

The proof-theoretic strength of an axiom system $\mathbf{Ax}$ formulated in the language $\mathcal{L}^*$ or a similar one containing the first-order language of PA, is generally measured in terms of its proof-theoretic ordinal. To introduce this notion we proceed as usual and set for any primitive recurisve relation $\sqsubset$ and any $\mathcal{L}^*$-formula φ:

$$\begin{aligned} \mathsf{field}(\sqsubset) &:= \{\, x \mid \exists y(x \sqsubset y) \vee \exists y(y \sqsubset x) \,\}, \\ \mathsf{Prog}(\sqsubset, \varphi) &:= (\forall x \in \mathsf{field}(\sqsubset))\Big(\forall y(y \sqsubset x \to \varphi(y)) \to \varphi(x)\Big), \\ \mathsf{TI}(\sqsubset, \varphi) &:= \mathsf{Prog}(\sqsubset, \varphi) \to (\forall x \in \mathsf{field}(\sqsubset))\varphi(x). \end{aligned}$$

DEFINITION 1.4.9. Let $\mathbf{Ax}$ be a theory formulated in the language $\mathcal{L}^*$.

1 An ordinal α is provable in $\mathbf{Ax}$ if there exists a primitive recursive well-ordering $\sqsubset$ of order-type α so that $\mathbf{Ax} \vdash (\forall x \subseteq \mathsf{N})\mathsf{TI}(\sqsubset, x)$.

2 The proof-theoretic ordinal of $\mathbf{Ax}$, denoted by $|\mathbf{Ax}|$, is the least ordinal which is not provable in $\mathbf{Ax}$.

THE PROOF-THEORETIC ORDINAL OF $\mathsf{sKPu}^{\mathsf{r}}_2\restriction$.

$$|\mathsf{sKPu}^{\mathsf{r}}_2\restriction| = |\mathsf{KPu}^{\mathsf{r}}| = |\mathsf{PA}| = \varepsilon_0.$$

CHAPTER 2

AUSSONDERUNGSAXIOM: FROM ADMISSIBLE TO POWER ADMISSIBLE SET THEORY

At this stage of our work, the schema of Δ_0-SEP is extended in as much as we also allow free class parameters to occur in its defining formulae. The separation schema is then reformulated as a single axiom which we call *Aussonderungsaxiom*. It will be shown that s-Π^1_1 RFN along with the Aussonderungsaxiom implies the existence of the power-set, determining then a significant increase in strength. The exact consistency strength of the corresponding extended theory will be established. The notion of power admissible sets goes back to Harvey Friedman [8]. They are the transitive standard models of admissible set theories augmented by the power-set axiom.

2.1 THE THEORIES $\mathsf{KPu}^{\mathsf{r}}$+P AND $\mathsf{sKPu}^{\mathsf{r}}_2$

Let the POWER SET axiom be (i.e. the universal closure of):

$$\exists y \forall z[z \in y \leftrightarrow \mathsf{S}(z) \land \forall x(x \in z \to x \in a)].$$

We write $\wp(a)$ for the power-set of a. The first-order theory $\mathsf{KPu}^{\mathsf{r}}$+P is just $\mathsf{KPu}^{\mathsf{r}}$ plus the POWER SET axiom.

Let AUS denote the Aussonderungsaxiom,

$$\exists x(\mathsf{S}(x) \land \forall z(z \in x \leftrightarrow z \in a \land z \in C).$$

The second-order theory of $\mathsf{sKPu}^{\mathsf{r}}_2$ is obtained from $\mathsf{sKPu}^{\mathsf{r}}_2{\restriction}$ through replacement of Δ_0-SEP by AUS.

REMARK 2.1.1. Note that, for any class C and any set term a, Δ^{c}_1-CA yields the class $\{\, z \mid z \in a \land z \in C \,\}$ consisting of exactly the same member as the set x whose existence being asserted by AUS.

Accordingly, using our definition of equality between sets and classes, the following two expressions are then derivable in the theory $\mathsf{sKPu}^{\mathsf{r}}_{2}$ as immediate consequences of AUS.

PROPOSITION 2.1.2. *The following are derivable in the theory* $\mathsf{sKPu}^{\mathsf{r}}_{2}$:

(a) $\exists x(\mathsf{S}(x) \land x = a \cap C)$,

(b) $\forall Y(Y \subseteq a \rightarrow \exists x(\mathsf{S}(x) \land x = Y))$.

Thus on account of AUS, we might say that the intersection of a class with any set is a set and that a subclass of a set is a set.

PROPOSITION 2.1.3. *For any formula* φ *and any set* a, *we have the following derivable in* $\mathsf{sKPu}^{\mathsf{r}}_{2}$:

(a) $\forall X(X \subseteq a \rightarrow \varphi) \leftrightarrow \forall x(x \subseteq a \rightarrow \varphi)$,

(b) $\exists X(X \subseteq a \land \varphi) \leftrightarrow \exists x(x \subseteq a \land \varphi)$.

Proof. The direction from left to right in (a) and the direction from right to left in (b) immediately follow follow from the fact that any set is a class (Proposition 1.2.3) and the full substitutivity of equality (Proposition 1.2.5). The remaining directions in (a) and (b) follow from the fact that any subclass of a set is a set (Proposition 2.1.2.(b)) and the full substitutivity of equality (Proposition 1.2.5). ☐

PROPOSITION 2.1.4. Δ_0-$\mathsf{I}_{\mathbb{N}}$ *and* $\mathsf{I}^{2}_{\mathbb{N}}$ *are provably equivalent in* $\mathsf{sKPu}^{\mathsf{r}}_{2}$.

Proof. That $\mathsf{I}^{2}_{\mathbb{N}}$ implies Δ_0-$\mathsf{I}_{\mathbb{N}}$ has already been proved in Proposition 1.3.5. The proof of the reverse implication is accomplished by the method of Specker presented by Bernays in [3]. Apply S-Π^1_1 RFN to the formula

$$0 \in A \land \forall x \forall y(x \in \mathsf{N} \land y \in \mathsf{N} \land x \in A \land \mathsf{Sc}(x,y) \rightarrow y \in A) \land \exists x(x \in \mathsf{N} \land x \notin A).$$

Denoting this formula by $\varphi(0, \mathsf{N}, A)$, thus

$$\varphi(0,\mathsf{N},A) \rightarrow \exists z \Big[\mathsf{Tran}(z) \land 0 \in z \land \mathsf{N} \in z \land \varphi^{(z)}(0,\mathsf{N},A)\Big].$$

From which we infer using $\mathsf{Tran}(z)$ and $\mathsf{N} \in z$

$$\begin{aligned}\varphi(0,\mathsf{N},A) \rightarrow \exists z\Big(&0 \in A \land 0 \in z \land \\ &\land \forall x \forall y(x \in \mathsf{N} \land y \in \mathsf{N} \land x \in A \land x \in z \land \mathsf{Sc}(x,y) \rightarrow \\ &\rightarrow y \in A \land y \in z) \land \\ &\land \exists x(x \in \mathsf{N} \land x \notin A \land x \in z)\Big).\end{aligned}$$

This last formula, along with Proposition 2.1.2.(a) and Proposition 1.2.5, logically entails the following

$$\varphi(0,\mathsf{N},A) \to \exists u\Big(0 \in u \wedge \forall x \forall y(x \in \mathsf{N} \wedge y \in \mathsf{N} \wedge x \in u \wedge \mathsf{Sc}(x,y) \to y \in u) \wedge \wedge \exists x(x \in \mathsf{N} \wedge x \notin u)\Big).$$

But the conclusion of this implication is the negation of Δ_0-$\mathsf{I}_{\mathbb{N}}$, hence by MODUS TOLLENDO TOLLENS, we have $\neg\varphi(0,\mathsf{N},A)$, that is $\mathsf{I}^2_{\mathbb{N}}$. ☐

PROPOSITION 2.1.5. *Δ_0-$\mathsf{I}_\in$ and $\mathsf{I}^2_\in$ are provably equivalent in $\mathsf{sKPu^r_2}$.*

Proof. That $\mathsf{I}^2_\in$ implies Δ_0-$\mathsf{I}_\in$ has already been proved in Proposition 1.3.4. Because of the presence of an unbounded universal set quantifier in the negation of $\mathsf{I}^2_\in$, we do not know how to apply the previous argument to the proof of the current implication. However the result can be established arguing as follows. Consider the contrapositive of $\mathsf{I}^2_\in$,

$$\forall y((\forall z \in y)(z \in A) \to y \in A) \to \forall y(y \in A)$$

and assume the premise holds and that $x \notin A$. By TRANSITIVE HULL, let t be a transitive set such that $\{x\} \subseteq t$ and consider the set

$$v = \{\, y \,|\, y \in t \wedge y \notin A \,\},$$

given by AUS. By Δ_0-$\mathsf{I}_\in$, since $x \in v$, there is a $y_0 \in v$ such that $y_0 \cap v = \emptyset$. If $z \in y_0$ then, by transitivity of t, $z \in t$ and $z \notin v$; so $z \in A$. By assumption then we have $y_0 \in A$, contradicting $y_0 \in v$. ☐

We show that proof-theoretic strength of $\mathsf{sKPu^r_2}$ significantly differs from that of $\mathsf{sKPu^r_2}\!\restriction$. It turns out, in fact, that $\mathsf{KPu^r}$+P and $\mathsf{sKPu^r_2}$ prove the same set-theoretic Π_2 sentences. In order to prove that all the theorems of $\mathsf{KPu^r}$+P are provable in $\mathsf{sKPu^r_2}$, it is enough to prove in $\mathsf{sKPu^r_2}$ all the axioms of $\mathsf{KPu^r}$+P.

2.2 $\mathsf{KPu^r}$+P SUBSYSTEM OF $\mathsf{sKPu^r_2}$

LEMMA 2.2.1. *Every instance of Δ_0-SEP is derivable in $\mathsf{sKPu^r_2}$.*

Proof. This is immediate by Δ^{c}_1-CA and AUS. ☐

In order to introduce the next argument, the following definition of supertransitivity is needed.

DEFINITION 2.2.2. For any set a, we let

$$\mathsf{Stran}(a) \;:=\; \mathsf{Tran}(a) \wedge \forall x(x \in a \to \forall y(y \subseteq x \to y \in a)).$$

In words, a super-transitive set is a transitive set closed under the subsets of its members.

Remark 2.2.3. Bernays [4], pp.138 and 139, proves that the full second-order schema of reflection (or even a schema of Π^1_1 reflection, as already noted by Gloede [9]) applied to the formula

$$\forall Y \forall a (Y \subseteq a \rightarrow \exists x (x = Y))$$

admits a self-strengthening to a schema with a super-transitive reflecting set. The latter schema is then showed to imply the existence of the Power set. Bernays' argument can be adapted to the current context in showing that the existence of the Power set is already derivable from a schema of reflection restricted to second-order set-theoretic formulae of logical complexity s-Π^1_1. Surprisingly enough, the subsequent simple observation does not seem, at least to our knowledge, to have been made before.

Lemma 2.2.4. *The* Power set *axiom is derivable in* $\mathsf{sKPu}^{\mathsf{r}}_2$.

Proof. In order to derive the Power set axiom we apply s-Π^1_1 Rfn to the derivable formula

$$\forall U \Big(U \subseteq a \rightarrow \exists x (\mathsf{S}(x) \wedge x = U \cap a) \Big). \tag{1}$$

Note that "$U \cap a$" in (1), is needed to keep the logical complexity of this formula down to s-Π^1_1. Let us briefly denote this formula by "$\varphi(a)$". We then get

$$\varphi(a) \rightarrow \exists w \Big[\mathsf{Tran}(w) \wedge a \in w \wedge \varphi^{(w)}(a) \Big],$$

which yields by Modus Ponendo Ponens

$$\exists w \Big[\mathsf{Tran}(w) \wedge a \in w \wedge \varphi^{(w)}(a) \Big]. \tag{2}$$

Before relativizing $\varphi(a)$ to the reflecting set w we have to replace within $\varphi(a)$ the symbols "$\subseteq$", "$\cap$" and "$=$" by their corresponding defining expressions. Accordingly, $\varphi(a)$ stands for the formula

$$\forall U \Bigg(\forall y (y \in U \rightarrow y \in a) \rightarrow$$

$$\rightarrow \exists x \Bigg(\mathsf{S}(x) \wedge \forall z \Big((z \in x \rightarrow z \in U \wedge z \in a) \wedge (z \in U \wedge z \in a \rightarrow z \in x) \Big) \Bigg) \Bigg).$$

By Proposition 2.1.3.(a), the relativization of this formula to the reflecting set w yields

$$\forall u \Bigg(u \subseteq w \rightarrow \forall y (y \in u \rightarrow y \in a) \rightarrow$$

$$\rightarrow \exists x \Bigg(x \in w \wedge \mathsf{S}(x) \wedge \forall z \Big((z \in x \rightarrow z \in u \wedge z \in a) \wedge (z \in u \wedge z \in a \rightarrow z \in x) \Big) \Bigg) \Bigg).$$

If, after doing this, we reinstate in this last expression the symbols for inclusion, intersection and equality, then we obtain along with (2),

$$\exists w\Big[\mathsf{Tran}(w) \wedge a \in w \wedge \forall u\Big(u \subseteq w \wedge u \subseteq a \rightarrow \exists x(x \in w \wedge \mathsf{S}(x) \wedge x = u \cap a)\Big)\Big].$$

From which we infer, using $\mathsf{Tran}(w)$ and $a \in w$,

$$\exists w\Big[\mathsf{Tran}(w) \wedge a \in w \wedge \forall u(u \subseteq a \rightarrow \exists x(x \in w \wedge \mathsf{S}(x) \wedge x = u))\Big].$$

From this, using the fact that

$$\exists x(x \in b \wedge \mathsf{S}(x) \wedge x = u) \rightarrow \mathsf{S}(u) \wedge u \in b$$

we infer,

$$\exists w\Big[\mathsf{Tran}(w) \wedge a \in w \wedge \forall u(u \subseteq a \rightarrow \mathsf{S}(u) \wedge u \in w)\Big].$$

Therefore, in particualr

$$\exists w \forall u(u \subseteq a \rightarrow \mathsf{S}(u) \wedge u \in w),$$

and obviously

$$\exists w \forall u(u \subseteq a \wedge \mathsf{S}(u) \rightarrow u \in w).$$

This last expression asserts that each subset of the set a is an element of w. The result is then obtained through an application of Δ_0-SEP. It follows that the POWER SET axiom is derivable in $\mathsf{sKPu}^{\mathsf{r}}_2$. □

COROLLARY 2.2.5. *Every theorem φ of $\mathsf{KPu}^{\mathsf{r}}$+P is also a theorem of $\mathsf{sKPu}^{\mathsf{r}}_2$,*

$$\mathsf{KPu}^{\mathsf{r}}+\mathrm{P} \vdash \varphi \quad \Longrightarrow \quad \mathsf{sKPu}^{\mathsf{r}}_2 \vdash \varphi.$$

2.3 A SELF-STRENGTHENING OF S-Π^1_1 RFN

The main concern of this Section is to show that, as for Π^1_1 RFN, also S-Π^1_1 RFN admits a self-strengthening to a schema with a super-transitive reflecting set. For Π^1_1 RFN such a strengthening is obtained by reflecting the formula of Remark 2.2.3. Things are not that easy with S-Π^1_1 RFN. The difficulty, in this respect, relies on the presence of the unbounded universal set-quantifer in the formula of Remark 2.2.3. In other words, the formula of Remark 2.2.3 is of logical complexity Π^1_1 and we cannot apply S-Π^1_1 RFN to it. Henceforth, we have to proceed in a different way. We begin by observing that the power-set of any transitive set is a super-transitive set.

LEMMA 2.3.1.

$$\mathsf{KPu}^{\mathsf{r}}+\mathrm{P} \vdash \forall a\Big(\mathsf{Tran}(a) \rightarrow \mathsf{Stran}(\wp(a))\Big).$$

Proof. We shall argue informally within the theory $\mathsf{KPu}^{\mathsf{r}}+\mathrm{P}$. Assume $\mathsf{Tran}(a)$, for an arbitrary set a. We have to show $\mathsf{Stran}(\wp(a))$, i.e.

(1) $\mathsf{Tran}(\wp(a))$ and

(2) $\forall x(x \in \wp(a) \to \forall y(y \subseteq x \to y \in \wp(a)))$.

(1) Assume $c \in \wp(a)$ and $d \in c$. Then, $c \subseteq a$ and $d \in a$. By transitivity of a, d is a subset of a. It follows that d is an element of $\wp(a)$.
(2) Assume $c \in \wp(a)$ and $d \subseteq c$. From $d \subseteq c$ and $c \subseteq a$, it follows $d \subseteq a$. Hence, $d \in \wp(a)$. ☐

The next result is a direct generalization of the Persistency Lemma of Section 1.4 to arbitrary transitive sets.

LEMMA 2.3.2. *For any* $[\text{s-}\Pi^1_1]^{\mathrm{E}}$ *formula* $\varphi(v_0, ..., v_n, C_0, ..., C_m)$ *of* $\mathcal{L}_2^*$, *with no free variables besides the displayed ones and not necessarily all of them we have:*

$$\begin{aligned} \mathsf{sKPu}^{\mathsf{r}}_2 \vdash \forall v_0 ... \forall v_n \forall C_0 ... \forall C_m \forall y \forall z \Bigg(\Bigg(& y \subseteq z \wedge \mathsf{Tran}(y) \wedge \mathsf{Tran}(z) \wedge \\ & \wedge v_0, ..., v_n \in y \wedge C_0, ..., C_m \subseteq z \wedge \\ & \wedge \varphi^{(y)}(v_0, ..., v_n, C_0 \cap y, ..., C_m \cap y) \Bigg) \to \\ & \to \varphi^{(z)}(v_0, ..., v_n, C_0, ..., C_m) \Bigg). \end{aligned}$$

Proof. Note that the implication above, by Proposition 2.1.3.(a) and Proposition 1.2.8, is provably equivalent to

$$\begin{aligned} \forall v_0 ... \forall v_n \forall c_0 ... \forall c_m \forall y \forall z \Bigg(\Bigg(& y \subseteq z \wedge \mathsf{Tran}(y) \wedge \mathsf{Tran}(z) \wedge \\ & \wedge v_0, ..., v_n \in y \wedge c_0, ..., c_m \subseteq z \wedge \\ & \wedge \varphi^{(y)}(v_0, ..., v_n, c_0, ..., c_m) \Bigg) \to \\ & \to \varphi^{(z)}(v_0, ..., v_n, c_0, ..., c_m) \Bigg). \end{aligned}$$

And this is established by a straightfoward inductive argument on the build-up of $\varphi(v_0, ..., v_n, C_0, ..., C_m)$ following exactly the same pattern as in the proof of the Persistency Lemma of Section 1.4. ☐

THEOREM 2.3.3. *For any* $\text{s-}\Pi^1_1$ *formula* $\varphi(v_0, ..., v_n, C_0, ..., C_m)$ *with no free variables besides the displayed ones and not necessarily all of them, the following is derivable within the theory* $\mathsf{sKPu}^{\mathsf{r}}_2$:

$$\begin{aligned} & \varphi(v_0, ..., v_n, C_0, ..., C_m) \to \\ & \to \exists z \, [\, \mathsf{Stran}(z) \wedge v_0, ..., v_n \in z \wedge \varphi^{(z)}(v_0, ..., v_n, C_0, ..., C_m) \,]. \end{aligned}$$

Proof. Let $\varphi(v_0, ..., v_n, C_0, ..., C_m)$ be a given S-Π^1_1 formula. Consider the corresponding instance of the schema of S-Π^1_1 RFN:

$$\begin{aligned}&\varphi(v_0, ..., v_n, C_0, ..., C_m) \to\\ &\to \exists y\,[\,\mathsf{Tran}(y) \wedge v_0, ..., v_n \in y \wedge \varphi^{(y)}(v_0, ..., v_n, C_0, ..., C_m)\,],\end{aligned}$$

which is, by Proposition 1.2.9, provably equivalent to

$$\begin{aligned}&\varphi(v_0, ..., v_n, C_0, ..., C_m) \to\\ &\to \exists y\,[\,\mathsf{Tran}(y) \wedge v_0, ..., v_n \in y \wedge \varphi^{(y)}(v_0, ..., v_n, C_0 \cap y, ..., C_m \cap y)\,].\end{aligned}$$

By Lemma 2.2.4, we obtain

$$\begin{aligned}&\varphi(v_0, ..., v_n, C_0, ..., C_m) \to\\ &\to \exists y \exists z\,[\,\mathsf{Tran}(y) \wedge z = \wp(y) \wedge v_0, ..., v_n \in y \wedge \varphi^{(y)}(v_0, ..., v_n, C_0 \cap y, ..., C_m \cap y)\,].\end{aligned}$$

And this, along with the observation that $y \subseteq z$, can be rewritten as follows,

$$\begin{aligned}&\varphi(v_0, ..., v_n, C_0, ..., C_m) \to \exists y \exists z\,[\,\mathsf{Tran}(y) \wedge z = \wp(y) \wedge v_0, ..., v_n \in y \wedge\\ &\qquad\qquad \wedge\, \varphi^{(y)}(v_0, ..., v_n, (C_0 \cap z) \cap y, ..., (C_m \cap z) \cap y)\,].\end{aligned}$$

Therefore, by Lemma 2.3.2 (instanciating $C_0, ..., C_m$ by $(C_0 \cap z), ..., (C_m \cap z)$) and Lemma 2.3.1, we get

$$\begin{aligned}&\varphi(v_0, ..., v_n, C_0, ..., C_m) \to\\ &\to \exists z\,[\,\mathsf{Stran}(z) \wedge v_0, ..., v_n \in z \wedge \varphi^{(z)}(v_0, ..., v_n, C_0 \cap z, ..., C_0 \cap z)\,],\end{aligned}$$

which is, by by Proposition 1.2.8, provably equivalent to

$$\begin{aligned}&\varphi(v_0, ..., v_n, C_0, ..., C_m) \to\\ &\to \exists z\,[\,\mathsf{Stran}(z) \wedge v_0, ..., v_n \in z \wedge \varphi^{(z)}(v_0, ..., v_n, C_0, ..., C_0)\,].\end{aligned}$$

□

Even though the schema of S-Π^1_1 RFN admits a self-strengthening to schema with a super-transitive reflecting set, the theory $\mathsf{sKPu}^{\mathsf{r}}_2$ remains "relatively" weak. As we shall have occasion to see in the next Section, $\mathsf{sKPu}^{\mathsf{r}}_2$ does not prove, for example, the existence of ω. By contrast, the schema of Π^1_1 RFN, along with AUS and Δ_0-$\mathsf{I}_\in$, already entails the existence of arbitrarily large Mahlo cardinals. This should also make the reader appreciating the "explosion" in strength we shall be getting, as soon as we shall replace Δ^{C}_1-CA by the full schema of Predicative Comprehension.

2.4 $\mathsf{sKPu}^{\mathsf{r}}_2$ CONSERVATIVE EXTENSION OF $\mathsf{KPu}^{\mathsf{r}}$+P

In order to prove that all the set-theoretic Π_2 sentences of $\mathsf{KPu}^{\mathsf{r}}$+P provable in $\mathsf{sKPu}^{\mathsf{r}}_2$ are also theorems of $\mathsf{KPu}^{\mathsf{r}}$+P, we proceed by carrying through an asymmetric interpretation of quasi normal $\mathsf{sKPu}^{\mathsf{r}}_2$ derivations into finite segments of the cumulative hierarchy. As in Section 1.4, we proceed into two steps. First, we provide a Tait-style reformulation of $\mathsf{sKPu}^{\mathsf{r}}_2$ that allows us to establish a partial cut elimination theorem yielding quasi-normal derivations. In a second step, quasi-normal derivations of such a Tait-style reformulation of $\mathsf{sKPu}^{\mathsf{r}}_2$ are then reduced to $\mathsf{KPu}^{\mathsf{r}}$+P by means of an asymmetric interpretation. We take up the first step.

A Tait-style reformulation of $\mathsf{sKPu}^{\mathsf{r}}_2$ is the same as for $\mathsf{sKPu}^{\mathsf{r}}_2\restriction$, where AUS reads as follows:

For all finite sets Γ of formulae of $\mathcal{L}^*_2$,

$$\Gamma, \underbrace{\exists x(\mathsf{S}(x) \land \forall z(z \in x \leftrightarrow z \in a \land z \in C))}_{[\text{S-}\Pi^1_1]^{\mathrm{E}}}.$$

The Tait-style reformulation of $\mathsf{sKPu}^{\mathsf{r}}_2$ is denoted by T_2.

EMBEDDING OF $\mathsf{sKPu}^{\mathsf{r}}_2$ INTO T_2. *Let φ be a $\mathcal{L}^*_2$ formula such that*

$$\mathsf{sKPu}^{\mathsf{r}}_2 \vdash \varphi.$$

Then there are two natural numbers n and k such that

$$\mathsf{T}_2 \vdash^n_k \varphi.$$

The non-logical axiom AUS has logical complexity $[\text{S-}\Pi^1_1]^{\mathrm{E}}$. We then establish a partial cut elimination theorem (up $[\text{S-}\Pi^1_1]^{\mathrm{E}}$ and $[\text{S-}\Sigma^1_1]^{\mathrm{E}}$ formulae), yielding quasi-normal T_2 derivations exactly as in Section 1.4.

PARTIAL CUT ELIMINATION FOR T_2. *For all finite set Γ of $\mathcal{L}^*_2$ formulae and all natural numbers n and k,*

$$\mathsf{T}_2 \vdash^n_{k+1} \Gamma \quad \Longrightarrow \quad \mathsf{T}_2 \vdash^{2_k(n)}_1 \Gamma$$

The following result concludes our first step.

COROLLARY 2.4.1. *Let φ be a $\mathcal{L}^*_2$ formula such that*

$$\mathsf{sKPu}^{\mathsf{r}}_2 \vdash \varphi.$$

Then there is a natural numbers n such that

$$\mathsf{T}_2 \vdash^n_1 \varphi.$$

The second step of reducing quasi-normal T_2 derivations to $\mathsf{KPu}^{\mathsf{r}}$+P consists in setting up a partial model for $\mathsf{sKPu}^{\mathsf{r}}_2$ (e.g. a model for the set-theoretic Π_2 sentences of $\mathsf{sKPu}^{\mathsf{r}}_2$) which will subsequently be used in order to provide an asymmetric interpretation theorem for quasi-normal T_2 derivations. It is argued that the whole procedure can be formalized within $\mathsf{KPu}^{\mathsf{r}}$+P. In particular, the partial models needed for an interpretation of $\mathsf{sKPu}^{\mathsf{r}}_2$ are available in $\mathsf{KPu}^{\mathsf{r}}$+P.

For any set a,

$$\bigcap a \ := \ \{\, z \,|\, (\forall v \in a)(z \in v)\,\}.$$

Whenever $a \neq \emptyset$, $\bigcap a$ is a set; it is a subset of any $v \in a$. (By our definition, $\bigcap \emptyset = \mathbf{V}$, but this is not a case that will ever concern us).

DEFINITION 2.4.2. For any set a, the transitive closure of a, denoted by $\mathsf{TC}(a)$, is the smallest transitive set including a. That is $\mathsf{TC}(a)$ is transitive, $a \subseteq \mathsf{TC}(a)$ and if b is any other transitive set such that $a \subseteq b$, then $\mathsf{TC}(a) \subseteq b$.

The existence of this set can be justified within $\mathsf{KPu}^{\mathsf{r}}$+P using the POWER SET axiom as follows.

PROPOSITION 2.4.3.

$$\mathsf{KPu}^{\mathsf{r}}\text{+P} \vdash \forall a \exists x (x = \mathsf{TC}(a)).$$

Proof. We argue informally within $\mathsf{KPu}^{\mathsf{r}}$+P and fix an arbitrary a. We need to prove the existence of a unique transitive set which includes a and is itself contained in every transitive set including a. The TRANSITIVE HULL axiom provides for any set a a set c such that

$$\mathsf{Tran}(c) \quad \text{and} \quad a \subseteq c.$$

By applying Δ_0-SEP to $\wp(c)$ we isolate the transitive sets containing a:

$$\exists z(\mathsf{S}(z) \wedge \forall y(y \in z \leftrightarrow y \in \wp(c) \wedge \mathsf{Tran}(y) \wedge a \subseteq y).$$

At this stage we consider the set $\bigcap z$. We aim to prove that $\bigcap z = \mathsf{TC}(a)$. Obviously,

$$a \subseteq \bigcap z \qquad \text{and} \qquad \mathsf{Tran}(\bigcap z).$$

What is required to prove is that the set $\bigcap z$ is included in any transitive set including a, that is

$$\forall v(\mathsf{Tran}(v) \wedge a \subseteq v \rightarrow \bigcap z \subseteq v).$$

For any term b, assume

$$\mathsf{Tran}(b) \wedge a \subseteq b.$$

By combining our assumption with the derivability of

$$\mathsf{Tran}(c) \quad \text{and} \quad a \subseteq c \quad \text{and} \quad c \in \wp(c),$$

we infer

$$\mathsf{Tran}(b \cap c) \quad \text{and} \quad a \subseteq (b \cap c) \quad \text{and} \quad (b \cap c) \in \wp(c).$$

By definition of the set z, we then obtain

$$\mathsf{Tran}(b) \wedge a \subseteq b \rightarrow (b \cap c) \in z.$$

and in particular

$$\mathsf{Tran}(b) \wedge a \subseteq b \rightarrow \bigcap z \subseteq b.$$

□

At this stage, working in $\mathsf{KPu}^{\mathsf{r}}$+P, let us introduce finite segments of the cumulative hierarchy which will subsequently be used in order to prove an asymmetric interpretation theorem for quasi-normal T_2 derivations.

For any set z, we define by recursion on n a finite hierarchy $\langle V_n^{\mathsf{N}}(z)\rangle_{n\in\mathbb{N}}$ of set terms $V_n^{\mathsf{N}}(z)$ as follows:

$$V_0^{\mathsf{N}}(z) := \mathsf{TC}(\{\mathsf{N}, z\}),$$

$$V_{n+1}^{\mathsf{N}}(z) := \wp(V_n^{\mathsf{N}}(z)).$$

We write V_n^{N} if $V_0^{\mathsf{N}}(z) = \mathsf{TC}(\{\mathsf{N}\})$ and V_n if $V_0^{\mathsf{N}}(z) = \emptyset$.

LEMMA 2.4.4. *For all natural numbers $n \in \mathbb{N}$,*

$$\mathsf{KPu}^{\mathsf{r}}\text{+P} \vdash \forall z \mathsf{Tran}(V_n^{\mathsf{N}}(z)).$$

Proof. By induction on n. We work informally within the theory $\mathsf{KPu}^{\mathsf{r}}$+P. Fix an arbitrary z.

$\underline{n = 0}$ We need to show $V_0^{\mathsf{N}}(z)$ is a transitive set. By definition of $V_0^{\mathsf{N}}(z)$, this reduces to showing that $\mathsf{TC}(\{\mathsf{N}, z\})$ is a transitive set. And this is so by definition of transitive closure.

$\underline{n \mapsto n+1}$ We need to prove that $V_{n+1}^{\mathsf{N}}(z)$ is a transitive set. We first show that $V_{n+1}^{\mathsf{N}}(z)$ is transitive. Assume for two arbitary sets a and w that $a \in V_{n+1}^{\mathsf{N}}(z)$ and $w \in a$. Since $a \in V_{n+1}^{\mathsf{N}}(z)$ we also have that $a \subseteq V_n^{\mathsf{N}}(z)$ and thus $w \in V_n^{\mathsf{N}}(z)$. By I.H., $w \subseteq V_n^{\mathsf{N}}(z)$. Hence $w \in V_{n+1}^{\mathsf{N}}(z)$. We are left with proving that $V_{n+1}^{\mathsf{N}}(z)$ is a set. By I.H., we have that $V_n^{\mathsf{N}}(z)$ is a set. Then so is $V_{n+1}^{\mathsf{N}}(z)$, by the POWER SET axiom. □

Sets and classes are interpreted, respectively, as elements and subsets of

$$\bigcup_{n\in\mathbb{N}} V_n^{\mathsf{N}}(z).$$

We keep the same notation as in Section 1.4. Let $\varphi(\vec{s}, \vec{C})$ be any formula of $\mathcal{L}_2^*$, whose all set and class parameters came from the lists $\vec{s}, \vec{C}$ respectively. We write $\varphi^{(V_n^{\mathsf{N}}(z))}(\vec{s}, \vec{c})$ to denote the result of replacing in $\varphi(\vec{s}, \vec{C})$

- every unbounded set quantifier $\mathcal{Q}x$ by $\mathcal{Q}x \in V_n^{\mathsf{N}}(z)$,
- every class quantifier $\mathcal{Q}Y$ by $\mathcal{Q}y \subseteq V_n^{\mathsf{N}}(z)$,
- every class variable C by a set variable c.

We avoid conflict of variables. It is worth noticing, however, that the translated formula $\varphi^{(V_n^{\mathsf{N}}(z))}(\vec{s},\vec{c})$ has logical complexity Δ_0, for any unbounded set quantifier $\mathcal{Q}y \subseteq V_n^{\mathsf{N}}(z)$ being in fact converted to a bounded set quantifer $\mathcal{Q}y \in V_{n+1}^{\mathsf{N}}(z)$.

LEMMA 2.4.5. *For any formula $\varphi(\vec{s},\vec{C},\vec{D})$ of $\mathcal{L}_2^*$, with no free variables besides the displayed ones and not necessarily all of them and for any set b wich does not occur free in the list $\vec{s}$ we have the following provable in* $\mathsf{KPu}^{\mathsf{r}}$+P :

$$\vec{s} \in b \to \Big(\varphi^{(b)}(\vec{s},\vec{c},\vec{d}) \leftrightarrow \varphi^{(b)}(\vec{s},\vec{c}\cap b,\vec{d})\Big).$$

The proof of Lemma 2.4.5 is obvious in virtue of Lemma 1.4.5 and the fact that $\mathsf{KPu}^{\mathsf{r}}$ is a subsystem of $\mathsf{KPu}^{\mathsf{r}}$+P. Persistence properties are obviously satisfied; we confine ourselves to stating the following result which will be often invoked in the subsequent asymmetric interpretation.

COROLLARY 2.4.6. *For any finite set $\Gamma_{\vec{s},\vec{C}}$ of $[\text{S-}\Pi^1_1]^{\mathrm{E}}$ and $[\text{S-}\Sigma^1_1]^{\mathrm{E}}$ formulae of $\mathcal{L}_2^*$, we have:*

$$\begin{aligned}\mathsf{KPu}^{\mathsf{r}}+\mathrm{P} \vdash \forall z\,\forall q\,\forall r\,\forall p\,\forall m\,\forall\vec{s}\,\forall\vec{c}\Bigg(\Bigg(&q>r\wedge r>p\wedge p>m\wedge m>0\,\wedge\\ &\wedge\,\vec{s}\in V_m^{\mathsf{N}}(z)\wedge\vec{c}\subseteq V_q^{\mathsf{N}}(z)\,\wedge\\ &\wedge\Big[\bigvee\Gamma_{\vec{s},\vec{c}\cap V_r^{\mathsf{N}}(z)}\Big[p,r\Big]\vee\bigvee\Delta\Big]\Bigg)\to\\ &\to\Big[\bigvee\Gamma_{\vec{s},\vec{c}}\Big[m,q\Big]\vee\bigvee\Delta\Big]\Bigg).\end{aligned}$$

As for the asymmetric interpretation of $\mathsf{T}_1{\restriction}_{\mathcal{C}_{\Delta_0}}$ into $\mathsf{KPu}^{\mathsf{r}}$, we interpret any given quasi-normal T_2 derivation of Γ (where Γ only contains $[\text{S-}\Pi^1_1]^{\mathrm{E}}$ and $[\text{S-}\Sigma^1_1]^{\mathrm{E}}$ formulae) by assigning bounds to existential set and universal class quantifiers occurring in the derivation, depending on any given bound for existential class and universal set quantifiers of the derivation.

ASYMMETRIC INTERPRETATION OF T_2 INTO $\mathsf{KPu}^{\mathsf{r}}+\mathrm{P}$. *Assume that $\Gamma_{\vec{s},\vec{C}}$ is a finite set of $[\text{S-}\Pi^1_1]^{\mathrm{E}}$ and $[\text{S-}\Sigma^1_1]^{\mathrm{E}}$ formulae of $\mathcal{L}_2^*$ so that*

$$\mathsf{T}_2 \vdash_1^n \Gamma_{\vec{s},\vec{C}}$$

for some natural number n. Then for all natural numbers $m>0$ we have

$$\mathsf{KPu}^{\mathsf{r}}+\mathrm{P} \vdash \forall z\,\forall\vec{s}\,\forall\vec{c}\Big(\vec{s}\in V_m^{\mathsf{N}}(z)\wedge\vec{c}\subseteq V_{m+2^n}^{\mathsf{N}}(z)\to\bigvee\Gamma_{\vec{s},\vec{c}}\Big[m,m+2^n\Big]\Big).$$

Proof. By induction on n. The subsequent asymmetric interpretation of T_2 into $\mathsf{KPu}^{\mathsf{r}}$+P is proved following the same pattern as for the asymmetric interpretation of $\mathsf{T}_1{\restriction}_{\mathcal{C}_{\Delta_0}}$ into $\mathsf{KPu}^{\mathsf{r}}$.

$\underline{n = 0}$ We content ourselves in showing how the asymmetric interpretation verifies AUS.

<u>AUS</u> Suppose that $\Gamma_{\vec{s},\vec{C}}$ is the non-logical axiom AUS. Then

$$\mathsf{T}_2 \vdash^0_1 \exists x(\mathsf{S}(x) \wedge \forall z(z \in x \leftrightarrow z \in a \wedge z \in C)).$$

Given an arbitrary $a \in V^{\mathsf{N}}_m(z)$, by transitivity of $V^{\mathsf{N}}_m(z)$, we have $a \subseteq V^{\mathsf{N}}_m(z)$. This means that for any set c, $(a \cap c) \subseteq V^{\mathsf{N}}_m(z)$. And this immediately provides us with the upper bound for the existential set quantifer, since

$$\mathsf{KPu}^{\mathsf{r}}\text{+P} \vdash (a \cap c) \in V^{\mathsf{N}}_{m+1}(z).$$

$\underline{n > 0}$ We content ourselves in showing how the asymmetric interpretation verifies Δ^{c}_1-CA.

$\underline{\Delta^{\mathrm{c}}_1\text{-CA}}$ Suppose that $\Gamma_{\vec{s},\vec{C}}$ is the conclusion of the non-logical inference rule for Δ^{c}_1-CA. Then there are two Σ^{c}_1 formulae $\varphi(a, \vec{s}, \vec{C})$ and $\psi(a, \vec{s}, \vec{C})$ and two natural numbers $n_0, n_1 < n$ such that

$$\mathsf{T}_2 \vdash^{n_0}_1 \Gamma_{\vec{s},\vec{C}}, \forall x(\varphi(x, \vec{s}, \vec{C}) \rightarrow \neg\psi(x, \vec{s}, \vec{C})),$$

$$\mathsf{T}_2 \vdash^{n_1}_1 \Gamma_{\vec{s},\vec{C}}, \forall x(\neg\psi(x, \vec{s}, \vec{C}) \rightarrow \varphi(x, \vec{s}, \vec{C})).$$

Let $p = \mathsf{max}(\{n_0, n_1\})$. Then we have

$$\mathsf{T}_2 \vdash^{p}_1 \Gamma_{\vec{s},\vec{C}}, \forall x(\varphi(x, \vec{s}, \vec{C}) \rightarrow \neg\psi(x, \vec{s}, \vec{C})), \tag{1}$$

$$\mathsf{T}_2 \vdash^{p}_1 \Gamma_{\vec{s},\vec{C}}, \forall x(\neg\psi(x, \vec{s}, \vec{C}) \rightarrow \varphi(x, \vec{s}, \vec{C})). \tag{2}$$

By inversion, we witness the universal quantifiers in (1) and (2) by some a such that $a \notin \mathrm{FV}\Big(\Gamma, \varphi, \psi\Big)$, obtaining then

$$\mathsf{T}_2 \vdash^{p}_1 \Gamma_{\vec{s},\vec{C}}, \neg\varphi(a, \vec{s}, \vec{C}), \neg\psi(a, \vec{s}, \vec{C}), \tag{3}$$

$$\mathsf{T}_2 \vdash^{p}_1 \Gamma_{\vec{s},\vec{C}}, \psi(a, \vec{s}, \vec{C}), \varphi(a, \vec{s}, \vec{C}). \tag{4}$$

The I.H. applied to (4) yields for all natural numbers $m > 0$,

$$\begin{aligned} \mathsf{KPu}^{\mathsf{r}}\text{+P} \vdash\ \forall z \forall \vec{s} \forall a \forall \vec{c} \Bigg(& \vec{s} \in V^{\mathsf{N}}_m(z) \wedge a \in V^{\mathsf{N}}_m(z) \wedge \vec{c} \subseteq V^{\mathsf{N}}_{m+2^p}(z) \rightarrow \\ & \rightarrow \Bigg[\bigvee \Gamma_{\vec{s},\vec{c}} \Big[m, m + 2^p \Big] \vee \\ & \vee \Bigg(\neg\psi^{(V^{\mathsf{N}}_{m+2^p}(z))}(a, \vec{s}, \vec{c}) \rightarrow \varphi^{(V^{\mathsf{N}}_{m+2^p}(z))}(a, \vec{s}, \vec{c}) \Bigg) \Bigg] \Bigg). \end{aligned} \tag{5}$$

And from this, we infer

$$\begin{aligned}\mathsf{KPu}^{\mathsf{r}}+\mathrm{P}\vdash\ \forall z\,\forall\vec{s}\,\forall a\Big(&\vec{s}\in V^{\mathsf{N}}_m(z)\wedge a\in V^{\mathsf{N}}_m(z)\wedge\vec{c}\cap V^{\mathsf{N}}_{m+2^p}(z)\subseteq V^{\mathsf{N}}_{m+2^p}(z)\to\\ &\to\Big[\bigvee\Gamma_{\vec{s},\vec{c}\cap V^{\mathsf{N}}_{m+2^p}(z)}\Big[m,m+2^p\Big]\vee\\ &\vee\Big(\neg\psi^{(V^{\mathsf{N}}_{m+2^p}(z))}(a,\vec{s},\vec{c}\cap V^{\mathsf{N}}_{m+2^p}(z))\to\\ &\to\varphi^{(V^{\mathsf{N}}_{m+2^p}(z))}(a,\vec{s},\vec{c}\cap V^{\mathsf{N}}_{m+2^p}(z))\Big)\Big]\Big).\end{aligned}\tag{6}$$

Since

$$\mathsf{KPu}^{\mathsf{r}}+\mathrm{P}\vdash\forall z(\vec{c}\cap V^{\mathsf{N}}_{m+2^p}(z)\subseteq V^{\mathsf{N}}_{m+2^p}(z)).\tag{7}$$

(6) and (7) along with Corollary 2.4.6 entail

$$\begin{aligned}\mathsf{KPu}^{\mathsf{r}}+\mathrm{P}\vdash\ \forall z\,\forall\vec{s}\,\forall a\,\forall\vec{c}\Big(&\vec{s}\in V^{\mathsf{N}}_m(z)\wedge a\in V^{\mathsf{N}}_m(z)\wedge\vec{c}\subseteq V^{\mathsf{N}}_{m+2^n}(z)\to\\ &\to\Big[\bigvee\Gamma_{\vec{s},\vec{c}}\Big[m,m+2^n\Big]\vee\\ &\vee\Big(\neg\psi^{(V^{\mathsf{N}}_{m+2^p}(z))}(a,\vec{s},\vec{c}\cap V^{\mathsf{N}}_{m+2^p}(z))\to\\ &\to\varphi^{(V^{\mathsf{N}}_{m+2^p}(z))}(a,\vec{s},\vec{c}\cap V^{\mathsf{N}}_{m+2^p}(z))\Big)\Big]\Big).\end{aligned}\tag{8}$$

The I.H. applied to (3) yields for all natural numbers $m>0$,

$$\begin{aligned}\mathsf{KPu}^{\mathsf{r}}+\mathrm{P}\vdash\ \forall z\,\forall\vec{s}\,\forall a\,\forall\vec{c}\Big(&\vec{s}\in V^{\mathsf{N}}_m(z)\wedge a\in V^{\mathsf{N}}_m(z)\wedge\vec{c}\subseteq V^{\mathsf{N}}_{m+2^p}(z)\to\\ &\to\Big[\bigvee\Gamma_{\vec{s},\vec{c}}\Big[m,m+2^p\Big]\vee\\ &\vee\Big(\varphi^{(V^{\mathsf{N}}_m(z))}(a,\vec{s},\vec{c})\to\neg\psi^{(V^{\mathsf{N}}_m(z))}(a,\vec{s},\vec{c})\Big)\Big]\Big).\end{aligned}\tag{9}$$

By instanciating m by $m+2^p$, we get

$$\begin{aligned}\mathsf{KPu}^{\mathsf{r}}+\mathrm{P}\vdash\ \forall z\,\forall\vec{s}\,\forall a\,\forall\vec{c}\Big(&\vec{s}\in V^{\mathsf{N}}_{m+2^p}(z)\wedge a\in V^{\mathsf{N}}_{m+2^p}(z)\wedge\vec{c}\subseteq V^{\mathsf{N}}_{m+2^p+2^p}(z)\to\\ &\to\Big[\bigvee\Gamma_{\vec{s},\vec{c}}\Big[m+2^p,m+2^p+2^p\Big]\vee\\ &\vee\Big(\varphi^{(V^{\mathsf{N}}_{m+2^p}(z))}(a,\vec{s},\vec{c})\to\neg\psi^{(V^{\mathsf{N}}_{m+2^p}(z))}(a,\vec{s},\vec{c})\Big)\Big]\Big).\end{aligned}\tag{10}$$

By Lemma 2.4.5, we have

$$\begin{aligned} \mathsf{KPu^r}{+}\mathrm{P} \vdash \forall \vec{s} \forall a \Bigg(& \vec{s} \in V^{\mathsf{N}}_{m+2^p}(z) \wedge a \in V^{\mathsf{N}}_{m+2^p}(z) \rightarrow \\ & \rightarrow \Bigg(\varphi^{(V^{\mathsf{N}}_{m+2^p}(z))}(a, \vec{s}, \vec{c}) \leftrightarrow \varphi^{(V^{\mathsf{N}}_{m+2^p}(z))}(a, \vec{s}, \vec{c} \cap V^{\mathsf{N}}_{m+2^p}(z)) \Bigg) \Bigg) \end{aligned} \tag{11}$$

and

$$\begin{aligned} \mathsf{KPu^r}{+}\mathrm{P} \vdash \forall \vec{s} \forall a \Bigg(& \vec{s} \in V^{\mathsf{N}}_{m+2^p}(z) \wedge a \in V^{\mathsf{N}}_{m+2^p}(z) \rightarrow \\ & \rightarrow \Bigg(\neg\psi^{(V^{\mathsf{N}}_{m+2^p}(z))}(a, \vec{s}, \vec{c}) \leftrightarrow \neg\psi^{(V^{\mathsf{N}}_{m+2^p}(z))}(a, \vec{s}, \vec{c} \cap V^{\mathsf{N}}_{m+2^p}(z)) \Bigg) \Bigg) \end{aligned} \tag{12}$$

Accordingly, by (11) and (12) we infer from (10),

$$\begin{aligned} \mathsf{KPu^r}{+}\mathrm{P} \vdash \forall z \, \forall \vec{s} \forall a \, \forall \vec{c} \Bigg(& \vec{s} \in V^{\mathsf{N}}_{m+2^p}(z) \wedge a \in V^{\mathsf{N}}_{m+2^p}(z) \wedge \vec{c} \subseteq V^{\mathsf{N}}_{m+2^p+2^p}(z) \rightarrow \\ & \rightarrow \Bigg[\bigvee \Gamma_{\vec{s},\vec{c}} \Big[m + 2^p, m + 2^p + 2^p \Big] \vee \\ & \vee \Bigg(\varphi^{(V^{\mathsf{N}}_{m+2^p}(z))}(a, \vec{s}, \vec{c} \cap V^{\mathsf{N}}_{m+2^p}(z)) \rightarrow \\ & \rightarrow \neg\psi^{(V^{\mathsf{N}}_{m+2^p}(z))}(a, \vec{s}, \vec{c} \cap V^{\mathsf{N}}_{m+2^p}(z)) \Bigg) \Bigg] \Bigg). \end{aligned} \tag{13}$$

By construction of $\langle V^{\mathsf{N}}_n(z) \rangle_{n \in \mathbb{N}}$ we have

$$\mathsf{KPu^r}{+}\mathrm{P} \vdash \forall z (b \in V^{\mathsf{N}}_m(z) \rightarrow b \in V^{\mathsf{N}}_{m+2^p}(z)). \tag{14}$$

Hence from (13) and (14) we infer

$$\begin{aligned} \mathsf{KPu^r}{+}\mathrm{P} \vdash \forall z \, \forall \vec{s} \forall a \, \forall \vec{c} \Bigg(& \vec{s} \in V^{\mathsf{N}}_m(z) \wedge a \in V^{\mathsf{N}}_m(z) \wedge \vec{c} \subseteq V^{\mathsf{N}}_{m+2^p+2^p}(z) \rightarrow \\ & \rightarrow \Bigg[\bigvee \Gamma_{\vec{s},\vec{c}} \Big[m + 2^p, m + 2^p + 2^p \Big] \vee \\ & \vee \Bigg(\varphi^{(V^{\mathsf{N}}_{m+2^p}(z))}(a, \vec{s}, \vec{c} \cap V^{\mathsf{N}}_{m+2^p}(z)) \rightarrow \\ & \rightarrow \neg\psi^{(V^{\mathsf{N}}_{m+2^p}(z))}(a, \vec{s}, \vec{c} \cap V^{\mathsf{N}}_{m+2^p}(z)) \Bigg) \Bigg] \Bigg). \end{aligned} \tag{15}$$

From this last expression we get

$$\begin{aligned}\mathsf{KPu}^{\mathsf{r}}+\mathrm{P}\vdash\ \forall z\,\forall\vec{s}\forall a\Biggl(&\vec{s}\in V^{\mathsf{N}}_m(z)\wedge a\in V^{\mathsf{N}}_m(z)\wedge\\ &\wedge\vec{c}\cap V^{\mathsf{N}}_{m+2^p+2^p}(z)\subseteq V^{\mathsf{N}}_{m+2^p+2^p}(z)\rightarrow\\ &\rightarrow\Biggl[\bigvee\Gamma_{\vec{s},\vec{c}\cap V^{\mathsf{N}}_{m+2^p+2^p}(z)}\Bigl[m+2^p,m+2^p+2^p\Bigr]\vee\\ &\vee\Biggl(\varphi^{(V^{\mathsf{N}}_{m+2^p}(z))}(a,\vec{s},(\vec{c}\cap V^{\mathsf{N}}_{m+2^p+2^p}(z))\cap V^{\mathsf{N}}_{m+2^p}(z))\rightarrow\\ &\rightarrow\neg\psi^{(V^{\mathsf{N}}_{m+2^p}(z))}(a,\vec{s},(\vec{c}\cap V^{\mathsf{N}}_{m+2^p+2^p}(z))\cap V^{\mathsf{N}}_{m+2^p}(z))\Biggr)\Biggr]\Biggr).\end{aligned}\tag{16}$$

By construction of $\langle V^{\mathsf{N}}_n(z)\rangle_{n\in\mathbb{N}}$ we have provable, within $\mathsf{KPu}^{\mathsf{r}}$+P, that

$$V^{\mathsf{N}}_{m+2^p}(z)\subseteq V^{\mathsf{N}}_{m+2^p+2^p}(z).$$

This obviously implies that

$$(\vec{c}\cap V^{\mathsf{N}}_{m+2^p+2^p}(z))\cap V^{\mathsf{N}}_{m+2^p}(z)=(\vec{c}\cap V^{\mathsf{N}}_{m+2^p}(z)).$$

Accordingly we obtain from (16) that

$$\begin{aligned}\mathsf{KPu}^{\mathsf{r}}+\mathrm{P}\vdash\ \forall z\,\forall\vec{s}\forall a\Biggl(&\vec{s}\in V^{\mathsf{N}}_m(z)\wedge a\in V^{\mathsf{N}}_m(z)\wedge\\ &\wedge\vec{c}\cap V^{\mathsf{N}}_{m+2^p+2^p}(z)\subseteq V^{\mathsf{N}}_{m+2^p+2^p}(z)\rightarrow\\ &\rightarrow\Biggl[\bigvee\Gamma_{\vec{s},\vec{c}\cap V^{\mathsf{N}}_{m+2^p+2^p}(z)}\Bigl[m+2^p,m+2^p+2^p\Bigr]\vee\\ &\vee\Biggl(\varphi^{(V^{\mathsf{N}}_{m+2^p}(z))}(a,\vec{s},\vec{c}\cap V^{\mathsf{N}}_{m+2^p}(z))\rightarrow\\ &\rightarrow\neg\psi^{(V^{\mathsf{N}}_{m+2^p}(z))}(a,\vec{s},\vec{c}\cap V^{\mathsf{N}}_{m+2^p}(z))\Biggr)\Biggr]\Biggr).\end{aligned}\tag{17}$$

Since

$$\mathsf{KPu}^{\mathsf{r}}+\mathrm{P}\vdash\ \forall z(\vec{c}\cap V^{\mathsf{N}}_{m+2^p+2^p}(z)\subseteq V^{\mathsf{N}}_{m+2^p+2^p}(z)).\tag{18}$$

(17) and (18) along with Corollary 2.4.6 entail

$$\begin{aligned}\mathsf{KPu}^{\mathsf{r}}+\mathrm{P}\vdash\ \forall z\,\forall\vec{s}\forall a\,\forall\vec{c}\Biggl(&\vec{s}\in V^{\mathsf{N}}_m(z)\wedge a\in V^{\mathsf{N}}_m(z)\wedge\vec{c}\subseteq V^{\mathsf{N}}_{m+2^n}(z)\rightarrow\\ &\rightarrow\Biggl[\bigvee\Gamma_{\vec{s},\vec{c}}\Bigl[m,m+2^n\Bigr]\vee\\ &\vee\Biggl(\varphi^{(V^{\mathsf{N}}_{m+2^p}(z))}(a,\vec{s},\vec{c}\cap V^{\mathsf{N}}_{m+2^p}(z))\rightarrow\\ &\rightarrow\neg\psi^{(V^{\mathsf{N}}_{m+2^p}(z))}(a,\vec{s},\vec{c}\cap V^{\mathsf{N}}_{m+2^p}(z))\Biggr)\Biggr]\Biggr).\end{aligned}\tag{19}$$

Hence from (8) and (19) we obtain

$$
\begin{aligned}
\mathsf{KPu}^{\mathsf{r}}{+}\mathrm{P} \vdash \forall z\,\forall\vec{s}\,\forall a\,\forall\vec{c}\biggl(&\vec{s}\in V^{\mathsf{N}}_{m}(z)\wedge a\in V^{\mathsf{N}}_{m}(z)\wedge\vec{c}\subseteq V^{\mathsf{N}}_{m+2^{n}}(z)\rightarrow\\
&\rightarrow\biggl[\bigvee\Gamma_{\vec{s},\vec{c}}\Bigl[m,m+2^{n}\Bigr]\vee\\
&\vee\Bigl(\varphi^{(V^{\mathsf{N}}_{m+2^{p}}(z))}(a,\vec{s},\vec{c}\cap V^{\mathsf{N}}_{m+2^{p}}(z))\leftrightarrow\\
&\leftrightarrow\neg\psi^{(V^{\mathsf{N}}_{m+2^{p}}(z))}(a,\vec{s},\vec{c}\cap V^{\mathsf{N}}_{m+2^{p}}(z))\Bigr)\biggr]\biggr).
\end{aligned}
\tag{20}
$$

Accordingly, we can form the set

$$
\begin{aligned}
b&=\{\,a\in V^{\mathsf{N}}_{m}(z)\mid\varphi^{(V^{\mathsf{N}}_{m+2^{p}}(z))}(a,\vec{s},\vec{c}\cap V^{\mathsf{N}}_{m+2^{p}}(z))\,\}\\
&=\{\,a\in V^{\mathsf{N}}_{m}(z)\mid\neg\psi^{(V^{\mathsf{N}}_{m+2^{p}}(z))}(a,\vec{s},\vec{c}\cap V^{\mathsf{N}}_{m+2^{p}}(z))\,\}.
\end{aligned}
$$

which is a subset of $V^{\mathsf{N}}_{m}(z)$. Therefore we get

$$
\begin{aligned}
\mathsf{KPu}^{\mathsf{r}}{+}\mathrm{P} \vdash \forall z\,\forall\vec{s}\,\forall\vec{c}\biggl(&\vec{s}\in V^{\mathsf{N}}_{m}(z)\wedge\vec{c}\subseteq V^{\mathsf{N}}_{m+2^{n}}(z)\rightarrow\biggl[\bigvee\Gamma_{\vec{s},\vec{c}}\Bigl[m,m+2^{n}\Bigr]\vee\\
&\vee\exists y\Bigl(y\in V^{\mathsf{N}}_{m+1}(z)\wedge\forall a\Bigl(a\in V^{\mathsf{N}}_{m}(z)\rightarrow\\
&\rightarrow\Bigl[\Bigl(a\in y\rightarrow\neg\psi^{(V^{\mathsf{N}}_{m+2^{p}}(z))}(a,\vec{s},\vec{c}\cap V^{\mathsf{N}}_{m+2^{p}}(z))\Bigr)\wedge\\
&\wedge\Bigl(\varphi^{(V^{\mathsf{N}}_{m+2^{p}}(z))}(a,\vec{s},\vec{c}\cap V^{\mathsf{N}}_{m+2^{p}}(z))\rightarrow a\in y\Bigr)\Bigr]\Bigr)\Bigr)\biggr]\biggr).
\end{aligned}
\tag{21}
$$

And from (21) by Corollary 2.4.6 we finally obtain

$$
\begin{aligned}
\mathsf{KPu}^{\mathsf{r}}{+}\mathrm{P} \vdash \forall z\,\forall\vec{s}\,\forall\vec{c}\biggl(&\vec{s}\in V^{\mathsf{N}}_{m}(z)\wedge\vec{c}\subseteq V^{\mathsf{N}}_{m+2^{n}}(z)\rightarrow\biggl[\bigvee\Gamma_{\vec{s},\vec{c}}\Bigl[m,m+2^{n}\Bigr]\vee\\
&\vee\exists y\Bigl(y\in V^{\mathsf{N}}_{m+1}(z)\wedge\forall a\Bigl(a\in V^{\mathsf{N}}_{m}(z)\rightarrow\\
&\rightarrow\Bigl[\Bigl(a\in y\rightarrow\neg\psi^{(V^{\mathsf{N}}_{m}(z))}(a,\vec{s},\vec{c}\cap V^{\mathsf{N}}_{m}(z))\Bigr)\wedge\\
&\wedge\Bigl(\varphi^{(V^{\mathsf{N}}_{m}(z))}(a,\vec{s},\vec{c}\cap V^{\mathsf{N}}_{m}(z))\rightarrow a\in y\Bigr)\Bigr]\Bigr)\Bigr)\biggr]\biggr).
\end{aligned}
$$

Since the formula

$$\begin{aligned} \exists y\Bigg(y \in V^{\mathsf{N}}_{m+1}(z) \wedge \forall a\Bigg(a \in V^{\mathsf{N}}_{m}(z) \rightarrow \\ \rightarrow \Bigg[\Bigg(a \in y \rightarrow \neg\psi^{(V^{\mathsf{N}}_{m}(z))}(a, \vec{s}, \vec{c} \cap V^{\mathsf{N}}_{m}(z))\Bigg) \wedge \\ \wedge \Bigg(\varphi^{(V^{\mathsf{N}}_{m}(z))}(a, \vec{s}, \vec{c} \cap V^{\mathsf{N}}_{m}(z)) \rightarrow a \in y \Bigg)\Bigg]\Bigg)\Bigg), \end{aligned}$$

is contained in $\Gamma_{\vec{s},\vec{c}}\Bigg[m, m + 2^n\Bigg]$, the asymmetric treatment of the non-logical inference rule for Δ^{C}_1-CA is complete. □

Π_2-CONSERVATIVITY. $\mathsf{sKPu}^{\mathsf{r}}_2$ *conservatively extends* $\mathsf{KPu}^{\mathsf{r}}$+P *for set-theoretic* Π_2 *sentences.*

Proof. Analogous to the proof of Π_2-Conservativity for $\mathsf{sKPu}^{\mathsf{r}}_2{\restriction}$. □

DEFINITION 2.4.7. The hierarchy $\langle V^{\mathsf{N}}_{\alpha} \rangle_{\alpha \in \mathbf{ON}}$ is defined by the following recursion on the class of all ordinals:

$$\begin{aligned} V^{\mathsf{N}}_{0} &:= \mathsf{TC}(\{\mathsf{N}\}) \\ V^{\mathsf{N}}_{\alpha+1} &:= \wp(V^{\mathsf{N}}_{\alpha}) \\ V^{\mathsf{N}}_{\lambda} &:= \bigcup_{\alpha<\lambda} V^{\mathsf{N}}_{\alpha}, \quad \text{for } \mathsf{Lim}(\lambda). \end{aligned}$$

For any class A and B and any binary relation E we let

$$E^{[A \times B]} := \{\langle x, y \rangle \mid x \in A \wedge y \in B \wedge xEy\}.$$

When $A = B$, we simply write $E^{[A]}$, instead of $E^{[A \times A]}$.

Let $\mathbf{Ax}$ be a theory formulated in the language $\mathcal{L}^*$ or $\mathcal{L}^*_2$. We make use of the following abbreviations:

$$\begin{aligned} \Big(V^{\mathsf{N}}_{\alpha}\Big)_{[\mathbf{Ax}]_{\Sigma_n}} &:= \langle V^{\mathsf{N}}_{\alpha}, \in^{[V^{\mathsf{N}}_{\alpha}]} \rangle \models \{\varphi \mid \varphi \text{ is a } \Sigma_n \text{ sentence and } \mathbf{Ax} \vdash \varphi\}, \\ \Big(V^{\mathsf{N}}_{\alpha}\Big)_{[\mathbf{Ax}]_{\Pi_n}} &:= \langle V^{\mathsf{N}}_{\alpha}, \in^{[V^{\mathsf{N}}_{\alpha}]} \rangle \models \{\varphi \mid \varphi \text{ is a } \Pi_n \text{ sentence and } \mathbf{Ax} \vdash \varphi\}. \end{aligned}$$

DEFINITION 2.4.8. Let $\mathbf{Ax}$ be a theory formulated in the language $\mathcal{L}^*$ or $\mathcal{L}^*_2$. We define

$$\begin{aligned} \|\mathbf{Ax}\|_{\Sigma_n} &:= \min\Bigg\{\alpha \,\Bigg|\, \Big(V^{\mathsf{N}}_{\alpha}\Big)_{[\mathbf{Ax}]_{\Sigma_n}}\Bigg\}, \\ \|\mathbf{Ax}\|_{\Pi_n} &:= \min\Bigg\{\alpha \,\Bigg|\, \Big(V^{\mathsf{N}}_{\alpha}\Big)_{[\mathbf{Ax}]_{\Pi_n}}\Bigg\}. \end{aligned}$$

COROLLARY 2.4.9.

$$\begin{aligned}\omega &= \|\mathsf{sKPu}^{\mathsf{r}}_{2}\|_{\Pi_2}\\ &= \|\mathsf{KPu}^{\mathsf{r}}{+}\mathrm{P}\|_{\Pi_2}\end{aligned}$$

Proof. Let us first show that $\omega = \|\mathsf{sKPu}^{\mathsf{r}}_{2}\|_{\Pi_2}$. Let φ be a set-theoretic Π_2 sentence derivable in $\mathsf{sKPu}^{\mathsf{r}}_{2}$. Write φ as $\forall x \exists y \psi(x,y)$, for ψ being Δ_0. Then we have, for an arbitary set term a, that

$$\mathsf{sKPu}^{\mathsf{r}}_{2} \vdash \exists y \psi(a,y).$$

By Corollary 2.4.1, there is a natural number n such that

$$\mathsf{T}_2 \vdash^{n}_{1} \exists y \psi(a,y).$$

Assume a to be an element of V^{N}_{ω}. This means that there exists an $0 < m < \omega$ such that $a \in V^{\mathsf{N}}_{m}$. The asymmetric interpretation of T_2 into $\mathsf{KPu}^{\mathsf{r}}{+}\mathrm{P}$ tells us for any $m > 0$,

$$\mathsf{KPu}^{\mathsf{r}}{+}\mathrm{P} \vdash \forall z \forall a \Big(a \in V^{\mathsf{N}}_{m}(z) \rightarrow \exists y (y \in V^{\mathsf{N}}_{m+2^n}(z) \wedge \psi^{(V^{\mathsf{N}}_{m+2^n}(z))}(a,y)) \Big).$$

Instanciating z by the set term N, we therefore obtain

$$\mathsf{KPu}^{\mathsf{r}}{+}\mathrm{P} \vdash \forall a \Big(a \in V^{\mathsf{N}}_{m} \rightarrow \exists y (y \in V^{\mathsf{N}}_{m+2^n} \wedge \psi^{(V^{\mathsf{N}}_{m+2^n})}(a,y)) \Big).$$

From this last line, using our assumption we then obtain

$$\mathsf{KPu}^{\mathsf{r}}{+}\mathrm{P} \vdash \exists y (y \in V^{\mathsf{N}}_{m+2^n} \wedge \psi^{(V^{\mathsf{N}}_{m+2^n})}(a,y)).$$

Since a was an arbitrary element of V^{N}_{ω}, this means that we have shown within the theory $\mathsf{KPu}^{\mathsf{r}}{+}\mathrm{P}$ that

$$\langle V^{\mathsf{N}}_{\omega}, \in^{[V^{\mathsf{N}}_{\omega}]} \rangle \models \varphi.$$

Concerning minimality, it is enough to note that the derivable set-theoretic Π_2 sentence $\forall x \exists y (x \in y)$ is such that for no $n < \omega$ we have

$$\langle V^{\mathsf{N}}_{n}, \in^{[V^{\mathsf{N}}_{n}]} \rangle \models \forall x \exists y (x \in y).$$

That $\omega = \|\mathsf{KPu}^{\mathsf{r}}{+}\mathrm{P}\|_{\Pi_2}$ follows from $\omega = \|\mathsf{sKPu}^{\mathsf{r}}_{2}\|_{\Pi_2}$ and the conservation result previously established. ☐

The next step we are going to undertake consists in replacing Δ^{c}_{1}-CA by a class existence axiom for any predicative formula. The argument used to justify this further strengthening of our axiom system, is contained in the following subsection.

2.5 ON THE DERIVABILITY OF Δ_1^{C}-CA

We make use of an itermediate theory which we denote by $\overline{\mathsf{sKPu}_2^{\mathsf{r}}}$. To the aim of presenting the theory $\overline{\mathsf{sKPu}_2^{\mathsf{r}}}$ we need to introduce the following axiom.

Let Δ_0^{C}-SEP denote the following second-order axiom schema:

$$\exists x(\mathsf{S}(x) \wedge \forall z(z \in x \leftrightarrow z \in a \wedge \varphi(z)).$$

for any Δ_0^{C} formula φ of $\mathcal{L}_2^*$.

The intermediate theory $\overline{\mathsf{sKPu}_2^{\mathsf{r}}}$ is obtained from $\mathsf{sKPu}_2^{\mathsf{r}}$ by dropping Δ_1^{C}-CA, adding the axiom $\forall x \exists Y(x = Y)$ and replacing AUS by Δ_0^{C}-SEP.

THEOREM 2.5.1. *The following is derivable in* $\overline{\mathsf{sKPu}_2^{\mathsf{r}}}$*:*

$$\forall x(\varphi(x) \leftrightarrow \neg\psi(x)) \rightarrow \exists Y \forall x(x \in Y \leftrightarrow \varphi(x)),$$

where φ *and* ψ *are* Σ_1^{C} *and do not contain the class variable* Y *free but may contain set and class parameters besides* x*.*

Proof. The argument is accomplished by the the method of Specker presented by Bernays in [4]. We shall argue informally within $\overline{\mathsf{sKPu}_2^{\mathsf{r}}}$. Assume

$$\forall x(\varphi(x) \leftrightarrow \neg\psi(x)),$$

and apply S-Π_1^1 RFN to the following S-Π_1^1 formula

$$\forall Y \exists x\Big((x \in Y \wedge \psi(x)) \vee (\varphi(x) \wedge x \notin Y)\Big),$$

which we denote by φ_0 and which we assume, without loss of generality, does not contain the variable w free. Therefore,

$$\varphi_0 \rightarrow \exists w[\mathsf{Tran}(w) \wedge \varphi_0^{(w)}].$$

By making explicit the relativization of φ_0 to the reflecting transitive set w and using the fact that $\forall x \exists Y(x = Y)$ along with the full substitutivity of equality (Proposition 1.2.5) we then obtain,

$$\begin{aligned}\varphi_0 \rightarrow \exists w\Bigg[\mathsf{Tran}(w) \wedge \forall y\Bigg(y \subseteq w \rightarrow \exists x\Bigg(& x \in w \wedge ((x \in y \wedge \psi^{(w)}(x)) \vee \\ & \vee (\varphi^{(w)}(x) \wedge x \notin y))\Bigg)\Bigg)\Bigg],\end{aligned}$$

which is logically equivalent to

$$\begin{aligned}\varphi_0 \rightarrow \exists w\Bigg[\mathsf{Tran}(w) \wedge \forall y\Bigg(y \subseteq w \rightarrow \exists x\Bigg(& (x \in w \wedge x \in y \wedge \psi^{(w)}(x)) \vee \\ & \vee (x \in w \wedge \varphi^{(w)}(x) \wedge x \notin y)\Bigg)\Bigg)\Bigg].\end{aligned}$$

In particular we can drop "$\mathsf{Tran}(w)$" and upon the premise "$y \subseteq w$" we can suppress "$x \in w$" within the first member of our disjunction. Hence,

$$\varphi_0 \to \exists w \forall y \Bigg(y \subseteq w \to \exists x \Big((x \in y \wedge \psi^{(w)}(x)) \vee (x \in w \wedge \varphi^{(w)}(x) \wedge x \notin y) \Big) \Bigg).$$

Denote this last implication by $\varphi_0 \to \psi_0$. Here $\psi^{(w)}(x)$ and $\varphi^{(w)}(x)$ are Δ_0^{C} formulae of $\mathcal{L}_2^*$ of the form $\psi_1(x, w)$ with no bound-class variables. By Δ_0^{C}-SEP we have

$$\exists y \forall x (x \in y \leftrightarrow x \in a \wedge \psi_1(x, a)).$$

This last formula is obviously equivalent to

$$\exists y \forall x \Big((x \in y \to x \in a \wedge \psi_1(x, a)) \wedge (x \in a \wedge \psi_1(x, a) \to x \in y) \Big),$$

and from this we infer in particular

$$\exists y \Bigg(y \subseteq a \wedge \forall x \Big((x \in y \to \psi_1(x, a)) \wedge (x \in a \wedge \psi_1(x, a) \to x \in y) \Big) \Bigg).$$

and trivially

$$\exists y \Bigg(y \subseteq a \wedge \forall x \Big((x \notin y \vee \psi_1(x, a)) \wedge (x \notin a \vee \neg\psi_1(x, a) \vee x \in y) \Big) \Bigg).$$

By generalizing with respect to a we then infer

$$\forall w \exists y \Bigg(y \subseteq w \wedge \forall x \Big((x \notin y \vee \psi_1(x, w)) \wedge (x \notin w \vee \neg\psi_1(x, w) \vee x \in y) \Big) \Bigg).$$

Instanciating "$\psi_1(x, w)$" by "$\neg\psi^{(w)}(x)$" we then get

$$\forall w \exists y \Bigg(y \subseteq w \wedge \forall x \Big((x \notin y \vee \neg\psi^{(w)}(x)) \wedge (x \notin w \vee \psi^{(w)}(x) \vee x \in y) \Big) \Bigg). \quad (1)$$

At this stage note that

$$\begin{aligned} \varphi(x) &\equiv \exists u \varphi_2(u, x) \\ \psi(x) &\equiv \exists u \psi_2(u, x), \end{aligned}$$

where φ_2 and ψ_2 are Δ_0^{C} formulae of $\mathcal{L}_2^*$. Note that the assumption

$$\forall x (\exists u \varphi_2(u, x) \leftrightarrow \forall u \neg\psi_2(u, x)),$$

logically entails the following

$$(\forall x \in w)(\exists u (u \in w \wedge \varphi_2(u, x)) \to \forall u (u \in w \to \neg\psi_2(u, x))).$$

By definition of relativization, this last expression obviously entails the following

$$(\forall x \in w)(\varphi^{(w)}(x) \rightarrow \neg\psi^{(w)}(x)). \tag{2}$$

And (1), along with (2), yields the following:

$$\forall w \exists y \Bigg(y \subseteq w \wedge \forall x \Big((x \notin y \vee \neg\psi^{(w)}(x)) \wedge (x \notin w \vee \neg\varphi^{(w)}(x) \vee x \in y) \Big) \Bigg).$$

But this is the negation of ψ_0. Therefore we obtain by MODUS TOLLENDO TOLLENS $\neg\varphi_0$, i.e.

$$\exists Y \forall x ((x \in Y \rightarrow \neg\psi(x)) \wedge (\varphi(x) \rightarrow x \in Y)).$$

And this, along with the assumption

$$\forall x (\varphi(x) \leftrightarrow \neg\psi(x)),$$

logically entails the following

$$\exists Y \forall x ((x \in Y \rightarrow \varphi(x)) \wedge (\varphi(x) \rightarrow x \in Y)).$$

That is

$$\exists Y \forall x (x \in Y \leftrightarrow \varphi(x)).$$

□

For more results on the derivability of Comprehension axiom shemata from second-order reflection principles the reader is reffered to Gloede [9].

COROLLARY 2.5.2. *For any formula φ of $\mathcal{L}^*_2$, we have*

$$\overline{\mathsf{sKPu}^{\mathsf{r}}_2} \vdash \varphi \quad \Longleftrightarrow \quad \mathsf{sKPu}^{\mathsf{r}}_2 \vdash \varphi.$$

Proof. From right to left. By proposition 1.2.3, we have derivable in $\mathsf{sKPu}^{\mathsf{r}}_2$ that every set is a class. The fact that any instance of Δ^{c}_0-SEP is derivable in $\mathsf{sKPu}^{\mathsf{r}}_2$ follows from AUS and Δ^{c}_1-CA.
From left to right. This is immediate by Theorem 2.5.1 and the fact that AUS is just a particular instance of Δ^{c}_0-SEP. □

Accordingly, we can regard $\overline{\mathsf{sKPu}^{\mathsf{r}}_2}$ as the same theory as $\mathsf{sKPu}^{\mathsf{r}}_2$.

COROLLARY 2.5.3.

$$\begin{aligned} \omega &= \|\mathsf{sKPu}^{\mathsf{r}}_2\|_{\Pi_2} \\ &= \|\mathsf{KPu}^{\mathsf{r}}{+}\mathrm{P}\|_{\Pi_2} \\ &= \|\overline{\mathsf{sKPu}^{\mathsf{r}}_2}\|_{\Pi_2} \end{aligned}$$

Chapter 3

Predicative Comprehension: From Power Admissible To Classical Set Theory

Given the strengthening of the axiom system $\mathsf{sKPu}^{\mathsf{r}}_2\!\restriction$ to $\mathsf{sKPu}^{\mathsf{r}}_2$, the result of Section 3.4 shows that it would be inadequate to keep the Comprehension schema restricted to $\mathcal{L}^*_2$ formulae of logical complexity Δ^{c}_1. Accordingly, the class existence axiom is extended in as much as we shall allow any predicative formula to occur in it. The extended class existence axiom is called Predicative Comprehension and denoted by PCA.

Definition 3.0.4. The Predicative Comprehension schema is formulated as follows:

$$\exists Y \forall x (x \in Y \leftrightarrow \varphi(x)) \qquad (\mathrm{PCA}),$$

where φ is any predicative formula of $\mathcal{L}^*_2$ not containing the class variable Y free but which may contain free set and class parameters besides x.

The question is now whether we are adding something which is genuinely new or whether, as for Δ^{c}_1-CA, it is already derivable in the theory $\overline{\mathsf{sKPu}^{\mathsf{r}}_2}$. Let Σ_1-$\mathsf{I}_{\mathbb{N}}$ be

$$\varphi(0) \wedge \forall x, y \in \mathsf{N}\,(\varphi(x) \wedge \mathsf{Sc}(x,y) \rightarrow \varphi(y)) \rightarrow \forall x \in \mathsf{N}\,\varphi(x),$$

for every $\mathcal{L}^*$ formula of logical complexity Σ_1. Further, Σ^{c}_1-$\mathsf{I}_{\mathbb{N}}$ is used to denote the above-mentioned schema but for any Σ^{c}_1 formula of $\mathcal{L}^*_2$. Let

$$\mathsf{KPu}^{\mathsf{r}} + \mathrm{P} + (\Sigma_1\text{-}\mathsf{I}_{\mathbb{N}})$$

be the theory obtained from $\mathsf{KPu}^{\mathsf{r}} + \mathrm{P}$ through the replacement of Δ_0-$\mathsf{I}_{\mathbb{N}}$ by Σ_1-$\mathsf{I}_{\mathbb{N}}$. Let us introduce the following abbreviations:

$$\begin{aligned} \mathsf{Lim}(a) &:= \mathsf{On}(a) \wedge a \neq 0 \wedge (\forall x \in a)(\exists z \in a)(z = x \cup \{x\}) \\ \exists!\, x\varphi(x) &:= \exists x(\varphi(x) \wedge \forall y(\varphi(y) \rightarrow x = y)). \end{aligned}$$

THEOREM 3.0.5.

$$\mathsf{KPu}^{\mathsf{r}} + \mathrm{P} + (\Sigma_1\text{-}\mathsf{I}_{\mathbb{N}}) \vdash \exists!\,\xi\Big(\mathsf{Lim}(\xi) \wedge \forall\eta(\eta < \xi \rightarrow \neg\mathsf{Lim}(\eta))\Big).$$

Proof. For the proof the reader is referred to Theorem 3.2 of Jäger [15] on page 69. □

DEFINITION 3.0.6. The Comprehension schema restricted to the formulae of $\mathcal{L}_2^*$ of logical complexity Σ_1^{c}, is formulated as follows:

$$\exists Y \forall x (x \in Y \leftrightarrow \varphi(x)) \qquad (\Sigma_1^{\mathrm{c}}\text{-CA}),$$

where φ is any Σ_1^{c} formula of $\mathcal{L}_2^*$ not containing the class variable Y free but which may contain free set and class parameters besides x.

THEOREM 3.0.7. *Not every instance of* Σ_1^{c}-CA *is derivable in* $\overline{\mathsf{sKPu}_2^{\mathsf{r}}}$ $(\mathsf{sKPu}_2^{\mathsf{r}})$.

Proof. Suppose not. Then in particular we would have any instance of $\Sigma_1^{\mathrm{c}}\text{-}\mathsf{I}_{\mathbb{N}}$ derivable in the theory $\overline{\mathsf{sKPu}_2^{\mathsf{r}}}$ $(\mathsf{sKPu}_2^{\mathsf{r}})$. But then every derivable statement of $\mathsf{KPu}^{\mathsf{r}} + \mathrm{P} + (\Sigma_1\text{-}\mathsf{I}_{\mathbb{N}})$ would also be a theorem of $\overline{\mathsf{sKPu}_2^{\mathsf{r}}}$ $(\mathsf{sKPu}_2^{\mathsf{r}})$. Once we have this then, by Theorem 3.0.5, the existence of ω become derivable in $\overline{\mathsf{sKPu}_2^{\mathsf{r}}}$ $(\mathsf{sKPu}_2^{\mathsf{r}})$. And this contradicts the result stated in the Corollary 2.5.3. □

It is at this point that the reader might be tempted to make a simplifying mistake, thinking that once we have PCA at our disposal and given the presence of class-parameters in the reflected S-Π_1^1 formulae, then the schema of S-Π_1^1 RFN does *immediately* imply Π_1^1 RFN. In order to clarify this and convince the reader that things are not that easy we need to introduce some notation.

If in a formula $\varphi(C)$ the class parameter C is to be replaced by a formula ψ, we write $\varphi([C/\boldsymbol{\lambda}x.\psi])$ for the formula obtained from φ by replacing every occurrence $t \in C$ by $\psi[x/t]$. Neither set nor class parameters of $\forall x\psi$ are allowed to become bound when substituting. It is worth remarking that ψ may contain other free variables besides x and "$\boldsymbol{\lambda}x$" is needed to indicate which terms are substituted for which variables.

We write $\Big(\varphi([B/\boldsymbol{\lambda}x.\psi])\Big)^{(b)}$ for the formula obtained from $\varphi^{(b)}$ by replacing every occurrence

$$t \in B \qquad \text{by} \qquad \psi^{(b)}[x/t].$$

In other words, $\Big(\varphi([B/\boldsymbol{\lambda}x.\psi])\Big)^{(b)}$ is used to denote the formula obtained from φ after performing the operation of first substituting and then relativizing. On the other side, $\varphi^{(b)}(B)[B/\boldsymbol{\lambda}x.\psi]$ is used to denote the formula obtained from φ after performing the operation of first relativizing and then substituting. It is worth mentioning that in general, even upon the premises "$\mathsf{Tran}(b)$" and "$a,\, x \in b$", the formula

$$\Big(\varphi(a, [B/\boldsymbol{\lambda}x.\psi])\Big)^{(b)}$$

is different from

$$\varphi^{(b)}(a, B)[B/\boldsymbol{\lambda} x.\psi].$$

Take, for example, $\varphi(a, B) \equiv a \in B$. Then

$$\Big(\varphi(a, [B/\boldsymbol{\lambda} x.\psi])\Big)^{(b)} \equiv \psi^{(b)}[x/a],$$

and

$$\varphi^{(b)}(a, B)[B/\boldsymbol{\lambda} x.\psi] \equiv \psi[x/a].$$

If we take the class variable B intersected with the reflecting transitive set b, then we would run in the same problem as before since in general

$$\begin{gathered} \mathbf{B} \cap b = \{\, x \in b \,|\, \varphi(x) \,\} \\ \neq \\ \mathbf{B}^{(b)} \cap b = \{\, x \in b \,|\, \varphi^{(b)}(x) \,\}. \end{gathered}$$

We will show however that once we have PCA at our disposal, S-Π^1_1 RFN and Π^1_1 RFN are, in a sense which will be made precise later on, "intimately connected". Further, as we have already occasion to see in the proof of Theorem 3.0.7, the theory $\mathsf{sKPu}^{\mathsf{r}}_2$ augumented by PCA proves any instance of Σ^{C}_1-$\mathsf{I}_{\mathbb{N}}$ and therefore the existence of ω. Accordingly we reformulate this theory, denoted in the following by sBL_1, in a slight different way without assuming the natural numbers as urelements and using a different language which we shall denote by $\mathcal{L}_2$.

3.1 THE THEORIES VNB AND sBL$_1$

Let $\mathcal{L}_\in$ denote the language of first order predicate calculus augumented by the binary predicate symbol $\in$. As in Section 2.2, the second-order language $\mathcal{L}_2$ is now obtained from $\mathcal{L}_\in$ by adjunction of an infinite stock of class variables $X, Y, Z, ...$, together with universal quantifiers binding them. All the notions introduced in Sections 1.1 and 1.2 (formulae, classifications of formulae, definitions of equality,...) are adapted to the current context in the obvious way. As for the previous part of our work, we freely make use of all standard set-theoretic notations and write

$$\begin{aligned} \langle\, a, b \,\rangle &:= \{\{a\}, \{a, b\}\}, \\ \mathsf{rel}(R) &:= \forall x(x \in R \rightarrow \exists y \exists z(x = \langle\, y, z \,\rangle)), \\ \mathsf{fun}(F) &:= \mathsf{rel}(F) \wedge \forall x \forall y \forall z(\langle\, x, y \,\rangle \in F \wedge \langle\, x, z \,\rangle \in F \rightarrow y = z), \\ \mathsf{dom}(F) &:= \{x : \exists y(\langle\, x, y \,\rangle \in F)\}, \\ \mathsf{rng}(F) &:= \{y : \exists x(\langle\, x, y \,\rangle \in F)\}. \end{aligned}$$

The theory VNB is formulated in the second-order language $\mathcal{L}_2$. The underlying logic of VNB is the classical second-order logic with first-order equality. The non-logical axioms of VNB are the following:

PAIR: $\forall a \forall b \exists y \forall x[x \in y \leftrightarrow (x = a \lor x = b)]$,

UNION: $\forall a \exists y \forall x[x \in y \leftrightarrow \exists z(x \in z \land z \in a)]$,

POWER SET: $\forall a \exists y \forall x(x \in y \leftrightarrow x \subseteq a)$,

AUS: $\forall C \forall a \exists y \forall x(x \in y \leftrightarrow x \in a \land x \in C)$,

INFINITY: $\exists z[\emptyset \in z \land \forall x(x \in z \to x \cup \{x\} \in z)]$,

REPLACEMENT: $\forall C[\mathsf{fun}(C) \land \exists x(x = \mathsf{dom}(C)) \to \exists x(x = \mathsf{rng}(C))]$,

$\mathsf{I}^2_\in$: $\forall C(\exists y(y \in C) \to \exists y(y \in C \land \forall x(x \in y \to x \notin C)))$,

PCA: $\exists C \forall x(x \in Y \leftrightarrow \varphi(x))$,

for any predicative formula φ, not
containing the class variable C free
but which may contain free set
and class parameters besides x.

The theory sBL_1 is formulated in the second-order language $\mathcal{L}_2$ of VNB. The underlying logic of VNB is the classical second-order logic plus the substitutivity axiom for set equality. The non-logical axioms of sBL_1 are the following:

$$\Delta_0\text{-}\mathsf{I}_\in,\ \text{AUS},\ \text{S-}\Pi^1_1\ \text{RFN},\ \text{INFINITY},\ \text{PCA}.$$

3.2 VNB SUBSYSTEM OF sBL$_1$

We show that every theorem of VNB is also a theorem of sBL_1. This is easily seen once we know that the second-order axiom of REPLACEMENT of VNB is derivable in sBL_1 since the axioms AUS, INFINITY, PCA of VNB are also axioms of sBL_1 and, as we have already seen, the axioms $\mathsf{I}^2_\in$, PAIR, UNION, POWER SET are all derivable in sBL_1. Before dealing with the derivability in sBL_1 of the second-order axiom of REPLACEMENT, let us first summarize the above-mentioned considerations in the following propositions.

PROPOSITION 3.2.1. $\mathsf{I}^2_\in$ *is derivable in* sBL_1.

Proof. By Proposition 2.1.5. □

PROPOSITION 3.2.2. PAIR *is derivable in* sBL_1.

Proof. By Proposition 1.3.2 and Proposition 1.1.4.(a). □

PROPOSITION 3.2.3. UNION *is derivable in* sBL_1.

Proof. By Proposition 1.3.3 and Proposition 1.1.3. ☐

PROPOSITION 3.2.4. POWER SET *is derivable in* sBL_1.

Proof. By Lemma 2.2.4. ☐

DEFINITION 3.2.5. The second-order axiom of COLLECTION reads as follows:

$$\begin{aligned}&\forall x(x \in a \to \exists z(\langle x, z \rangle \in B)) \to \\ &\exists y \forall x(x \in a \to \exists z(z \in y \wedge \langle x, z \rangle \in B)).\end{aligned}$$

LEMMA 3.2.6. *The axioms of* REPLACEMENT *and* COLLECTION *are shown to be provably equivalent in* VNB.

Actually for the result we are aiming to show it is enough to know that COLLECTION implies REPLACEMENT; a detailed proof of such an implication can be found in Bernays [4] pp. 133-134, where COLLECTION is called the second-order version of *Thiele's Replacement axiom*. For a proof of the other direction the reader is referred to Gloede in [9] p. 293.

PROPOSITION 3.2.7. COLLECTION *is a theorem of* sBL_1 *that is*

$$\begin{aligned}\mathsf{sBL}_1 \vdash &\forall x(x \in a \to \exists z(\langle x, z \rangle \in B)) \to \\ &\to \exists y \forall x(x \in a \to \exists z(z \in y \wedge \langle x, z \rangle \in B)).\end{aligned}$$

Proof. Apply S-Π^1_1 RFN to the Σ^{C} formula

$$\forall x(x \in a \to \exists z(\langle x, z \rangle \in B)).$$

☐

REMARK 3.2.8. In the proof of Proposition 3.2.7, we rely on the fact that ordered pairing is a provably Δ_0 function. Hence in the process of relativization we make use of the absoluteness property of Δ_0 notions for transitive sets. In other words, we treat it as it were an atomic symbol of the base language $\mathcal{L}_2$. This observation will be often tacitly invoked in the remaining part of our work.

PROPOSITION 3.2.9. REPLACEMENT *is derivable in* sBL_1.

Proof. By Lemma 3.2.6 and Proposition 3.2.7. ☐

COROLLARY 3.2.10. *Every theorem* φ *of* VNB *is also a theorem of* sBL_1,

$$\mathsf{VNB} \vdash \varphi \quad \Longrightarrow \quad \mathsf{sBL}_1 \vdash \varphi.$$

3.3 VNB PROPER SUBSYSTEM OF sBL$_1$

So far we have seen that any theorem of VNB is also a theorem of sBL$_1$. The next question is whether we can prove in VNB, everything that can be proved in sBL$_1$. Since Δ_0-$\mathsf{I}_\in$ and the schema of S-Π^1_1 RFN are not among the axioms of VNB and Δ_0-$\mathsf{I}_\in$ is derivable in VNB (cf. Proposition 1.3.4) this reduces to asking whether each instance of S-Π^1_1 RFN is derivable in VNB. The answer to this question is no: The schema of S-Π^1_1 RFN is, in fact, independent from the axiom system of VNB.

To the aim of proving the above-mentioned result and all of the results contained in Section 3.6 we need to introduce the notions of "*Indescribability*" and "*Tree*". Before starting, we should emphasize that, with the exception of the so-called "*Strong Upward Persistency Property*" of $[\text{S-}\Pi^1_1]^{\mathrm{E}}$ formulae, the material we present in this part of our work is known in the literature (the reader is referred, for example, to Kanamori [16], Kunen [17], Lévy [20] and Barwise [2]), so we do not have any claims to originality except possibly regarding the presentation of the material itself and the way in which standard results are used and adapted to achieve the current task. We take up the notion of "*Indescribability*" first, and this in turn requires the presentation of some preliminary material. For the following, we fix ZFC as our metatheory. But there is an important caveat: By the Gödel-Tarski undefinability of truth argument the general satisfaction relation for proper-class structures is formally indefinable in ZFC. This is the source of possible unformalizability in our work, and the issue is discussed as it arises (see, for example, Appendix B)

DEFINITION 3.3.1. By a *full structure for* $\mathcal{L}_2$ we mean a ordered 4-tuple

$$\langle\, A,\, E^{[A]},\, \wp(A),\, \in^{[A\times\wp(A)]}\,\rangle$$

with

- $A \neq \emptyset$ being either a set or a class (possibly proper class) and serving as the range of the set variables (we call A the *domain* of the structure);
- $\wp(A)$ serving as the range of class variables;
- $E^{[A]}$ interpreting the membership relation $\in$ between sets and sets;
- $\in^{[A\times\wp(A)]}$ interpreting the relation $\in$ between sets and classes.

REMARK 3.3.2. Hence by "full" we mean the intended interpretation of second-order variables as ranging over arbitrary subcollections of the domain of the structure. Formulae of $\mathcal{L}_2$ are interpreted in $\langle\, A,\, E^{[A]},\, \wp(A),\, \in^{[A\times\wp(A)]}\,\rangle$ in the obvious way.

Some abbreviation is introduced. Let $\varphi(v_0, ..., v_n, C_0, ..., C_m)$ be any formula of $\mathcal{L}_2$ with free variables as indicated. We write

$$\langle\, A,\, E^{[A]}\,\rangle \models^2 \varphi[a_0, ..., a_n, B_0, ..., B_m]$$

to indicate that the formula φ of $\mathcal{L}_2$ is satisfied in the structure

$$\langle\, A,\, E^{[A]},\, \wp(A),\, \in^{[A\times\wp(A)]}\,\rangle$$

with the variable assignment taking v_i to $a_i \in A$ and C_i to $B_i \in \wp(A)$.

DEFINITION 3.3.3. When A is an $\in$-transitive class, we call the corresponding full structure for $\mathcal{L}_2$ of the form

$$\langle\, A,\, \in^{[A]},\, \wp(A),\, \in^{[A\times\wp(A)]}\,\rangle,$$

the *intended* or *standard model for* $\mathcal{L}_2$.

REMARK 3.3.4. It is also worth noticing that, when dealing with interpretation of formulae of $\mathcal{L}_2$ in the standard model for $\mathcal{L}_2$, then any free set-variable might also be regarded as a free-class variable. Further, when A is an $\in$-transitive set of the form V_α for some ordinal α then, we have the corresponding well-known structute of the form

$$\langle\, V_\alpha,\, \in^{[V_\alpha]},\, \wp(V_\alpha),\, \in^{[V_\alpha\times\wp(V_\alpha)]}\,\rangle.$$

The structure above is a very particular example of the standard models for $\mathcal{L}_2$.

To reiterate, with full models for $\mathcal{L}_2$, by fixing a domain A we thereby fix the range of both the set and class variables. There is no further "interpreting" to be done. This is not the case with the next models we are going to introduce. As we will see, we must separately determine a range for the set variables and a range for the class variables.

DEFINITION 3.3.5. By an *Henkin structure for* $\mathcal{L}_2$ we mean a ordered 4-tuple

$$\langle\, A,\, E^{[A]},\, \mathcal{S}_A,\, \in^{[A\times\mathcal{S}_A]}\,\rangle$$

where the items A, $E^{[A]}$ and $\in^{[A\times\mathcal{S}_A]}$ are explained for the Henkin structures as for the full structures for $\mathcal{L}_2$ but where

$$\emptyset \neq \mathcal{S}_A \subseteq \wp(A).$$

REMARK 3.3.6. Hence the central facet of any given Henkin structure for $\mathcal{L}_2$ is that the class variables range over a fixed collection of subcollections of the domain A which may not include all the subcollections of A. To reiterate, an Henkin structure for $\mathcal{L}_2$ differs from the full structure for $\mathcal{L}_2$ by having a possibly smaller collection $\mathcal{S}_A$ of subcollections of elements from A to serve as the range of the class variables.

For any formula $\varphi(v_0,...,v_n,C_0,...,C_m)$ of $\mathcal{L}_2$ with free variables as indicated, we write

$$\langle\, A,\, E^{[A]},\, \mathcal{S}_A,\, \in^{[A\times\mathcal{S}_A]}\,\rangle \models \varphi[a_0,...,a_n,B_0,...,B_m]$$

to indicate that the formula φ of $\mathcal{L}_2$ is satisfied in the structure

$$\langle\, A,\ E^{[A]},\ \mathcal{S}_A,\ \in^{[A\times\mathcal{S}_A]}\,\rangle$$

with the variable assignment taking v_i to $a_i \in A$ and C_i to $B_i \in \mathcal{S}_A$.

INDESCRIBABILITY. *For Ξ being either* S-Π^1_1 *or* Π^1_1.

(-) *An ordinal α is Ξ-indescribable if and only if $\alpha > 0$ and for each formula $\varphi(v_0, ..., v_n, C_0, ..., C_m)$ in Ξ, in which z does not occur free and with no free variable besides the displayed ones free and not necessarily all of them, for any set $a_0, ..., a_n \in V_\alpha$ and any $B_0, ..., B_m \subseteq V_\alpha$,*

$$\langle\, V_\alpha,\ \in^{[V_\alpha]}\,\rangle \models^2 \varphi[a_0, ..., a_n, B_0, ..., B_m] \to$$
$$\to \exists z[\mathsf{Tran}(z) \wedge a_0, ..., a_n \in z \wedge \varphi^{(z)}[a_0, ..., a_n, B_0, ..., B_m]].$$

(-) *α is Ξ-describable if and only if α is not Ξ-indescribable.*

(-) *A structure $\langle\, A,\ E^{[A]},\ \wp(A),\ \in^{[A\times\wp(A)]}\,\rangle$ is Ξ-indescribable if and only if A is non-void and for each formula $\varphi(v_0, ..., v_n, C_0, ..., C_m)$ in Ξ, in which z does not occur free and with no free variable besides the displayed ones free and not necessarily all of them, for any set $a_0, ..., a_n \in A$ and any $B_0, ..., B_m \in \wp(A)$,*

$$\langle\, A,\ E^{[A]}\,\rangle \models^2 \varphi[a_0, ..., a_n, B_0, ..., B_m] \to$$
$$\to \exists z[\mathsf{Tran}(z) \wedge a_0, ..., a_n \in z \wedge \varphi^{(z)}[a_0, ..., a_n, B_0, ..., B_m]].$$

(-) *A structure $\langle\, A,\ E^{[A]},\ \wp(A),\ \in^{[A\times\wp(A)]}\,\rangle$ is Ξ-describable if and only if $\langle\, A,\ E^{[A]},\ \wp(A),\ \in^{[A\times\wp(A)]}\,\rangle$ is not Ξ-indescribable.*

(-) *A structure $\langle\, A,\ E^{[A]},\ \wp(A),\ \in^{[A\times\wp(A)]}\,\rangle$ satisfies the schema of Ξ* RFN *without class-parameters if and only if A is non-void and the full structure $\langle\, A,\ E^{[A]},\ \wp(A),\ \in^{[A\times\wp(A)]}\,\rangle$ satisfies each instance of Ξ* RFN *where class-parameters are not allowed to appear in the corresponding defining formula.*

REMARK 3.3.7. As in the remaining part of our work we shall be quoting Barwise [2], it is worth pointing out the following differences between the current approach and his approach:

- Barwise introduces the notion of "Ξ-indescribability" by using instead of Ξ formulae $\varphi(v_0, ..., v_n, C_0, ..., C_m)$ of $\mathcal{L}_2$ containing free class-variables "$C_0, ..., C_m$", the corresponding formula $\varphi(v_0, ..., v_n, \mathsf{R}_0, ..., \mathsf{R}_m)$ containing unary predicate constants "$\mathsf{R}_0, ..., \mathsf{R}_m$" and considering, instead of the full structure $\langle\, A,\ E^{[A]},\ \wp(A),\ \in^{[A\times\wp(A)]}\,\rangle$ for $\mathcal{L}_2$, the extended first-order structure $\langle\, A,\ E^{[A]},\ R_0, ..., R_m\,\rangle$ with arbitrary $R_i \subseteq A$ $(0 \leq i \leq m)$ interpreting the unary predicate constant R_i. Moreover, the extended first-order structures considered by Barwise are always admissible sets of the form $\langle\, A,\ \in^{[A]},\ R\,\rangle$ where (as A is closed under PAIR) $R_0, ..., R_m$ are coded up into a single $R \subseteq A$.

- With respect to the above-mentioned structures, Barwise defines an admissible set A to be Ξ-indescribable if and only if $\langle\, A, \in^{[A]}, R\,\rangle$ satisfies each instance of the schema of Ξ RFN for any $R \subseteq A$.
- Further, Barwise introduces the notion of "α-indescribability" with respect to the H_α's and not for the V_α's as in our case. However, this is of no harm, as in the following we will only be concerned with "α-indescribability" for $\alpha = \omega$ or for α being a strongly inaccessible cardinal (see Definition 3.3.9). In these cases $H_\alpha = V_\alpha$ (for a proof we referr to Kunen [17] Lemma 6.3 p.131).
- We also warn the reader that "satisfying the schema of Ξ RFN without class-parameters" corresponds (up to the above-mentioned differences) to the Barwise expression "satisfying Ξ RFN".

EXAMPLE 3.3.8. An ordinal α is S-Π^1_1-indescribable if and only if for any S-Π^1_1 formula $\varphi(v_0, ..., v_n, C_0, ..., C_m)$ and any set $a_0, ..., a_n \in V_\alpha$ and any $B_0, ..., B_m \subseteq V_\alpha$, if

$$\langle\, V_\alpha, \in^{[V_\alpha]}\,\rangle \models^2 \varphi[a_0, ..., a_n, B_0, ..., B_m],$$

then there is a transitive set $a \in V_\alpha$ such that $a_0, ..., a_n \in a$ and by Proposition 1.2.9,

$$\langle\, V_\alpha, \in^{[V_\alpha]}\,\rangle \models^2 \varphi^{(a)}[a_0, ..., a_n, B_0 \cap a, ..., B_m \cap a].$$

That is

$$\varphi^{(a\cap V_\alpha)}(a_0, ..., a_n, B_0 \cap a, ..., B_m \cap a).$$

But since $a \in V_\alpha$ and V_α is transitive, this means that $a \cap V_\alpha = a$. Therefore

$$\varphi^{(a)}(a_0, ..., a_n, B_0 \cap a, ..., B_m \cap a).$$

Which is again equivalent to

$$\langle\, a, \in^{[a]}\,\rangle \models^2 \varphi[a_0, ..., a_n, B_0 \cap a, ..., B_m \cap a].$$

In connection with the presentation of the set-theoretical notion of "*Tree*", it is worth introducing also the following notions.

Given a function f and $a \subseteq \mathsf{dom}(f)$, we define the image of a under f to be

$$f[a] := \{\, f(x) \,|\, x \in a\,\}.$$

DEFINITION 3.3.9. Let $C \subseteq \mathbf{ON}$. We say that C is cofinal in $\mathbf{ON}$ or unbounded in $\mathbf{ON}$, denoted by $\mathsf{unbounded}(C)$, if and only if for any ordinal α, $C \nsubseteq \alpha$, e.g. C is a proper class of ordinals. Let β be a limit ordinal, and let $C \subseteq \beta$. We say that C is cofinal in β or unbounded in β, denoted by $\mathsf{unbounded}(C, \beta)$, if and only if $\forall\alpha(\alpha \in \beta \rightarrow \exists\gamma(\gamma \in C \wedge \alpha \leq \gamma))$. The cofinality of an ordinal β, denoted by $\mathsf{cf}(\beta)$, is the least ordinal α such that there is a function $f : \alpha \longrightarrow \beta$ with range cofinal in β. A limit ordinal β is regular, denoted by $\mathsf{reg}(\beta)$, if and only if $\mathsf{cf}(\beta) = \beta$ and singular, denoted by $\mathsf{sing}(\beta)$, otherwise. An ordinal β is strongly inaccessible, denoted by $\mathsf{inacc}(\beta)$, if and only if β is an uncountable regular ordinal and closed under cardinal exponentiation, e.g. $\forall\lambda(\lambda < \beta \rightarrow 2^\lambda < \beta)$.

Note that the definition of $\mathsf{unbounded}(C, \beta)$ is just the relativization of the definition of $\mathsf{unbounded}(C)$ to the set V_β, for $\mathsf{Lim}(\beta)$. For any ordinal β, $\mathsf{cf}(\beta) \leq \beta$. So, a limit ordinal β is singular if and only if $\mathsf{cf}(\beta) < \beta$. On a formal level, the definitions of unboundedness, regularity and singularity can be applied to any ordinal and not only to limit ordinals. We confined ourselves to the limit ordinals just because these notions turn out to be trivial in the case of successor ordinals. For example, let $A \subseteq \xi + 1$ for some ordinal ξ. Then, whenever $\xi \in A$, we obviously have $\mathsf{unbounded}(A, \xi + 1)$. Further, for any successor ordinal α, $\mathsf{cf}(\alpha) = 1$. To see this, let $\alpha = \gamma + 1$, for some ordinal γ. Then the map $f : 1 \longrightarrow \gamma + 1$, defined by $f(0) = \gamma$, is such that $f[1]$ is cofinal in $\gamma + 1$. Hence $\mathsf{reg}(1)$ and any other successor ordinal is singular. It is a triviality that $\mathsf{reg}(0)$. And $\mathsf{reg}(\omega)$ since for every $n \in \omega$ and every function on n into ω, $f[n]$ is a strictly bounded subset of ω.

Definition 3.3.10. An ordinal α is a cardinal if and only if for no $\beta < \alpha$ there is function $f : \beta \xrightarrow{\text{onto}} \alpha$.

Note that the regularity of an ordinal α directly implies the α is a cardinal, altough the converse does not hold. Hence in the following we will always speak of regular cardinal as also of strongly inaccessible cardinals. Further, any infinite successor cardinal (i.e. cardinal of the form $\aleph_{\alpha+1}$) is regular (the proof of this last assertion requires the Axiom of Choice (AC)). Towards Definition 3.3.9, we also remark that the requirement of closure under cardinal exponentiation, used in the definition of strong inaccessibility, requires AC. Without AC, we do not even know that $2^{\aleph_0}$ is an aleph. For an alternative definition of strong inaccessibility dispensing AC and equivalent to our definition in presence of AC, the reader is referred, for example, to Bernays [4], p. 157.

Next is the set-theoretical notion of "*Tree*".

Definition 3.3.11. A tree is a partially ordered set $\langle T, <_T \rangle$ such that for any $t \in T$ the set $\{s \in T \mid s <_T t\}$ is well-ordered by the relation $<_T$.

Sometimes we shall blur the distinction between a tree and its underlying node-set, referring to T when we mean $\langle T, <_T \rangle$.

Definition 3.3.12. Let T be a tree.

(-) The order-type (ot) of the set $\{s \in T \mid s <_T t\}$ under $<_T$ is called the height of t in T, denoted by $ht(t)$.

(-) For any ordinal α, the α-th level of of $\langle T, <_T \rangle$, denoted by $T_{(\alpha)}$, is

$$\begin{aligned} T_{(\alpha)} &= \{t \in T \mid ht(t) = \alpha\} \\ &= \{t \in T \mid \mathsf{ot}(\langle \{s \in T \mid s <_T t\}, <_T \rangle) = \alpha\}. \end{aligned}$$

(-) The height of T, denoted by $ht(T)$, is the least α such that $T_{(\alpha)} = \emptyset$.

(-) A chain C of T, denoted by $\mathsf{chain}(C)$, is a linearly ordered subset of T.

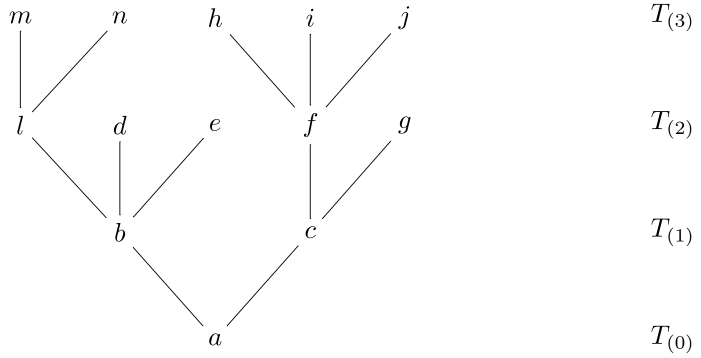
(-) A branch B of T is a maximal chain of T (i.e., a chain B such that for no $x \in T \setminus B$, is $B \cup \{x\}$ a chain). The length of a branch B of T is its order-type under $<_T$.

(-) A cofinal branch B of T is a branch with members at every non-empty level of T:
$$\forall\alpha(\alpha < ht(T) \rightarrow B \cap T_{(\alpha)} \neq \emptyset).$$

Remark 3.3.13. Associated to each chain C of T is its order-type under $<_T$. By definition, we know that C is a linearly ordered subset of T. Hence all we are left with is to show every non-empty subset C_0 of C has a $<_T$-minimal element. Let t be an element of C_0. If t is not $<_T$-minimal in C_0, then the set $\{s \in C_0 \mid s <_T t\}$ is a non-empty subset of the well-ordered set $\{s \in T \mid s <_T t\}$. Hence $\{s \in C_0 \mid s <_T t\}$ has a $<_T$-minimal element, say y. We claim that y is also a $<_T$-minimal element of C_0. Suppose not. Then there would be an $x \in C_0$ such that $x <_T y <_T t$. It follows that $x <_T t$ and $x \in \{s \in C_0 \mid s <_T t\}$, contradicting the $<_T$-minimality of y.

To get used to this terminology, let us consider a simple example. It is customary to represent a tree $\langle T, <_T \rangle$ pictorially using vertical (near vertical) connecting lines to denote the ordering $<_T$ in the upward direction and drawing the levels of T on horizontal lines.

Example 3.3.14. The tree T pictured below has 4 non zero levels. Hence T is a tree of height 4.



- $T_{(0)} = \{a\}$;
- $T_{(1)} = \{b, c\}$;
- $T_{(2)} = \{l, d, e, f, g\}$;
- $T_{(3)} = \{m, n, h, i, j\}$.

- The set $\{a, b, c\}$ is not a chain;
- The set $\{a, b, l\}$ is a chain but not a branch;
- The set $\{a, b, d\}$ is a branch but not a cofinal branch;
- The set $\{a, c, f, j\}$ is a cofinal branch.

PROPOSITION 3.3.15. *Let $\langle T, <_T \rangle$ be a tree of height ξ.*

(a) *For any node $t \in T$, T has a branch containing t;*

(b) *For any $\nu < \xi$, T has a branch of length bigger or equal to ν.*

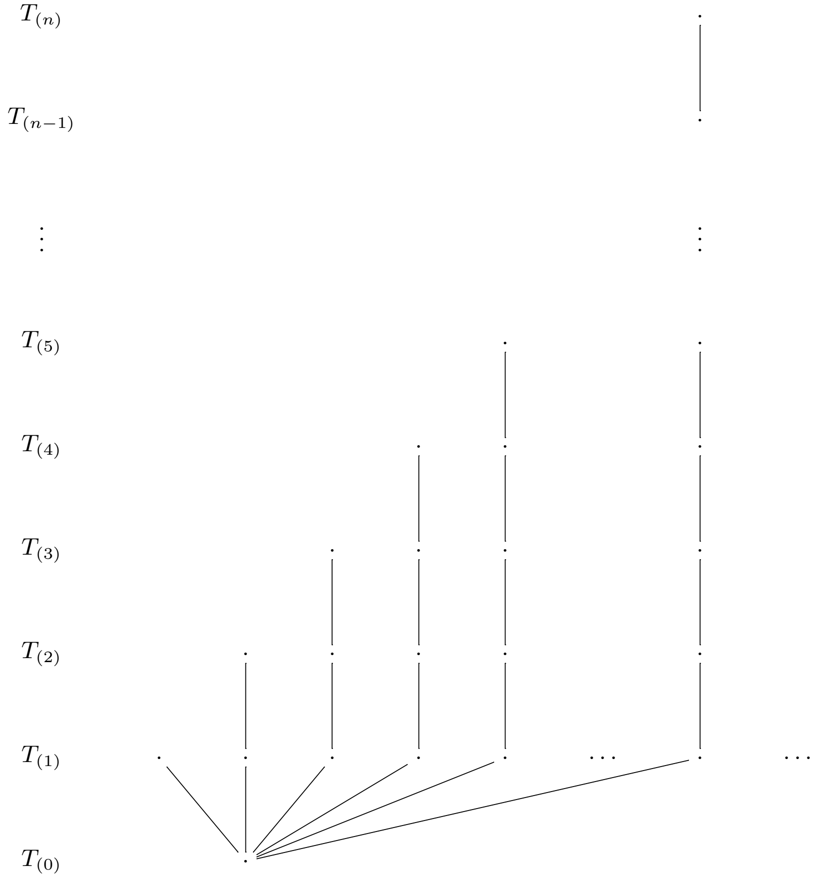
Proof. The reader is referred to Lévy [20], Proposition 2.6, p.294. The proof of both point (a) and (b) requires AC. Point (a) essentially tells us that, using AC, every chain can be extended to a maximal chain; point (b) gives us a lower bound on the length of the branches a tree can go along. □

To reiterate, according to Proposition 3.3.15.(b), any tree of height ξ has branches of length bigger or equal to ν, for any $\nu < \xi$. But since a tree is, in fact, a branching process we are interested in knowing not only the minimal length of all the branches the process can go along, but also the existence of branches of length ξ, e.g. cofinal branches.

When $\xi = \pi + 1$ for some ordinal π, then every branch through a node of $T_{(\pi)}$ is a cofinal branch. This is a direct consequence of Proposition 3.3.15.(a). However, at least for this particular simple case, AC can be dispensed with arguing as follows. Since $ht(T) = \pi + 1$, we have that $\forall \varrho(\varrho < \pi + 1 \rightarrow T_{(\varrho)} \neq \emptyset)$. Let $\varrho = \pi$ and $t \in T_{(\pi)}$. It is easy to check (more details concerning this point, however, can be found in the proof of Theorem 3.6.6.(3)) that the set $\{ s \in T \mid s <_T t \}$ is a chain such that for any $\sigma < \pi$ there is a unique node $s \in T_{(\sigma)}$ such that $s <_T t$. Since $T_{(\pi+1)} = \emptyset$, there is no $v \in T$ such that $t <_T v$, e.g. t has no successor node in T. Therefore the set $\{ s \in T \mid s <_T t \} \cup \{t\}$ is a chain intersecting every non void level of T, e.g. a cofinal branch.

On the other hand, if ξ is a limit ordinal, then it is not guaranteed that such a cofinal branch exists: see Figure 3.1 on the next page. With regard to this example, we might cogently argue that the reason for which T fails to have a cofinal branch relies on the fact that this tree is infinitely branching or, in a looser way, too wide. The above-mentioned tree is, in fact, such that $|T_{(1)}| = \omega$, for example. Therefore we could think to impose a narrowness condition on T by requiring that for any n, $|T_{(n)}| < \omega$. And indeed any finitely branching tree of height ω has a cofinal branch (König's Lemma). This narrowness condition, however, is not sufficient to guarantee in general the existence of cofinal branches. There exists, in fact, a tree of height ω_1 such that $\forall \alpha(\alpha < \omega_1 \rightarrow |T_{(\alpha)}| < \omega_1)$ but with no cofinal branch (see for example Kunen [17], Theorem 5.6, p.70). As already remarked, the question concerning the existence of cofinal branches for trees of height ξ where ξ is a successor ordinal, has an immediate answer.

Figure 3.1: A tree T of height ω where every branch is finite.



Accordingly, we might content ourselves to the case of limit ordinals. Further, as long as singular ordinals ξ are concerned, trees of height ξ such that $\forall\alpha(\alpha < \xi \to |T_{(\alpha)}| < |\xi|)$ and with no cofinal branch are known to exist (see, for example, Kanamori [16], p.78). Hence the subsequent definition will be stated only for regular cardinals.

Definition 3.3.16. For any regular κ, a κ-tree is a tree T of height κ such that

$$\forall\alpha(\alpha < \kappa \to |T_{(\alpha)}| < \kappa).$$

A κ-Aronszajn tree is a κ-tree with no cofinal branch. A regular cardinal κ has the tree-property if and only if there are no κ-Aronszajn trees.

In other words, a regular cardinal κ has the tree-property if and only if every

κ-tree has a cofinal branch. Therefore, ω has the tree-property and there exists an ω_1-Aronszajn tree. The tree-property under discussion trascends inaccessibility: the existence of a κ-Aronszajn tree is, in fact, known to be true for the first, second and many more strongly inaccessible cardinals. It is also known that the first strongly inaccessible cardinals κ for which this property fails is a lot bigger than the first strongly inaccessible cardinal.

Weakly Compact Cardinals. The weakly compact cardinals are those cardinals that are strongly inaccessible with the tree property.

Remark 3.3.17. As well-known the weakly comapct cardinals have many diverse model-theoretic characterizations. We have chosen the tree-property characterization of weak compactness as our base definition. For an equivalent alternative definition we referr to Barwise [2]. We also warn the reader that our definition of a weakly compact cardinal κ rules out the possiblity that $\kappa = \omega$. This is not the case with Bariwise: ω is the only countable example of a weakly compact cardinal! Towards a detailed analysis of the relative size of a weakly compact cardinal with respect to the strongly inaccessible cardinals, as well as Mahlo cardinals, the reader is referred, for example, to Lévy [20] pp. 303-304.

Before stating the next result, we remind the reader that class parameters are allowed in the definition of "s-Π^1_1-indescribability". The rôle played by the class-parameters in the notion of "s-Π^1_1-indescribability" will be brought out in Section 3.6.

Theorem 3.3.18. *An ordinal α is* s-Π^1_1*-indescribable if and only if either it is ω or is a weakly compact cardinal.*

A proof of Theorem 3.3.18, appealing to compactness properties of infinitary languages, can be found in Barwise [2], Theorem VIII.9.10, p.361. An alternative proof (exploiting the connection between s-Π^1_1 Rfn and the tree-property) of the necessary conditions needed to be satisfied by an ordinal α for being s-Π^1_1-indescribable, will be presented in Section 3.6. (see Theorem 3.6.6). Before stating the subsequent result we remind the reader that if μ is the first strongly inaccessible cardinal then $\langle V_\mu, \in^{[V_\mu]} \rangle \models^2 \mathsf{VNB}$; for a proof the reader is referred, for example, to Kanamori [16] p. 19.

Theorem 3.3.19. *The schema of* s-Π^1_1 Rfn *is independent from* VNB.

Proof. If μ is the first strongly inaccessible cardinal, then

$$\langle V_\mu, \in^{[V_\mu]} \rangle \models^2 \mathsf{VNB} \qquad \text{and} \qquad \mu \text{ is s-}\Pi^1_1\text{-describable.}$$

This means that there is some instance of the schema of s-Π^1_1 Rfn which is not derivable in VNB. On the other hand, if μ is the first weakly compact cardinal then

$$\langle V_\mu, \in^{[V_\mu]} \rangle \models^2 \mathsf{VNB} \qquad \text{and} \qquad \mu \text{ is s-}\Pi^1_1\text{-indescribable.}$$

And this, in turn, entails that there is also some instance of the schema of s-Π^1_1 Rfn whose negation is not derivable in VNB. □

To reiterate, there are instances of $\text{S-}\Pi^1_1$ RFN which cannot be proved in VNB. Accordingly we may regard our theory sBL_1 as being $\mathsf{VNB} + \text{S-}\Pi^1_1$ RFN . Summing up, Theorem 3.3.19 tells us that sBL_1 is a theory stronger than VNB. But how much stronger? The exact consistency strength of the theory sBL_1 remains an open problem.

We conclude this section by stating and proving the so-called "*Strong Upward Persistency Property*" of $[\text{S-}\Pi^1_1]^{\text{E}}$ formulae. The subsequent preliminary notions are needed:

Let $\langle\, A,\, E^{[A]},\, \mathcal{S}_A,\, \in^{[A\times\mathcal{S}_A]}\,\rangle$ be a Henkin structure for $\mathcal{L}_2$. For any set $a \in A$, we define

$$a_{E^{[A]}} \;:=\; \{\, x \in A \,|\, xE^{[A]}a \,\}.$$

Let $\mathcal{A}$ and $\mathcal{B}$ be two Henkin structure for $\mathcal{L}_2$:

$$\mathcal{A} = \langle\, A,\, E^{[A]},\, \mathcal{S}_A,\, \in^{[A\times\mathcal{S}_A]}\,\rangle,$$
$$\mathcal{B} = \langle\, B,\, F^{[B]},\, \mathcal{S}_B,\, \in^{[B\times\mathcal{S}_B]}\,\rangle.$$

We define $\mathcal{B}$ to be an *end extension* of $\mathcal{A}$ or $\mathcal{A}$ to be an *initial substructure* of $\mathcal{B}$ if and only if $A \subseteq B$ and for any $a \in A$, $a_{E^{[A]}} = a_{F^{[B]}}$. We also define $\mathcal{B}$ to be a *proper end extension* of $\mathcal{A}$ or $\mathcal{A}$ to be a *proper initial substructure* of $\mathcal{B}$ if, in addition, $A \neq B$. When $\mathcal{B}$ is an *end extension* of $\mathcal{A}$ ($\mathcal{A}$ is an *initial substructure* of $\mathcal{B}$), we write

$$\mathcal{A} \subseteq_{\mathsf{end}} \mathcal{B}.$$

When $\mathcal{B}$ is a *proper end extension* of $\mathcal{A}$ ($\mathcal{A}$ is a *proper initial substructure* of $\mathcal{B}$), we write

$$\mathcal{A} \subseteq_{\mathsf{pend}} \mathcal{B}.$$

PROPOSITION 3.3.20. *Let $\mathcal{A}$ and $\mathcal{B}$ be two Henkin structure for $\mathcal{L}_2$:*

$$\mathcal{A} = \langle\, A,\, \in^{[A]},\, \mathcal{S}_A,\, \in^{[A\times\mathcal{S}_A]}\,\rangle,$$
$$\mathcal{B} = \langle\, B,\, \in^{[B]},\, \mathcal{S}_B,\, \in^{[B\times\mathcal{S}_B]}\,\rangle.$$

Assume that $A \subseteq B$ and $\mathsf{Tran}(A)$. Then $\mathcal{A} \subseteq_{\mathsf{end}} \mathcal{B}$.

Now, one of foundamental properties of $[\text{S-}\Pi^1_1]^{\text{E}}$ formulae is their upward persistency under end extensions with the intended interpretation of second-order variables as ranging over arbitrary subsets of the domain

UPWARD PERSISTENCY. *Let $\varphi(v_0, ..., v_n, C_0, ..., C_m)$ be a $[\text{S-}\Pi^1_1]^{\text{E}}$ formula of $\mathcal{L}_2$ with no free variables besides the displayed ones and not necessarily all of them. Let $\mathcal{A}$ and $\mathcal{S}$ be two full structures for $\mathcal{L}_2$*

$$\mathcal{A} = \langle\, A,\, E^{[A]},\, \wp(A),\, \in^{[A\times\wp(A)]}\,\rangle$$

$$\mathcal{S} = \langle\, S,\, F^{[S]},\, \wp(S),\, \in^{[S\times\wp(S)]}\,\rangle$$

such that $\mathcal{A} \subseteq_{\mathsf{end}} \mathcal{S}$*. Then for any* $a_0, ..., a_n \in A$ *and any* $B_0, ..., B_m \subseteq S$ *if*

$$\langle\, A,\, E^{[A]}\,\rangle \models^2 \varphi[a_0, ..., a_n, B_0 \cap A, ..., B_m \cap A]$$

then

$$\langle\, S,\, F^{[S]}\,\rangle \models^2 \varphi[a_0, ..., a_n, B_0, ..., B_m].$$

Cf. Barwise [2], Lemma VIII.2.2, p. 317 as also our persistency result proved in Section 1.4. Such a property, however, admits a strengthening in the following sense: Under end extensions and the same intended interpretation as above, $[\mathrm{S}\text{-}\Pi^1_1]^{\mathrm{E}}$ formulae are shown to persist upward while keeping all the existential set quantifiers relativized to the domain of the initial substructure. And it is indeed this strengthening that we shall refer to as the "*Strong Upward Persistency Property*". Let us start by proving the following:

Absoluteness. *Let* $\varphi(v_0, ..., v_n, C_0, ..., C_m)$ *be a* Δ^{C}_0 *formula of* $\mathcal{L}_2$ *with no free variables besides the displayed ones and not necessarily all of them. Let* $\mathcal{A}$ *and* $\mathcal{S}$ *be two full structures for* $\mathcal{L}_2$

$$\mathcal{A} = \langle\, A,\, E^{[A]},\, \wp(A),\, \in^{[A\times\wp(A)]}\,\rangle$$

$$\mathcal{S} = \langle\, S,\, F^{[S]},\, \wp(S),\, \in^{[S\times\wp(S)]}\,\rangle$$

such that $\mathcal{A} \subseteq_{\mathsf{end}} \mathcal{S}$*. Then for any* $a_0, ..., a_n \in A$ *and any* $B_0, ..., B_m \subseteq S$*,*

$$\langle\, A,\, E^{[A]}\,\rangle \models^2 \varphi[a_0, ..., a_n, B_0 \cap A, ..., B_m \cap A]$$

if and only if

$$\langle\, S,\, F^{[S]}\,\rangle \models^2 \varphi[a_0, ..., a_n, B_0, ..., B_m].$$

Proof. The proof proceeds by induction on the build-up of the Δ^{C}_0 formula $\varphi(v_0, ..., v_n, C_0, ..., C_m)$.

$\underline{\varphi(v_0, ..., v_n, C_0, ..., C_m) \equiv v_0 \in v_1}$: For any $a_0, a_1 \in A$, we have

$$a_0 E^{[A]} a_1 \Longleftrightarrow a_0 F^{[S]} a_1.$$

$\underline{\varphi(v_0, ..., v_n, C_0, ..., C_m) \equiv v_0 \in C_0}$: For any $a_0 \in A$ and any $B_0 \subseteq S$, we trivially have

$$a_0 \in^{[A\times\wp(A)]} B_0 \cap A \Longleftrightarrow a_0 \in^{[S\times\wp(S)]} B_0.$$

$\underline{\varphi(v_0, ..., v_n, C_0, ..., C_m) \equiv \neg\varphi_0(v_0, ..., v_n, C_0, ..., C_m)}$: For any $a_0, ..., a_n \in A$ and any $B_0, ..., B_m \subseteq S$,

$$\langle\, A,\, E^{[A]}\,\rangle \models^2 \neg\varphi_0[a_0, ..., a_n, B_0 \cap A, ..., B_m \cap A]$$

if and only if

$$\langle\, A,\, E^{[A]}\,\rangle \not\models^2 \varphi_0[a_0, ..., a_n, B_0 \cap A, ..., B_m \cap A].$$

If and only if (by I.H.)

$$\langle\, S,\, F^{[S]}\,\rangle \not\models^2 \varphi_0[a_0, ..., a_n, B_0, ..., B_m].$$

If and only if

$$\langle\, S,\, F^{[S]}\,\rangle \models^2 \neg\varphi_0[a_0, ..., a_n, B_0, ..., B_m].$$

$\underline{\varphi(v_0, ..., v_n, C_0, ..., C_m) \equiv \varphi_0(v_0, ..., v_n, C_0, ..., C_m) \wedge \varphi_1(v_0, ..., v_n, C_0, ..., C_m)}$:

For any $a_0, ..., a_n \in A$ and any $B_0, ..., B_m \subseteq S$,

$$\langle\, A,\, E^{[A]}\,\rangle \models^2 \varphi_0[a_0, ..., a_n, B_0 \cap A, ..., B_m \cap A] \wedge \varphi_1[a_0, ..., a_n, B_0 \cap A, ..., B_m \cap A]$$

if and only if

$$\langle\, A,\, E^{[A]}\,\rangle \models^2 \varphi_0[a_0, ..., a_n, B_0 \cap A, ..., B_m \cap A]$$

and

$$\langle\, A,\, E^{[A]}\,\rangle \models^2 \varphi_1[a_0, ..., a_n, B_0 \cap A, ..., B_m \cap A].$$

If and only if (by I.H.)

$$\langle\, S,\, F^{[S]}\,\rangle \models^2 \varphi_0[a_0, ..., a_n, B_0, ..., B_m]$$

and

$$\langle\, S,\, F^{[S]}\,\rangle \models^2 \varphi_1[a_0, ..., a_n, B_0, ..., B_m].$$

If and only if

$$\langle\, S,\, F^{[S]}\,\rangle \models^2 \varphi_0[a_0, ..., a_n, B_0, ..., B_m] \wedge \varphi_1[a_0, ..., a_n, B_0, ..., B_m].$$

Similarly for disjunction.

$\underline{\varphi(v_0, ..., v_n, C_0, ..., C_m) \equiv \exists x(x \in v_n \wedge \varphi_0(v_0, ..., v_n, x, C_0, ..., C_m))}$:

For any $a_0, ..., a_n \in A$ and any $B_0, ..., B_m \subseteq S$,

$$\langle\, A,\, E^{[A]}\,\rangle \models^2 \exists x(x \in a_n \wedge \varphi_0[a_0, ..., a_n, x, B_0 \cap A, ..., B_m \cap A])$$

if and only if for some $e \in A$,

$$\langle\, A,\, E^{[A]}\,\rangle \models^2 e \in a_n \wedge \varphi_0[a_0, ..., a_n, e, B_0 \cap A, ..., B_m \cap A].$$

If and only if for some $e \in A$,

$$\langle\, A,\, E^{[A]}\,\rangle \models^2 e \in a_n$$

and

$$\langle\, A,\, E^{[A]}\,\rangle \models^2 \varphi_0[a_0, ..., a_n, e, B_0 \cap A, ..., B_m \cap A].$$

If and only if (by I.H. and the fact that ${a_n}_{E^{[A]}} = {a_n}_{F^{[S]}}$) for some $e \in S$,

$$\langle\, S,\, F^{[S]}\,\rangle \models^2 e \in a_n$$

and

$$\langle\, S,\, F^{[S]}\,\rangle \models^2 \varphi_0[a_0, ..., a_n, e, B_0, ..., B_m].$$

If and only if for some $e \in S$,

$$\langle\, S,\, F^{[S]}\,\rangle \models^2 e \in a_n \wedge \varphi_0[a_0, ..., a_n, e, B_0, ..., B_m].$$

If and only if

$$\langle\, S,\, F^{[S]}\,\rangle \models^2 \exists x(x \in a_n \wedge \varphi_0[a_0, ..., a_n, x, B_0, ..., B_m]).$$

$$\underline{\varphi(v_0, ..., v_n, C_0, ..., C_m) \equiv \forall x(x \in v_n \to \varphi_0(v_0, ..., v_n, x, C_0, ..., C_m))}:$$

For any $a_0, ..., a_n \in A$, any $B_0, ..., B_m \subseteq S$,

$$\langle\, A,\, E^{[A]}\,\rangle \models^2 \forall x(x \in a_n \to \varphi_0[a_0, ..., a_n, x, B_0 \cap A, ..., B_m \cap A]).$$

If and only if for any $e \in A$,

$$\langle\, A,\, E^{[A]}\,\rangle \models^2 e \notin a_n \vee \varphi_0[a_0, ..., a_n, e, B_0 \cap A, ..., B_m \cap A].$$

If and only if for any $e \in A$,

$$\langle\, A,\, E^{[A]}\,\rangle \models^2 e \notin a_n$$

or

$$\langle\, A,\, E^{[A]}\,\rangle \models^2 \varphi_0[a_0, ..., a_n, e, B_0 \cap A, ..., B_m \cap A].$$

If and only if (by I.H. and the fact that ${a_n}_{E^{[A]}} = {a_n}_{F^{[S]}}$) for any $e \in S$,

$$\langle\, S,\, F^{[S]}\,\rangle \models^2 e \notin a_n$$

or

$$\langle\, S,\, F^{[S]}\,\rangle \models^2 \varphi_0[a_0, ..., a_n, e, B_0, ..., B_m].$$

If and only if for any $e \in S$,

$$\langle\, S,\, F^{[S]}\,\rangle \models^2 e \notin a_n \vee \varphi_0[a_0, ..., a_n, e, B_0, ..., B_m].$$

If and only if

$$\langle\, S,\, F^{[S]}\,\rangle \models^2 \forall x(x \in a_n \to \varphi_0[a_0, ..., a_n, x, B_0, ..., B_m]).$$

□

In order to state and prove the strong upward persistency property of $[\text{s-}\Pi^1_1]^{\text{E}}$ formulae, some notation is introduced. If φ is a a $[\text{s-}\Pi^1_1]^{\text{E}}$ formula then we denote by $\varphi^{\|b\|}$ the formula φ with only the existential-set quantifiers relativized to b.

STRONG UPWARD PERSISTENCY. *Let* $\varphi(v_0, ..., v_n, C_0, ..., C_m)$ *be a* $[\text{S-}\Pi^1_1]^{\mathrm{E}}$ *formula of* $\mathcal{L}_2$ *with no free variables besides the displayed ones and not necessarily all of them. Let* $\mathcal{A}$ *and* $\mathcal{S}$ *be two full structures for* $\mathcal{L}_2$

$$\mathcal{A} = \langle\, A,\ E^{[A]},\ \wp(A),\ \in^{[A \times \wp(A)]}\,\rangle$$

$$\mathcal{S} = \langle\, S,\ F^{[S]},\ \wp(S),\ \in^{[S \times \wp(S)]}\,\rangle$$

such that $\mathcal{A} \subseteq_{\mathsf{end}} \mathcal{S}$. *Then for any* $a_0, ..., a_n \in A$ *and any* $B_0, ..., B_m \subseteq S$ *if*

$$\langle\, A,\ E^{[A]}\,\rangle \models^2 \varphi[a_0, ..., a_n, B_0 \cap A, ..., B_m \cap A]$$

then

$$\langle\, S,\ F^{[S]}\,\rangle \models^2 \varphi^{\|A\|}[a_0, ..., a_n, B_0, ..., B_m].$$

Proof. The proof proceeds by induction on the build-up of the $[\text{S-}\Pi^1_1]^{\mathrm{E}}$ formula $\varphi(v_0, ..., v_n, C_0, ..., C_m)$.

$\underline{\Delta^{\mathrm{C}}_0}$: This is immediate by the previous result.

Concerning the induction step we need only to consider the following two cases, since the other cases [$\wedge$, $\vee$, $(\forall x \in v)$ and $(\exists x \in v)$] are treated as for the previous result.

$\underline{\varphi(v_0, ..., v_n, C_0, ..., C_m) \equiv \exists x \varphi_0(v_0, ..., v_n, x, C_0, ..., C_m)}$:

Assume for any $a_0, ..., a_n \in A$, any $B_0, ..., B_m \subseteq S$ that

$$\langle\, A,\ E^{[A]}\,\rangle \models^2 \exists x \varphi_0[a_0, ..., a_n, x, B_0 \cap A, ..., B_m \cap A].$$

This means that for some $e \in A$, we have

$$\langle\, A,\ E^{[A]}\,\rangle \models^2 \varphi_0[a_0, ..., a_n, e, B_0 \cap A, ..., B_m \cap A].$$

By I.H.

$$\langle\, S,\ F^{[S]}\,\rangle \models^2 \varphi_0^{\|A\|}[a_0, ..., a_n, e, B_0, ..., B_m].$$

Further

$$\langle\, S,\ F^{[S]}\,\rangle \models^2 e \in A.$$

Hence

$$\langle\, S,\ F^{[S]}\,\rangle \models^2 \exists x(x \in A \wedge \varphi_0^{\|A\|}[a_0, ..., a_n, x, B_0, ..., B_m]).$$

That is

$$\langle\, S,\ F^{[S]}\,\rangle \models^2 \varphi^{\|A\|}[a_0, ..., a_n, B_0, ..., B_m].$$

$\underline{\varphi(v_0, ..., v_n, C_0, ..., C_m) \equiv \forall X \varphi_0(v_0, ..., v_n, C_0, ..., C_m, X)}$:

Assume for any $a_0, ..., a_n \in A$, any $B_0, ..., B_m \subseteq S$ that

$$\langle\, A,\ E^{[A]}\,\rangle \models^2 \forall X \varphi_0[a_0, ..., a_n, B_0 \cap A, ..., B_m \cap A, X].$$

This means that for any $D \subseteq S$, we have

$$\langle\, A,\, E^{[A]}\, \rangle \models^2 \varphi_0[a_0, ..., a_n, B_0 \cap A, ..., B_m \cap A, D \cap A]. \tag{1}$$

By I.H.

$$\langle\, S,\, F^{[S]}\, \rangle \models^2 \varphi_0^{\|A\|}[a_0, ..., a_n, B_0, ..., B_m, D].$$

Hence

$$\langle\, S,\, F^{[S]}\, \rangle \models^2 \varphi^{\|A\|}[a_0, ..., a_n, B_0, ..., B_m].$$

□

If we were to allow Henkin structures instead of full structures point (1), for example, would fail: just because D is an arbitrary element of $\mathcal{S}_S$, there is no reason to suppose that $D \cap A$ is an element of $\mathcal{S}_A$ at all!

To reiterate, the Strong Upward Persistency property tells us that the relativization of a s-Π^1_1 formula to some transitive set b for example, will be indifferent to the replacement of $\forall X[X \subseteq b \to \dots]$ by $\forall X \dots$ and to the replacement of $C \cap b$ by C. As a result, the notion of s-Π^1_1-indescribablity can be recasted as follows. An ordinal α is s-Π^1_1-indescribable if and only if for any s-Π^1_1 formula $\varphi(v_0, ..., v_n, C_0, ..., C_m)$ with free variables as indicated and any set $a_0, ..., a_n \in V_\alpha$ and any $B_0, ..., B_m \subseteq V_\alpha$, if

$$\langle\, V_\alpha, \in^{[V_\alpha]}\, \rangle \models^2 \varphi[a_0, ..., a_n, B_0, ..., B_m]$$

then there is a transitive set $a \in V_\alpha$ such that $a_0, ..., a_n \in a$ and

$$\langle\, V_\alpha, \in^{[V_\alpha]}\, \rangle \models^2 \varphi^{\|a\|}[a_0, ..., a_n, B_0, ..., B_m].$$

3.4 THE THEORY BL_1

The Bernays-Lévy theory BL_1 is formulated in the second-order language $\mathcal{L}_2$ of VNB and it consists of the following three axioms:

$$\mathsf{BL}_1 := \Delta_0\text{-}\mathsf{I}_\in,\ \text{AUS},\ \Pi^1_1\ \text{RFN}.$$

REMARK 3.4.1. Actually, the theory BL_1 as known in the literature (the reader is referred to Gloede [22] on page 303) includes also the axiom of EXTENSIONALITY. Our approach dispenses with EXTENSIONALITY by introducing an explicit definition of equality between sets. This is, of course, of no harm as in the process of relativization we make use of the above-mentioned absoluteness property of Δ_0 notions for transitive sets (cf. Reamark 3.2.8), allowing us to treat “equality” as it were an atomic symbol of the base language $\mathcal{L}_2$.

The theory VNB + Π^1_1 RFN is known in the literature as BL_1. To see why we state and quickly sketch the proof of the following result.

THEOREM 3.4.2. PAIR, UNION, INFINITY, POWER SET, REPLACEMENT, $\mathsf{I}^2_\in$ *and each instance of* PCA *are all derivable in* BL_1.

Proof. PAIR, UNION, POWER SET, $\mathsf{I}^2_\in$ and REPLACEMENT, as we had already occasion to see, are derivable using S-Π^1_1 RFN and essentially the same proofs apply here. Concerning the derivability of INFINITY and PCA (i.e. each instance thereof) the reader is referred to Bernays [4] on p. 128 and Gloede [9] on p.305, respectively. □

Therefore in virtue of this result we can indeed regard $\mathsf{BL_1}$ as being $\mathsf{VNB}+\Pi^1_1$ RFN. We also remark that $\mathsf{BL_1}$ proves the consistency of VNB.

THEOREM 3.4.3. *For any Π^1_1 formula $\varphi(v_0, ..., v_n, C_0, ..., C_m)$ with no free variables besides the displayed ones and not necessarily all of them, the following is derivable within the theory* $\mathsf{BL_1}$:

$$\begin{aligned}&\varphi(v_0, ..., v_n, C_0, ..., C_m) \to \\ &\quad\to \exists\beta\,[\,\mathsf{inacc}(\beta) \wedge v_0, ..., v_n \in V_\beta \wedge \varphi^{(V_\beta)}(v_0, ..., v_n, C_0, ..., C_m)\,].\end{aligned}$$

And, in turn, this strengthened schema of Π^1_1 RFN admits a further self-strengthening to a schema entailing the existence of arbitrarily large Mahlo cardinals. All the details of this argument are discussed at length in Bernays [4] and Gloede [9].

Unfortunately, due to the low logical complexity of S-Π^1_1 formulae none of these self-strengthening is known (at least to the author) to hold, within $\mathsf{sBL_1}$, for the S-Π^1_1 RFN axiom schema. established. As next step, we make a comparison between the theories $\mathsf{sBL_1}$ and $\mathsf{BL_1}$. It will also be shown that these two theories admit the same standard models.

3.5 COMPARING $\mathsf{sBL_1}$ WITH $\mathsf{BL_1}$

To the aim of pointing out resemblances and differences between $\mathsf{sBL_1}$ and $\mathsf{BL_1}$, it will be convenient to list simultaneously their correspondig set of axioms, in the following synoptic way:

$$\begin{aligned}\mathsf{BL_1} &:= \Delta_0\text{-}\mathsf{I}_\in,\ \text{AUS},\ \Pi^1_1\ \text{RFN}.\\ \mathsf{sBL_1} &:= \Delta_0\text{-}\mathsf{I}_\in,\ \text{AUS},\ \text{S-}\Pi^1_1\ \text{RFN},\ \text{INFINITY},\ \text{PCA}.\end{aligned}$$

As we had already occasion to see in the previous section, INFINITY and each instance of PCA are derivable in $\mathsf{BL_1}$. Further every instance of S-Π^1_1 RFN is also an instance of Π^1_1 RFN. The following observation is therefore obvious.

COROLLARY 3.5.1. *Every theorem φ of $\mathsf{sBL_1}$ is also a theorem of $\mathsf{BL_1}$,*

$$\mathsf{sBL_1} \vdash \varphi \quad\Longrightarrow\quad \mathsf{BL_1} \vdash \varphi.$$

To reiterate, INFINITY and each instance of PCA are derivable in $\mathsf{BL_1}$. At this stage let us consider the following intermediate theory

$$\mathsf{strictBL_1} := \Delta_0\text{-}\mathsf{I}_\in,\ \text{AUS},\ \text{S-}\Pi^1_1\ \text{RFN}.$$

Contrary to BL_1, neither INFINITY nor each instance of PCA are derivable in $\mathsf{strictBL}_1$. Indeed we shall prove that

- $\mathsf{strictBL}_1 \cup \{\mathrm{PCA}\} \nvdash$ INFINITY,
- $\mathsf{strictBL}_1 \cup \{\mathrm{PCA}\} \nvdash \neg$INFINITY.

And

- $\mathsf{strictBL}_1 \cup \{\text{INFINITY}\} \nvdash$ PCA,
- $\mathsf{strictBL}_1 \cup \{\text{INFINITY}\} \nvdash \neg$PCA.

Before starting, let us remark the following.

PROPOSITION 3.5.2. *Every theorem φ of $\mathsf{strictBL}_1$ is also a theorem of $\mathsf{sKPu}_2^{\mathsf{r}}$,*

$$\mathsf{strictBL}_1 \vdash \varphi \quad \Longrightarrow \quad \mathsf{sKPu}_2^{\mathsf{r}} \vdash \varphi.$$

COROLLARY 3.5.3. *We have*

$$\mathsf{strictBL}_1 \nvdash \text{INFINITY},$$

and

$$\mathsf{strictBL}_1 \nvdash \mathrm{PCA}.$$

Proof. These two facts are entailed by Proposition 3.5.2, Corollary 2.4.9 and Theorem 3.0.7, respectively. □

3.6 THE INDEPENDENCE OF INFINITY

Let

$$(\mathsf{strictBL}_1)^+ := \mathsf{strictBL}_1 \cup \{\mathrm{PCA}\}.$$

LEMMA 3.6.1. INFINITY *is not derivable in* $(\mathsf{strictBL}_1)^+$.

Proof. Let us show that

$$\langle V_\omega, \in^{[V_\omega]} \rangle \models^2 (\mathsf{strictBL}_1)^+ \qquad \text{and} \qquad \langle V_\omega, \in^{[V_\omega]} \rangle \not\models^2 \text{INFINITY}.$$

AUS and Δ_0-$\mathsf{I}_\in$ are readily seen to hold in this model. PCA holds due to the particular choice of our satisfaction relation which interprets classes as arbitrary subsets of V_ω. By Theorem 3.3.18, ω is S-Π^1_1-indescribable. Clearly INFINITY does not hold. □

COROLLARY 3.6.2. *There are instances of the schema of* PCA *which are independent from* $\mathsf{strictBL}_1$.

Proof. By the proof of Lemma 3.6.1 we know that

$$\langle V_\omega, \in^{[V_\omega]} \rangle \models^2 \mathsf{strictBL}_1 \qquad \text{and} \qquad \langle V_\omega, \in^{[V_\omega]} \rangle \models^2 \mathrm{PCA}.$$

And this implies that there are instances of the schema of PCA whose negation is not derivable in $\mathsf{strictBL}_1$. The result is obtained along with Corollary 3.5.3. □

LEMMA 3.6.3. *The negation of* INFINITY *is not derivable in* $(\mathsf{strictBL}_1)^+$*.*

Proof. Let μ be the first weakly compact cardinal. By Theorem 3.3.18, μ is S-Π^1_1-indescribable. Clearly INFINITY does hold, for $\omega \in V_\mu$. Therefore,

$$\langle V_\mu, \in^{[V_\mu]} \rangle \models^2 (\mathsf{strictBL}_1)^+ \qquad \text{and} \qquad \langle V_\mu, \in^{[V_\mu]} \rangle \models^2 \text{INFINITY}.$$

□

We have then established the independence of the axiom of INFINITY from our theory $(\mathsf{strictBL}_1)^+$. As obvious consequence we also have that

COROLLARY 3.6.4. *The axiom of* INFINITY *is independent from* $\mathsf{strictBL}_1$*.*

Proof. By Lemma 3.6.3 and Corollary 3.5.3. □

According to Theorem VIII.3.3 of Barwise [2], every countable admissible set *satisfies* the schema of S-Π^1_1 RFN (for a proof of this result the reader is referred to Barwise [2], pp. 322-323). We warn the reader of the striking difference between "satisfying the schema of S-Π^1_1 RFN" and "S-Π^1_1-indescribability". Satisfying S-Π^1_1 RFN means, according to our terminology, satisfying S-Π^1_1 RFN (i.e. each instance thereof) without class-parameters. And indeed Theorem VIII.3.3 can be restated as follows

> *Every countable admissible set satisfies the schema of* S-Π^1_1 RFN *without class-parameters.*

As for any other schema of reflection, it is worth emphasizing that "satisfying the schema of S-Π^1_1 RFN without class-parameters" is a much weaker notion than "S-Π^1_1-indescribability" and indeed as long as class-parameters are allowed to occur in the schema of S-Π^1_1 RFN, Theorem VIII.3.3 fails.

LEMMA 3.6.5. $L_{\omega_1^{\mathrm{CK}}}$ *is* S-Π^1_1*-describable.*

Proof. ω_1^{CK} is the least countable ordinal which cannot be represented by a recursive well-ordering on the natural numbers. Let us work informally within ZFC where the existence and countability of ω_1^{CK} can be proved. It is a folklore result that ω_1^{CK} is the least admissible ordinal above ω. Since

$$\left| L_{\omega_1^{\mathrm{CK}}} \right| = |\omega_1^{\mathrm{CK}}| = \aleph_0,$$

(see Lemma 3.8.7.(vi) on page 97) we have that $L_{\omega_1^{\mathrm{CK}}}$ is a countable admissible set. Furthermore, from $|\omega_1^{\mathrm{CK}}| = \aleph_0$, it follows that $\mathsf{sing}(\omega_1^{\mathrm{CK}})$ and $\mathsf{cf}(\omega_1^{\mathrm{CK}}) = \omega$. This means there must be a function F such that

$$F : \omega \longrightarrow \omega_1^{\mathrm{CK}} \qquad \text{and} \qquad \omega_1^{\mathrm{CK}} = \bigcup_{n \in \omega} F(n).$$

We claim that $L_{\omega_1^{\text{CK}}}$ is S-Π^1_1-describable. If S-Π^1_1 RFN held in

$$\langle\, L_{\omega_1^{\text{CK}}},\, \in^{[L_{\omega_1^{\text{CK}}}]},\, \wp(L_{\omega_1^{\text{CK}}}),\, \in^{[L_{\omega_1^{\text{CK}}}\times\wp(L_{\omega_1^{\text{CK}}})]} \rangle$$

then since

$$\langle\, L_{\omega_1^{\text{CK}}},\, \in^{[L_{\omega_1^{\text{CK}}}]}\,\rangle \models^2 \forall n\Big(n\in\omega \to \exists\gamma(\gamma\in \mathbf{ON}\wedge\langle n,\gamma\rangle\in F)\Big),$$

there would be a transitive reflecting set $b \in L_{\omega_1^{\text{CK}}}$ such that $\omega \in b$ and

$$\langle\, b, \in^{[b]}\,\rangle \models^2 \forall n\Big(n\in\omega \to \exists\gamma(\gamma\in \mathbf{ON}\wedge\langle n,\gamma\rangle\in F\cap b)\Big).$$

By the Strong Upward Persistency property we shall have

$$\langle\, L_{\omega_1^{\text{CK}}},\, \in^{[L_{\omega_1^{\text{CK}}}]}\,\rangle \models^2 \Bigg(\forall n\Big(n\in\omega \to \exists\gamma(\gamma\in \mathbf{ON}\wedge\langle n,\gamma\rangle\in F)\Big)\Bigg)^{\|b\|},$$

which is equivalent to

$$\langle\, L_{\omega_1^{\text{CK}}},\, \in^{[L_{\omega_1^{\text{CK}}}]}\,\rangle \models^2 \forall n\Big(n\in\omega \to \exists\gamma(\gamma\in b\wedge\gamma\in \mathbf{ON}\wedge\langle n,\gamma\rangle\in F)\Big).$$

Let $b \cap \mathbf{ON} = \alpha$. Hence we obtain

$$\exists\alpha\Big(\alpha<\omega_1^{\text{CK}}\wedge\forall n(n\in\omega\to F(n)<\alpha)\Big).$$

But this contradicts the fact that

$$\omega_1^{\text{CK}} = \bigcup_{n\in\omega} F(n),$$

that is

$$\forall\alpha\Big(\alpha<\omega_1^{\text{CK}}\to\exists n(n\in\omega\wedge\alpha\leq F(n))\Big).$$

□

Thus in general countable admissible sets fail to satisfy the schema of S-Π^1_1 RFN with second-order parameters. Hence the main question needed to be addressed: is there any countable admissible set which is S-Π^1_1-indescribable? By Theorem 3.3.18, we already know that this question has a positive answer, for the countable admissible set V_ω is the only such an example. Having (hopefully) convinced the reader of the relavance of the class-parameters in the notion of "S-Π^1_1-indescribability", we now turn to a detailed analysis of Theorem 3.3.18. As already mentioned in Section 3.3 (cf. paragraph following Theorem 3.3.18 itself), we want to give here a different proof of the necessary conditions needed to be satisfied by an ordinal α for being S-Π^1_1-indescribable.

THEOREM 3.6.6. *If α is* s-Π^1_1*-indescribable, then α is a regular infinite cardinal closed under cardinal exponentiation and with the tree-property.*

Proof. We shall work informally within ZFC and assume that our ordinal α is s-Π^1_1-indescribable. The argument breaks up into the following three cases:

(1) α is regular (hence a cardinal): If not, there would be a $\mu < \alpha$ and a functional class F such that

$$F : \mu \longrightarrow \alpha \qquad \text{and} \qquad \alpha = \bigcup_{\xi<\mu} F(\xi).$$

We argue as in the proof of Lemma 3.6.5. It is worth noticing that 0 and 1 are both regular cardinal. However, under the assumption that α is a s-Π^1_1-indescribable, the possiblity that either $\alpha = 0$ or $\alpha = 1$ is trivially ruled out. Since any other finite cardinal is singular, α must be a regular infinite cardinal.

(2) α is closed under cardinal exponentiation: If not, then there would be a $\lambda < \alpha$ such that $\alpha \leq 2^\lambda$ and a functional class $G : \wp(\lambda) \longrightarrow \alpha$ being surjective. By (1), α is an infinite cardinal, hence a limit ordinal. Hence, for $\lambda < \alpha$ and $\mathsf{Lim}(\alpha)$, $V_{\lambda+2} \subset V_\alpha$. Since $\wp(\lambda) \in V_{\lambda+2}$, then $\wp(\lambda) \in V_\alpha$. Therefore

$$\langle V_\alpha, \in^{[V_\alpha]} \rangle \models^2 \forall y \Big(y \in \wp(\lambda) \to \exists\gamma(\gamma \in \mathbf{ON} \wedge \langle y, \gamma\rangle \in G) \Big).$$

By hypothesis there is a transitive set $b \in V_\alpha$ such that $\wp(\lambda) \in b$ and

$$\langle b, \in^{[b]} \rangle \models^2 \forall y \Big(y \in \wp(\lambda) \to \exists\gamma(\gamma \in \mathbf{ON} \wedge \langle y, \gamma\rangle \in G \cap b) \Big).$$

By the Strong Upward Persistency property we shall have

$$\langle V_\alpha, \in^{[V_\alpha]} \rangle \models^2 \Bigg(\forall y \Big(y \in \wp(\lambda) \to \exists\gamma(\gamma \in \mathbf{ON} \wedge \langle y, \gamma\rangle \in G) \Big) \Bigg)^{\|b\|},$$

which is equivalent to

$$\langle V_\alpha, \in^{[V_\alpha]} \rangle \models^2 \forall y \Big(y \in \wp(\lambda) \to \exists\gamma(\gamma \in b \wedge \gamma \in \mathbf{ON} \wedge \langle y, \gamma\rangle \in G) \Big).$$

Let $b \cap \mathbf{ON} = \beta$. Hence we obtain

$$\exists\beta \Big(\beta < \alpha \wedge \forall y (y \in \wp(\lambda) \to G(y) < \beta) \Big).$$

But this violates the assumption that the range of G is all of α, that is

$$\forall\beta \Big(\beta < \alpha \to \exists y (y \in \wp(\lambda) \wedge \beta = G(y)) \Big).$$

(3) α has the tree-property: Suppose not. Then there is an α-Aronszajn tree. Let $\langle S, <_S \rangle$ be such a α-Aronszajn tree. By definition of "α-Aronszajn tree", we know that $ht(S) = \alpha$. Hence we have the following assertion to hold true of $\mathbf{V}$:

$$\neg\left(\exists C\Big[C \subseteq S \wedge \mathsf{chain}(C) \wedge (\forall \gamma < \alpha)(S_{(\gamma)} \cap C \neq \emptyset)\Big]\right).$$

The first step of our argument consists in finding an isomorphic copy $\langle T, <_T \rangle$ of $\langle S, <_S \rangle$ such that $T \subseteq V_\alpha$, $<_T \subseteq V_\alpha$ and (since $ht(T)$ will be α and $\mathbf{ON} \cap V_\alpha = \alpha$)

$$\langle V_\alpha, \in^{[V_\alpha]} \rangle \models^2 \neg\left(\exists C\Big[C \subseteq T \wedge \mathsf{chain}(C) \wedge (\forall \gamma \in \mathbf{ON})(T_{(\gamma)} \cap C \neq \emptyset)\Big]\right).$$

We claim that $|\, S \,| = \alpha$. Since $ht(S) = \alpha$ then for every $\gamma < \alpha$, the γ-th level $S_{(\gamma)}$ of S is non-empty, and we certainly have that $\alpha \leq |\, S \,|$. To obtain the reverse inequality we first note that $S = \bigcup\{\, S_{(\gamma)} \mid \gamma < \alpha \,\}$. So

$$|\, S \,| = \left| \bigcup_{\gamma<\alpha} S_{(\gamma)} \right| \leq \sum_{\gamma<\alpha} |\, S_{(\gamma)} \,| \leq \sum_{\gamma<\alpha} \alpha = \alpha \otimes \alpha = \alpha.$$

Hence $|\, S \,| = \alpha$ and by definition of cardinality this implies the existence of a bijection g of S onto α. Then a partial ordering $\prec_\alpha$ ca be defined on α by $<_S$ and g setting

$$\prec_\alpha = \{\, \langle g(s),\, g(t) \rangle \mid s \in S \wedge t \in S \wedge \langle s,\, t \rangle \in <_S \,\}.$$

Therefore we obtain an order preserving function g, e.g.

$$(\forall s, t \in S)(s <_T t \leftrightarrow g(s) \prec_\alpha g(t))$$

mapping S one-to-one and onto α. That is an order isomorphism g from $\langle S, <_S \rangle$ onto $\langle \alpha, \prec_\alpha \rangle$. Accordingly we have established that the p.o.'s $\langle S, <_S \rangle$ and $\langle \alpha, \prec_\alpha \rangle$ are isomorphic and we write

$$\langle S, <_S \rangle \cong \langle \alpha, \prec_\alpha \rangle.$$

Let us introduce the following abbreviation. For any tree $\langle S, <_S \rangle$ and for any $t \in S$ we let

$$\mathsf{pr}_S(t) := \{\, s \in S \mid s <_S t \,\}.$$

Let us show that $\langle \alpha, \prec_\alpha \rangle$ is a α-tree with no cofinal branch. The argument breaks down to proving the following points:

(a) $\langle \alpha, \prec_\alpha \rangle$ is a partially ordered set such that for any $\gamma < \alpha$ the set $\mathsf{pr}_\alpha(\gamma)$ is well-ordered by the relation $\prec_\alpha$;

(b) $ht(\alpha) = \alpha$;

(c) $\forall\gamma(\gamma < \alpha \rightarrow |\, \alpha_{(\gamma)} \,| < \alpha)$;

(d) $\langle\, \alpha, \prec_\alpha \,\rangle$ has no cofinal branch.

Point (a) in turn reduces down to prove the following points

(a1) $(\forall\gamma < \alpha)(\langle\, \gamma,\, \gamma \,\rangle \notin \prec_\alpha)$,

(a2) $(\forall\gamma, \beta, \delta < \alpha)(\langle\, \gamma,\, \beta \,\rangle \in \prec_\alpha \wedge \langle\, \beta,\, \delta \,\rangle \in \prec_\alpha \rightarrow \langle\, \gamma,\, \delta \,\rangle \in \prec_\alpha)$,

(a3) $(\forall\gamma < \alpha)(\prec_\alpha \restriction \mathsf{pr}_\alpha(\gamma)$ is a partial order relation$)$,

(a4) $(\forall\gamma < \alpha)(\forall\beta, \delta \in \mathsf{pr}_\alpha(\gamma))(\beta = \delta \vee \langle\, \beta,\, \delta \,\rangle \in \prec_\alpha \vee \langle\, \delta,\, \beta \,\rangle \in \prec_\alpha)$,

(a5) $(\forall\gamma < \alpha)(\forall z \subseteq \mathsf{pr}_\alpha(\gamma))(z = \emptyset \vee \exists v(v \in z \wedge \neg\exists y(y \in z \wedge \langle\, y\,, v \,\rangle \in \prec_\alpha))$.

We sketch the proof of the following points:

(a1) (a2) These two points are immediate, for g is a bijection of S onto α such that $(\forall s, t \in S)(s <_T t \leftrightarrow g(s) \prec_\alpha g(t))$.

(a3) (a4) (a5) For any $t \in S$ we know, by definition of tree, that $<_S \restriction \mathsf{pr}_S(t)$ is a well-ordering relation. Since g is an order isomorphism and order properties are invariant under order isomorphism (order invariant) then, for any $\gamma < \alpha$, $\prec_\alpha \restriction \mathsf{pr}_\alpha(\gamma)$ is a well-ordering relation too.

(b) Suppose not. Then there would be a node β in α such that

$$\mathsf{ot}(\langle\, \mathsf{pr}_\alpha(\beta), \prec_\alpha \,\rangle) = \alpha.$$

But this means that there is a $\beta < \alpha$ with the same order-type as α. A contradiction.

(c) Suppose for some $\gamma < \alpha$,

$$|\alpha_{(\gamma)}| = |\{\, \gamma < \alpha \mid \mathsf{ot}(\langle\, \mathsf{pr}_\alpha(\gamma), \prec_\alpha \,\rangle) = \gamma \,\}| \geq \alpha.$$

And since isomorphic p.o.'s have the same order-type and g is a bijection of S onto μ, then

$$|S_{(\gamma)}| = |\{\, t \in S \mid \mathsf{ot}(\langle\, \mathsf{pr}_S(t), <_T \,\rangle) = \gamma \,\}| \geq \alpha.$$

A contradiction.

(d) If C were a cofinal branch of $\langle\, \alpha, \prec_\alpha \,\rangle$ then since g is an order isomorphism from $\langle\, S, <_S \,\rangle$ onto $\langle\, \alpha, \prec_\alpha \,\rangle$, then $g^{-1}(C)$ would be a cofinal branch of $\langle\, S, <_S \,\rangle$ as well, contradicting the fact $\langle\, S, <_S \,\rangle$ does not have cofinal branch.

Obviously, $\alpha = \mathbf{ON} \cap V_\alpha \subseteq V_\alpha$. Further, $\prec_\alpha \subseteq V_\alpha \times V_\alpha$. And since V_α is closed under PAIR (α is a limit ordinal), $\prec_\alpha \subseteq V_\alpha$. Therefore, $\langle\, \alpha, \prec_\alpha \,\rangle$ is the isomorphic copy $\langle\, T, <_T \,\rangle$ of $\langle\, S, <_S \,\rangle$ we were looking for. And, we certainly have

$$\langle\, V_\alpha, \in^{[V_\alpha]} \,\rangle \models^2 \neg\left(\exists C\left[C \subseteq T \wedge \mathsf{chain}(C) \wedge (\forall\gamma \in \mathbf{ON})(T_{(\gamma)} \cap C \neq \emptyset)\right]\right).$$

This concludes the first step of our argument. At this point it is worth noticing that the expression "$t \in T_{(\gamma)}$" stands for the following formula:

$$\exists f \Big[f \text{ is a bijection } \wedge \mathsf{dom}(f) = \gamma \wedge \forall s (s \in \mathsf{rng}(f) \leftrightarrow s <_T t) \wedge \\ \wedge (\forall \xi, \eta \in \gamma)(\xi \in \eta \leftrightarrow f(\xi) <_T f(\eta)) \Big].$$

And this, in turn, makes the whole assertion

$$\neg \left(\exists C \Big[C \subseteq T \wedge \mathsf{chain}(C) \wedge (\forall \gamma \in \mathbf{ON})(T_{(\gamma)} \cap C \neq \emptyset) \Big] \right), \qquad (1)$$

of logical complexity Π^1_1. Therefore, the second step of our argument, consists in showing that the formula (1) can be rendered by a set-theoretical formula of logical complexity s-Π^1_1. In order to achieve this we proceed as follows. We first prove that for every $\gamma < \alpha$, $T_{(\gamma)}$ is a set in V_α. Fix an arbitrary $\gamma < \alpha$. We already know that $T_{(\gamma)} \subseteq V_\alpha$ and by definition of α-tree, $|T_{(\gamma)}| < \alpha$. Hence for some cardinal $\nu < \alpha$ we shall have that $|T_{(\gamma)}| = \nu$. This implies, in particular, the existence of a surjective map $f : \nu \longrightarrow T_{(\gamma)}$ and by Replacement (α is regular infinite cardinal) the range of this map is a set. Thus, having shown that each $T_{(\gamma)}$ is actually an element of V_α, we set

$$LEV = \{ \langle \gamma, x \rangle \mid \gamma < \alpha \wedge x = T_{(\gamma)} \}.$$

LEV is a relation and as directly involved in the definition itself

$$\forall \gamma, x, y \Big(\langle \gamma, x \rangle \in LEV \wedge \langle \gamma, y \rangle \in LEV \rightarrow x = y \Big).$$

Therefore LEV is a functional class (in V_α) mapping each ordinal γ to the γ-th level $T_{(\gamma)}$ of T. Hence the assertion (1) can indeed be rendered by the following s-Π^1_1 formula holding in V_α:

$$\langle V_\alpha, \in^{[V_\alpha]} \rangle \models^2 \forall C \Bigg[\Bigg(C \subseteq T \wedge \forall x \forall y \Big(x \in C \wedge y \in C \rightarrow \\ \rightarrow \Big(\langle x, y \rangle \in <_T \vee x = y \vee \langle y, x \rangle \in <_T \Big) \Big) \Bigg) \rightarrow \\ \rightarrow \exists \gamma \exists w \Big(\langle \gamma, w \rangle \in LEV \wedge \forall z (z \in w \rightarrow z \notin C) \Big) \Bigg].$$

We are, of course, using T, $<_T$ and LEV as class-parameters. Under the assumption that α is s-Π^1_1-indescribable there exists a transitive reflecting set $b \in V_\alpha$ such that

$$\langle\, V_\alpha,\, \in^{[V_\alpha]}\,\rangle \models^2 \Bigg(\forall C\,\bigg[\Big(C \subseteq T \cap b \wedge \forall x \forall y \Big(x \in C \wedge y \in C \to \\ \to \Big(\langle x, y\rangle \in ({<_T} \cap b) \vee x = y \vee \langle y, x\rangle \in ({<_T} \cap b)\Big)\Big)\Big) \to \\ \to \exists\gamma\exists w\Big(\langle\, \gamma,\, w\,\rangle \in (LEV \cap b) \wedge \forall z (z \in w \to z \notin C)\Big)\bigg]\Bigg)^{(b)}.$$

Transitivity of V_α together with $b \in V_\alpha$ implies that $b \cap V_\alpha = b$. Hence we obtain

$$\langle\, b,\, \in^{[b]}\,\rangle \models^2 \forall C\,\bigg[\Big(C \subseteq T \cap b \wedge \forall x \forall y \Big(x \in C \wedge y \in C \to \\ \to \Big(\langle x, y\rangle \in ({<_T} \cap b) \vee x = y \vee \langle y, x\rangle \in ({<_T} \cap b)\Big)\Big)\Big) \to \\ \to \exists\gamma\exists w\Big(\langle\, \gamma,\, w\,\rangle \in (LEV \cap b) \wedge \forall z (z \in w \to z \notin C)\Big)\bigg].$$

By the Strong Upward Persistency property, then we obtain

$$\langle\, V_\alpha,\, \in^{[V_\alpha]}\,\rangle \models^2 \Bigg(\forall C\,\bigg[\Big(C \subseteq T \wedge \forall x \forall y \Big(x \in C \wedge y \in C \to \\ \to \Big(\langle x, y\rangle \in {<_T} \vee x = y \vee \langle y, x\rangle \in {<_T}\Big)\Big)\Big) \to \\ \to \exists\gamma\exists w\Big(\langle\, \gamma,\, w\,\rangle \in LEV \wedge \forall z (z \in w \to z \notin C)\Big)\bigg]\Bigg)^{\|b\|}.$$

Since T, $<_T$ and LEV are subclasses of V_α and all the set-quantifiers of this formula are relativized to b and $b \cap V_\alpha = b$, we have that

$$\forall C\,\bigg[C \subseteq V_\alpha \to \bigg(\Big(\forall x (x \in b \wedge x \in C \to x \in T) \wedge \\ \wedge \forall x \forall y \Big(x \in b \wedge y \in b \wedge x \in C \wedge y \in C \to \\ \to \Big(\langle x, y\rangle \in {<_T}\ \vee x = y \vee \langle y, x\rangle \in {<_T}\Big)\Big)\Big) \to \\ \to \exists\gamma\exists w\Big(\gamma \in b \wedge w \in b \wedge \langle\, \gamma,\, w\,\rangle \in LEV \wedge \\ \wedge \forall z (z \in w \to z \notin C)\Big)\bigg)\bigg].$$

From this, using the fact that $b \subseteq V_\alpha$, we obtain

$$\begin{aligned}\forall C \Bigg[C \subseteq V_\alpha \to \Bigg(\Bigg(& \forall x (x \in V_\alpha \wedge x \in C \to x \in T) \wedge \\ & \wedge \forall x \forall y \Bigg(x \in V_\alpha \wedge y \in V_\alpha \wedge x \in C \wedge y \in C \to \\ & \to \Big(\langle x, y \rangle \in <_T \vee x = y \vee \langle y, x \rangle \in <_T \Big) \Bigg) \Bigg) \to \\ & \to \exists \gamma \exists w \Bigg(\gamma \in b \wedge w \in V_\alpha \wedge \langle\, \gamma,\, w \,\rangle \in LEV \wedge \\ & \wedge \forall z (z \in w \to z \notin C) \Bigg) \Bigg) \Bigg].\end{aligned}$$

Thus

$$\begin{aligned}\langle\, V_\alpha,\, \in^{[V_\alpha]} \,\rangle \models^2 \ \forall C \Bigg[\Bigg(& C \subseteq T \wedge \forall x \forall y \Bigg(x \in C \wedge y \in C \to \\ & \to \Big(\langle x, y \rangle \in <_T \vee x = y \vee \langle y, x \rangle \in <_T \Big) \Bigg) \Bigg) \to \\ & \to \exists \gamma \exists w \Bigg(\gamma \in b \wedge \langle\, \gamma,\, w \,\rangle \in LEV \wedge \\ & \wedge \forall z (z \in w \to z \notin C) \Bigg) \Bigg].\end{aligned}$$

From which we get by definition of LEV, since $\alpha = \mathbf{ON} \cap V_\alpha$,

$$\begin{aligned}\langle\, V_\alpha,\, \in^{[V_\alpha]} \,\rangle \models^2 \ \forall C \Bigg[\Bigg(& C \subseteq T \wedge \forall x \forall y \Bigg(x \in C \wedge y \in C \to \\ & \to \Big(x <_T y \vee x = y \vee y <_T x \Big) \Bigg) \Bigg) \to \\ & \to \exists \gamma \exists w \Bigg(\gamma \in b \wedge \gamma \in \mathbf{ON} \wedge w = T_{(\gamma)} \wedge \\ & \wedge \forall z (z \in w \to z \notin C) \Bigg) \Bigg].\end{aligned}$$

Let $\xi = b \cap \mathbf{ON}$. We then have

$$\begin{aligned}\langle\, V_\alpha,\, \in^{[V_\alpha]} \,\rangle \models^2 \ \forall C \Bigg[\Bigg(& C \subseteq T \wedge \forall x \forall y \Bigg(x \in C \wedge y \in C \to \\ & \to \Big(x <_T y \vee x = y \vee y <_T x \Big) \Bigg) \Bigg) \to \\ & \to \exists \gamma \exists w \Bigg(\gamma < \xi \wedge w = T_{(\gamma)} \wedge \forall z (z \in w \to z \notin C) \Bigg) \Bigg].\end{aligned}$$

By definition of α-tree we know that $ht(T) = \alpha$ and, again, since $\mathbf{ON} \cap V_\alpha = \alpha$, we have

$$\langle V_\alpha, \in^{[V_\alpha]} \rangle \models^2 \forall\delta(\delta \in \mathbf{ON} \rightarrow T_{(\delta)} \neq \emptyset).$$

And so since $\xi < \alpha$, we have in particular

$$\langle V_\alpha, \in^{[V_\alpha]} \rangle \models^2 T_{(\xi)} \neq \emptyset.$$

Accordingly let $t \in V_\alpha$ be such that

$$\langle V_\alpha, \in^{[V_\alpha]} \rangle \models^2 t \in T_{(\xi)}.$$

For any ordinal β and for any node u such that $ht(u) = \beta$, the definition of "height of u" implies the existence of a bijection of $\{v \in T \mid v <_T u\}$ onto β. Hence for every $\delta < \beta$ there is a unique $<_T$-predecessor v of u such that $ht(v) = \delta$:

$$\langle V_\alpha, \in^{[V_\alpha]} \rangle \models^2 \forall\beta\,\forall u\,\forall\delta\Big[\beta \in \mathbf{ON} \wedge u \in T_{(\beta)} \wedge \delta < \beta \rightarrow \\ \rightarrow \exists!\, v\,(v \in T_{(\delta)} \wedge v <_T u)\Big].$$

But then for $t \in T_{(\xi)}$ there would be a chain

$$C = \{s \in T \mid s <_T t\}$$

such that for every $\gamma < \xi$ there exists exactly one $s \in T_{(\gamma)}$ such that $s <_T t$. Hence we would have that

$$\langle V_\alpha, \in^{[V_\alpha]} \rangle \models^2 \exists C\,\Bigg[C \subseteq T \wedge \forall x\forall y\Big(x \in C \wedge y \in C \rightarrow \\ \rightarrow \Big(x <_T y \vee x = y \vee y <_T x\Big)\Big) \wedge \\ \wedge \forall\gamma\forall w\Big(\gamma < \xi \wedge w = T_{(\gamma)} \rightarrow w \cap C \neq \emptyset\Big)\Bigg],$$

a contradiction.

□

It must be reported, however, that if V_α is an admissible set satisfying the schema of s-Π^1_1 RFN without class-parameters then the tree-property can be shown to hold, at least, for V_α-recursive trees (see Barwise [2], Theorem VIII.7.1, pp. 344-345).

COROLLARY 3.6.7. *If α is s-Π^1_1-indescribable, then either $\alpha = \omega$ or α is a weakly compact cardinal.*

Next is the Hanf-Scott characterization result [12]of a weakly compact cardinal:

THEOREM 3.6.8. *α is a weakly compact cardinal iff α is Π^1_1-indescribable.*

For a proof the reader is referred to Kanamori [16], Theorem 6.4, pp. 59-60. It follows that being a weakly compact cardinal is also a sufficient condition for being S-Π^1_1-indescribable. As well known, ω is Π^1_1-describable, indeed ω is described by the Π_2 sentence $\forall x \exists y (x \in y)$. By contrast, in view of Theorem 3.3.18, we have that

LEMMA 3.6.9. *ω is S-Π^1_1-indescribable.*

We have reached the end of this story. All the above-mentioned observations are synthesized in the statement of Theorem 3.3.18.

Let us conclude this subsection with the following observation:

THEOREM 3.6.10. *The theories sBL_1 and BL_1 admit the same standard models:*

$$\langle V_\kappa, \in^{[V_\kappa]} \rangle \models^2 \mathsf{BL}_1 \quad \Longleftrightarrow \quad \langle V_\kappa, \in^{[V_\kappa]} \rangle \models^2 \mathsf{sBL}_1.$$

Proof. The direction from left to right is trivial. The other direction follows from Corollary 3.6.7 and Theorem 3.6.8. The possibility that $\kappa = \omega$ is ruled out by INFINITY. $\square$

3.7 THE INDEPENDENCE OF PCA

Let

$$(\mathsf{strictBL}_1)^{++} := \mathsf{strictBL}_1 \cup \{\text{INFINITY}\}.$$

LEMMA 3.7.1. *There are instances of the schema of* PCA *whose negation is not derivable in* $(\mathsf{strictBL}_1)^{++}$.

Proof. By the proof of Lemma 3.6.3 we know that if μ is the first weakly compact cardinal then

$$\langle V_\mu, \in^{[V_\mu]} \rangle \models^2 (\mathsf{strictBL}_1)^{++} \qquad \text{and} \qquad \langle V_\mu, \in^{[V_\mu]} \rangle \models^2 \text{PCA}.$$

$\square$

We are left with showing that there are instances of the schema of PCA which are not derivable in $(\mathsf{strictBL}_1)^{++}$.

Let

$$\mathsf{sKPu}^{\mathsf{r}}_2 + \text{INFINITY} \qquad \text{and} \qquad \mathsf{KPu}^{\mathsf{r}} + \text{P} + \text{INFINITY}$$

be the theories obtained from $\mathsf{sKPu}^{\mathsf{r}}_2$ and $\mathsf{KPu}^{\mathsf{r}} + \text{P}$ respectively through the adjunction of the axiom of INFINITY.

LEMMA 3.7.2. *Every theorem* φ *of* $\mathsf{KPu^r} + \mathrm{P} + \textsc{Infinity}$ *is also a theorem of* $\mathsf{sKPu}^\mathsf{r}_2 + \textsc{Infinity}$*,*

$$\mathsf{KPu^r} + \mathrm{P} + \textsc{Infinity} \vdash \varphi \quad \Longrightarrow \quad \mathsf{sKPu}^\mathsf{r}_2 + \textsc{Infinity} \vdash \varphi.$$

From Section 2.4, we know that $\mathsf{sKPu}^\mathsf{r}_2$ conservatively extends $\mathsf{KPu^r} + \mathrm{P}$ for set-theoretic Π_2 sentences. The key point of the present argument consists in showing that such a conservation result also holds when replacing $\mathsf{sKPu}^\mathsf{r}_2$ and $\mathsf{KPu^r} + \mathrm{P}$ by $\mathsf{sKPu}^\mathsf{r}_2 + \textsc{Infinity}$ and $\mathsf{KPu^r} + \mathrm{P} + \textsc{Infinity}$, respectively.

The Tait-style reformulation T_3 of $\mathsf{sKPu}^\mathsf{r}_2 + \textsc{Infinity}$ is the same as for $\mathsf{sKPu}^\mathsf{r}_2$ where the non-logical axiom of INFINITY reads as follows:

For all finite sets Γ of formulae of $\mathcal{L}_2^*$,

$$\Gamma,\ \underbrace{\exists u\,[\,\emptyset \in u \land \forall x(x \in u \rightarrow (x \cup \{x\}) \in u)\,]}_{[\text{S-}\Pi^1_1]^{\mathrm{E}}}.$$

EMBEDDING OF $\mathsf{sKPu}^\mathsf{r}_2 + \textsc{Infinity}$ INTO T_3. *Let* φ *be a* $\mathcal{L}_2^*$ *formula such that*

$$\mathsf{sKPu}^\mathsf{r}_2 + \textsc{Infinity} \vdash \varphi.$$

Then there are two natural numbers n *and* k *such that*

$$\mathsf{T}_3 \vdash^n_k \varphi.$$

Since the non-logical axiom of INFINITY is of logical complexity $[\text{S-}\Pi^1_1]^{\mathrm{E}}$, we then establish a partial cut elimination theorem (up to $[\text{S-}\Pi^1_1]^{\mathrm{E}}$ and $[\text{S-}\Sigma^1_1]^{\mathrm{E}}$ formulae) yielding quasi-normal T_3 derivations exactly as in Section 2.4.

PARTIAL CUT ELIMINATION FOR T_3. *For all finite sets* Γ *of* $\mathcal{L}_2^*$ *formulae and all natural numbers* n *and* k*,*

$$\mathsf{T}_3 \vdash^n_{k+1} \Gamma \quad \Longrightarrow \quad \mathsf{T}_3 \vdash^{2_k(n)}_1 \Gamma.$$

COROLLARY 3.7.3. *Let* φ *be a* $\mathcal{L}_2^*$ *formula such that*

$$\mathsf{sKPu}^\mathsf{r}_2 + \textsc{Infinity} \vdash \varphi.$$

Then there exists a natural number n *such that*

$$\mathsf{T}_3 \vdash^n_1 \varphi.$$

The next step of reducing $\mathsf{sKPu}^\mathsf{r}_2 + \textsc{Infinity}$ to $\mathsf{KPu^r} + \mathrm{P} + \textsc{Infinity}$ consists in setting up a partial model for $\mathsf{sKPu}^\mathsf{r}_2 + \textsc{Infinity}$ (e.g. a model for the set-theoretic Π_2 sentences of $\mathsf{sKPu}^\mathsf{r}_2 + \textsc{Infinity}$) which will subsequently be used in order to prove an asymmetric interpretation theorem for quasi-normal T_3 derivations. It is argued that the whole procedure can be formalized in $\mathsf{KPu^r} + \mathrm{P} + \textsc{Infinity}$. In particular, the partial models needed for such an interpretation are available within the theory $\mathsf{KPu^r} + \mathrm{P} + \textsc{Infinity}$.

For any set z, we define by recursion on n a finite hierarchy $\langle V_n^{\mathsf{N},\omega}(z)\rangle_{n\in\mathbb{N}}$ of set terms $V_n^{\mathsf{N},\omega}(z)$ as follows:

$$V_0^{\mathsf{N},\omega}(z) := \mathsf{TC}(\{\mathsf{N},\omega,z\}),$$

$$V_{n+1}^{\mathsf{N},\omega}(z) := \wp(V_n^{\mathsf{N},\omega}(z)).$$

LEMMA 3.7.4. *For all natural numbers $n \in \mathbb{N}$,*

$$\mathsf{KPu^r} + \mathrm{P} + \textsc{Infinity} \vdash \forall z \mathsf{Tran}(V_n^{\mathsf{N},\omega}(z)).$$

Sets and classes are interpreted, respectively, as elements and subsets of

$$\bigcup_{n\in\mathbb{N}} V_n^{\mathsf{N},\omega}(z).$$

We keep the same notation as in Section 2.4. Let $\varphi(\vec{s},\vec{C})$ be any formula of $\mathcal{L}_2^*$, whose all set and class parameters came from the lists $\vec{s},\vec{C}$ respectively. We write $\varphi^{(V_n^{\mathsf{N},\omega}(z))}(\vec{s},\vec{c})$ to denote the result of replacing in $\varphi(\vec{s},\vec{C})$

- every unbounded set quantifier $\mathcal{Q}x$ by $\mathcal{Q}x \in V_n^{\mathsf{N},\omega}(z)$,
- every class quantifier $\mathcal{Q}Y$ by $\mathcal{Q}y \subseteq V_n^{\mathsf{N},\omega}(z)$,
- every class variable C by a set variable c.

We avoid conflict of variables. Persistence properties are obviously satisfied; we confine ourselves to stating the following result.

COROLLARY 3.7.5. *For any finite set $\Gamma_{\vec{s},\vec{C}}$ of $[\text{S-}\Pi_1^1]^{\mathrm{E}}$ and $[\text{S-}\Sigma_1^1]^{\mathrm{E}}$ formulae of $\mathcal{L}_2^*$, we have:*

$$\begin{aligned}\mathsf{KPu^r} + \mathrm{P} + \textsc{Infinity} \vdash \forall z \forall q \forall r \forall p \forall m \forall \vec{s} \forall \vec{c} \Bigg(\Bigg(& q > r \wedge r > p \wedge p > m \wedge m > 0 \wedge \\ & \wedge \vec{s} \in V_m^{\mathsf{N},\omega}(z) \wedge \vec{c} \subseteq V_q^{\mathsf{N},\omega}(z) \wedge \\ & \wedge \Big[\bigvee \Gamma_{\vec{s},\vec{c}\cap V_r^{\mathsf{N},\omega}(z)}\Big[p,r\Big] \vee \bigvee \Delta\Big]\Bigg) \rightarrow \\ & \rightarrow \Big[\bigvee \Gamma_{\vec{s},\vec{c}}\Big[m,q\Big] \vee \bigvee \Delta\Big]\Bigg).\end{aligned}$$

As for the asymmetric interpretation of T_2 into $\mathsf{KPu^r} + \mathrm{P}$, we interpret any given quasi-normal T_3 derivation of Γ (where Γ only contains $[\text{S-}\Pi_1^1]^{\mathrm{E}}$ and $[\text{S-}\Sigma_1^1]^{\mathrm{E}}$ formulae) by assigning bounds to existential set and universal class quantifiers occurring in the derivation, depending on any given bound for existential class and universal set quantifiers of the derivation.

ASYMMETRIC INTERPRETATION OF T_3 INTO $\mathsf{KPu}^{\mathsf{r}} + \mathrm{P} + \textsc{Infinity}$. *Assume that $\Gamma_{\vec{s},\vec{C}}$ is a finite set of $[\text{s-}\Pi^1_1]^{\mathrm{E}}$ and $[\text{s-}\Sigma^1_1]^{\mathrm{E}}$ formulae of $\mathcal{L}^*_2$ so that*

$$\mathsf{T}_3 \vdash^n_1 \Gamma_{\vec{s},\vec{C}}$$

for some natural number n. Then for all natural numbers $m > 0$ we have

$$\mathsf{KPu}^{\mathsf{r}} + \mathrm{P} + \textsc{Infinity} \vdash \forall z \forall \vec{s} \forall \vec{c} \Big(\vec{s} \in V^{\mathsf{N},\omega}_m(z) \wedge \vec{c} \subseteq V^{\mathsf{N},\omega}_{m+2^n}(z) \rightarrow$$
$$\rightarrow \bigvee \Gamma_{\vec{s},\vec{c}} \Big[m,\, m+2^n \Big] \Big).$$

Proof. By induction on n. Apart from the non-logical axiom of INFINITY, all axioms and rules of inference are treated in exactly the same way as for the asymmetric interpretation of $\mathsf{sKPu}^{\mathsf{r}}_2$.

INFINITY Suppose $\Gamma_{\vec{s},\vec{C}}$ is the non-logical axiom of INFINITY. Then

$$\mathsf{T}_3 \vdash^0_1 \exists u \Big[\exists y (y \in u \wedge \forall w (w \in y \rightarrow w \neq w)) \wedge$$
$$\wedge \forall x (x \in u \rightarrow \exists y (y \in u \wedge \forall w (w \in y \leftrightarrow w = x \vee w \in x))) \Big].$$

Let $m > 0$ be given. We work within $\mathsf{KPu}^{\mathsf{r}} + \mathrm{P} + \textsc{Infinity}$ informally. We have to show, for any z, that

$$\exists u \Big[u \in V^{\mathsf{N},\omega}_{m+1} \wedge \exists y (y \in u \wedge \forall w (w \in y \rightarrow w \neq w)) \wedge$$
$$\wedge \forall x (x \in u \rightarrow \exists y (y \in u \wedge \forall w (w \in y \leftrightarrow w = x \vee w \in x))) \Big].$$

By construction of $\langle V^{\mathsf{N},\omega}_n(z) \rangle_{n \in \mathbb{N}}$, we have that $\omega \in V^{\mathsf{N},\omega}_0(z) \subseteq V^{\mathsf{N},\omega}_{m+1}(z)$. Hence this formula is seen to be true by taking $u = \omega$. □

Π_2-CONSERVATIVITY. $\mathsf{sKPu}^{\mathsf{r}}_2 + \textsc{Infinity}$ *conservatively extends* $\mathsf{KPu}^{\mathsf{r}} + \mathrm{P} + \textsc{Infinity}$ *for set-theoretic Π_2 sentences.*

Proof. Mutatis mutandis analogous to the proof of Π_2-Conservativity for $\mathsf{sKPu}^{\mathsf{r}}_2$. □

THEOREM 3.7.6. *Not every instance of* PCA *is derivable in* $\mathsf{sKPu}^{\mathsf{r}}_2 + \textsc{Infinity}$.

Proof. Suppose not. Let $\mathsf{sKPu}^{\mathsf{r}}_2 + \textsc{Infinity} + \mathrm{PCA}$ denote the augmented theory of $\mathsf{sKPu}^{\mathsf{r}}_2 + \textsc{Infinity}$ obtained by adding any instance of the schema of Predicative Comprehension. By Corollary 3.2.10, we know that VNB is a subsystem of sBL_1. Further, sBL_1 is in turn a subsystem of $\mathsf{sKPu}^{\mathsf{r}}_2 + \textsc{Infinity} + \mathrm{PCA}$. Then we would have VNB as subsystem of $\mathsf{sKPu}^{\mathsf{r}}_2 + \textsc{Infinity} + \mathrm{PCA}$. Arguing along the same line as in the proof of Corollary 2.4.9, then we can show that $V_{\omega}^{\mathsf{N},\omega}$ is a model of all the set-theoretic Π_2 sentences derivable in $\mathsf{sKPu}_2^{\mathsf{r}} + \text{INFINITY} + \text{PCA}$. Henceforth, $V_{\omega}^{\mathsf{N},\omega}$ would be also a model of all the set-theoretic Π_2 sentences derivable in VNB. But in VNB we can prove the existence, for example, of $\omega + \omega$. It would follow that $\omega + \omega \in V_{\omega}^{\mathsf{N},\omega}$. A contradiction. □

LEMMA 3.7.7. *Every theorem φ of $(\mathsf{strictBL}_1)^{++}$ is also a theorem of $\mathsf{sKPu}_2^{\mathsf{r}} + \text{INFINITY}$,*

$$(\mathsf{strictBL}_1)^{++} \vdash \varphi \quad \Longrightarrow \quad \mathsf{sKPu}_2^{\mathsf{r}} + \text{INFINITY} \vdash \varphi.$$

Proof. By Proposition 3.5.2. □

THEOREM 3.7.8. *Not every instance of* PCA *is derivable in* $(\mathsf{strictBL}_1)^{++}$.

Proof. By Lemma 3.7.7 and Theorem 3.7.6. □

COROLLARY 3.7.9. *The schema of* PCA *is independent from* $(\mathsf{strictBL}_1)^{++}$.

Proof. By Theorem 3.7.8 and Lemma 3.7.1. □

3.8 THE CONSISTENCY OF GÖDEL'S AXIOM OF CONSTRUCTIBILITY WITH sBL_1

In this section, the consistency of Gödel's Axiom of Constructibility with the theory sBL_1 will be established. The current task is to show that $\mathsf{sBL}_1 + \mathrm{V{=}L}$ is conservative over sBL_1 for set-theoretic Σ_1 sentences. Although previous conservation results relied on proof-theoretic methods, involving a direct analysis of the structure of the derivations, the present conservation result will be obtained by semantical, i.e. model-theoretic methods. The main technique that is going to be used belongs to *inner model theory*.

DEFINITION 3.8.1. Let **Ax** be a theory formulated in the language $\mathcal{L}_{\in}$. For a proper class A: A is an *inner model* of **Ax** if and only if A is a transitive class, $\mathbf{ON} \subseteq A$ and, for each axiom Ax of **Ax**, $\mathbf{Ax} \vdash (\mathsf{Ax})^A$.

Note that ZF has a trivial inner model, namely **V**. Roughly speaking, inner models are constructed by identifying a certain property of sets and reinterpreting the notion of "set" as "set with that property". What we are going to do is to assume that that the axioms of VNB together with every instance of the schema of S-Π_1^1 RFN (sBL_1) hold true of the universe **V**, and to construct under this assumption an inner model such that the axioms of VNB together with every instance of the schema of S-Π_1^1 RFN plus V=L hold in this inner model. Since we are concerned with class-set theories, in constructing our inner model, we must separately identify both a property φ_0 of sets and a property φ_1 of classes. And then we have to reinterpret the notions of "set" and "class" as "set with the property φ_0" and "class with the property φ_1", respectively. As it will appear clear later, we are going to construct (so to speak) a second-order inner Henkin model. We begin by constructing the first-order part of our model.

Definition 3.8.2. A set y is said to be *first-order definable over a structure* $\mathcal{A} = \langle A, \in^{[A]} \rangle$ *allowing parameters from* A if and only if there exists a first-order formula $\varphi(v_0)$ in the language of $\mathcal{A}$ and with parameters from A and no free variables other than v_0, such that

$$y = \{ a \in A \,|\, \langle A, \in^{[A]} \rangle \models \varphi[a] \}.$$

For any set z,

$$\mathsf{def}(z) := \{ y \subseteq z \,|\, y \text{ is first-order definable over } \langle z, \in^{[z]} \rangle \}.$$

Remark 3.8.3. To be precise, in order for the definition above to make sense we need to know that the syntax and the semantics of the language of $\mathcal{A}$ are formalizable within set theory itself. But since we only need to know that this is possible, not how it may be done, we do not emphasize this for the time being. We have also not bothered to distinguish between an element $a \in A$ and the constant of the language of $\mathcal{A}$ denoting it in the structure $\mathcal{A}$.

Definition 3.8.4. It is well-known that within ZF the notion of "constructible set" is defined in terms of an auxiliary hierarchy of sets, the L_α's, which are defined for all ordinals α, by transfinite recursion in the usual way:

$$\begin{aligned} L_0 &:= \emptyset \\ L_{\alpha+1} &:= \mathsf{def}(L_\alpha) \\ L_\lambda &:= \bigcup_{\alpha<\lambda} L_\alpha, \quad \text{for } \mathsf{Lim}(\lambda). \end{aligned}$$

$\langle L_\alpha \rangle_{\alpha \in \mathbf{ON}}$ is the *constructible hierarchy*.

Definition 3.8.5. The *constructible universe* is the class

$$L := \bigcup_{\alpha \in \mathbf{ON}} L_\alpha.$$

A set is a *constructible set* if and only if it belongs to L. And the assertion $\forall x(x \in L)$ is the *Axiom of Constructibility*, denoted by V=L.

Remark 3.8.6. The property of being constructible is first-order definable in ZF. Therefore L is, by PCA, a class of sBL_1.

We will take L to be the first-order part of our model, i.e. we will reinterpret the notion of "set" as "set with the property of being constructible".

The subsequent Lemma establishes results about the constructible hierarchy and the constructible universe which will be often invoked in the remaining part of our work.

Lemma 3.8.7. *We have:*

(i) $\forall\alpha\forall\beta(\alpha \leq \beta \rightarrow L_\alpha \subseteq L_\beta)$,

(ii) $\forall\alpha(\mathsf{Tran}(L_\alpha))$,

(iii) $\forall\alpha(L_\alpha \subseteq V_\alpha)$,

(iv) $\forall\alpha\forall\beta(\alpha < \beta \rightarrow (\alpha \in L_\beta \wedge L_\alpha \in L_\beta))$,

(v) $\forall\alpha(\alpha \leq \omega \rightarrow L_\alpha = V_\alpha)$,

(vi) $\forall\alpha(\alpha \geq \omega \rightarrow |L_\alpha| = |\alpha|)$,

(vii) *For any axiom* Ax *of* ZF, $\mathsf{ZF} \vdash (\mathsf{Ax})^L$.

For a proof, the reader is referred, for example, to Kunen [17].

REMARK 3.8.8. We remind the reader that we already made use of point (vi) in the proof of Lemma 3.6.5 on page 83. By (ii), we have that $\mathsf{Tran}(L)$ and, by (iv), $\mathbf{ON} \subseteq L$. This two facts along with (vii), tell us that L is an inner model of ZF. In this respect, it worth noticing that the first-order part of the second-order inner Henkin model we are constructing is itself an inner model of ZF.

We now turn to the range of class variables. We will follow Gödel's definition [10], of constructible classes, which are nowadays customarily called "amenable classes".

DEFINITION 3.8.9. We say that a class C is an *amenable class*, denoted by $\mathsf{amenable}(C, L)$, if and only if all its elements are constructible sets and if the intersection of C with any constructible set is also a constructible set, that is

$$\mathsf{amenable}(C, L) := C \subseteq L \wedge \forall u \forall y(u \in L \wedge y = u \cap C \rightarrow y \in L).$$

REMARK 3.8.10. Note that "amenable class" is a well-defined notion in sBL_1, for the intersection of a class with a set is again a set (AUS). Hence, it is already obvious why AUS is going to hold under this particular choice of the interpretation for the class variables (see Lemma 3.8.15 on page 101). Note also that if we were to interpret classes as ranging over arbitrary subclasses of L, (in other words, if we were to adopt a full-interpretation), then AUS would fail: just because C is a subclass of L, there is no reason to suppose that the intersection of C with a constructible set is an element of L at all! Note that $\omega \in L_{\omega+1}$. If $\wp(\omega) \nsubseteq L$, then there is a non-constructible set c of positive integers, hence a subclass of L, and a constructible set, namely ω, such that their intersection, i.e. c itself, is not constructible.

LEMMA 3.8.11. *The following are derivable in* sBL_1:

(i) $\mathsf{amenable}(L, L)$,

(ii) $\forall a(a \in L \rightarrow \mathsf{amenable}(a, L))$,

(iii) $\forall C \forall a(a \in L \wedge \mathsf{amenable}(C, L) \rightarrow \mathsf{amenable}(C \cap a, L))$.

Proof. These are immediate consequences of the definition of amenability.
(i) Obviously, $L \subseteq L$. Let $u \in L$ and $y = u \cap L$ be given. By transitivity of L, $u \subseteq L$. Hence, $y = u \cap L = u$ and $y \in L$.
(ii) For $a \in L$, by transitivity of L, $a \subseteq L$. Let $u \in L$ and $y = u \cap a$ be given. Hence $y \in L$, by Lemma 3.8.7.(vii), i.e. the corresponding instance of the Separation schema of ZF in L.
(iii) Let $a \in L$ and $\mathsf{amenable}(C, L)$ be given. From the amenability of C, we obviously have that $(C \cap a) \subseteq L$. Let $u \in L$ and $y = u \cap (C \cap a)$ be given. We need to show that $y \in L$. Clearly, $u \cap (C \cap a) = C \cap (u \cap a)$. By Lemma 3.8.7.(vii), $(u \cap a)$ is an element of L. Therefore, from the amenability of C itself, it follows that $y \in L$. □

More relevant closure properties of the amenable classes will be analized in the proof of Lemma 3.8.26 on page 109.

We denote the collection of amenable classes by $\mathsf{ac}(L)$:

$$\mathsf{ac}(L) \;:=\; \{\, C \mid \mathsf{amenable}(C, L) \,\}$$

REMARK 3.8.12. Actually, we are being a bit sloppy here: $\mathsf{ac}(L)$ is a family of classes and not a class of sBL_1. Therefore the expression "$B \in \mathsf{ac}(L)$" is, in sBL_1, merely *une façon de parler* for $\mathsf{amenable}(B, L)$ which is a perfectly meaningfull formula of the language $\mathcal{L}_2$.

We will reinterpret the notion of "class" as "class with the property of being amenable". Roughly speaking, we will take $\mathsf{ac}(L)$ to be the second-order part of our inner model.

The classes and sets of our inner model form a subfamily of the classes and sets of the theory sBL_1, and the $\in$-relations of the model are the original $\in$-relations of sBL_1 but restricted to the classes and sets of of our model:

$$\langle\, L,\, \in^{[L]},\, \mathsf{ac}(L),\, \in^{[L \times \mathsf{ac}(L)]} \,\rangle.$$

We adopt the following convention. Let φ be any formula of $\mathcal{L}_2$. We write $(\varphi)^{L,\, \mathsf{ac}(L)}$ to denote the result of replacing in φ

- every unbounded set quantifier $\mathcal{Q}x$, occurring in φ, by $\mathcal{Q}x(x \in L ...)$,
- every class quantifier $\mathcal{Q}Y$, occurring in φ, by $\mathcal{Q}Y(\mathsf{amenable}(Y, L)...)$.

The first step we are going to undertake, and which will be foundamental to all of our next work, consists in establishing the following result:

$$\mathsf{sBL}_1 \vdash \Big(\mathsf{sBL}_1\Big)^{L,\, \mathsf{ac}(L)}.$$

This will be Theorem 3.8.28 on page 111. For each axiom and axiom schema (i.e. each instance thereof) Ax of $\mathsf{sBL_1}$ in turn, we argue in $\mathsf{sBL_1}$ to prove $(\mathsf{Ax})^{L,\,\mathsf{ac}(L)}$. The next two lemmata are standard (see Lemma 3.8.7.(vii)); their corresponding proofs have been included for completeness sake only.

LEMMA 3.8.13.

$$\mathsf{sBL_1} \vdash \Big(\Delta_0\text{-}\mathsf{I}_\in\Big)^{L,\mathsf{ac}(L)}.$$

Proof. We must show that

$$\mathsf{sBL_1} \vdash \Big(\forall a[\exists y(y \in a) \to \exists y(y \in a \wedge \forall z(z \in y \to z \notin a))]\Big)^{L,\mathsf{ac}(L)}.$$

Let us argue informally within the theory $\mathsf{sBL_1}$. Let $a \in L$ be given, $a \neq \emptyset$. We must show that $\Big(\exists y(y \in a \wedge \forall z(z \in y \to z \notin a))\Big)^{L,\mathsf{ac}(L)}$. Note that $\Big(\underbrace{\exists y(y \in a \wedge \forall z(z \in y \to z \notin a))}_{\Delta_0}\Big)^{L,\mathsf{ac}(L)}$. Since $a \in L$, by transitivity of L, $y \in L$ too. Therefore $\Big(\exists y(y \in a \wedge \forall z(z \in y \to z \notin a))\Big)^{L,\mathsf{ac}(L)}$ is the same as $\exists y(y \in a \wedge \forall z(z \in y \to z \notin a))$. Upon the assumptions that $a \in L$ and $a \neq \emptyset$, by Δ_0-$\mathsf{I}_\in$ itself, there is a $y \in a$ such that $\forall z(z \in y \to z \notin a)$. Obviously, y is as required. □

LEMMA 3.8.14.

$$\mathsf{sBL_1} \vdash \Big(\textsc{Infinity}\Big)^{L,\mathsf{ac}(L)}.$$

Proof. We must show that

$$\mathsf{sBL_1} \vdash \Big(\exists z[\exists y(y \in z \wedge \forall w(w \in y \to w \neq w)) \wedge \\ \wedge \forall x(x \in z \to \exists y(y \in z \wedge \forall w(w \in y \leftrightarrow w = x \vee w \in x)))]\Big)^{L,\mathsf{ac}(L)}.$$

Note that

$$\mathsf{sBL_1} \vdash \Big(\exists z[\underbrace{\exists y(y \in z \wedge \forall w(w \in y \to w \neq w))}_{\Delta_0} \wedge \\ \wedge \underbrace{\forall x(x \in z \to \exists y(y \in z \wedge \forall w(w \in y \leftrightarrow w = x \vee w \in x)))}_{\Delta_0}]\Big)^{L,\mathsf{ac}(L)}.$$

Let us argue informally within the theory sBL_1. Since Δ_0-formulae are closed under conjunction, all we are left with is finding a $z \in L$ such that

$$\begin{aligned} &\exists y(y \in z \wedge \forall w(w \in y \to w \neq w)) \wedge \\ &\wedge \forall x(x \in z \to \exists y(y \in z \wedge \forall w(w \in y \leftrightarrow w = x \vee w \in x))). \end{aligned}$$

And, for $\omega \in L_{\omega+1} \subseteq L$, this is seen to be true of L by taking $z = \omega$. ☐

LEMMA 3.8.15.

$$\mathsf{sBL}_1 \vdash \Big(\text{AUS}\Big)^{L,\mathsf{ac}(L)}.$$

Proof. We must show that

$$\mathsf{sBL}_1 \vdash \Big(\forall C \forall a \exists y \forall x(x \in y \leftrightarrow x \in a \wedge x \in C)\Big)^{L,\mathsf{ac}(L)}.$$

Let us argue informally within the theory sBL_1. Let $C \in \mathsf{ac}(L)$, $a \in L$ be given. We seek a $y \in L$ such that $\Big(\forall x(x \in y \leftrightarrow x \in a \wedge x \in C)\Big)^{L,\mathsf{ac}(L)}$. From $C \in \mathsf{ac}(L)$ it follows, in particular, that $\forall u \forall y(u \in L \wedge y = u \cap C \to y \in L)$. From this, using the assumption $a \in L$, we obtain $\forall y(y = a \cap C \to y \in L)$. By AUS itself, there is a y such that $\forall x(x \in y \leftrightarrow x \in a \wedge x \in C)$. Therefore $y \in L$. ☐

We are left with showing that the schemata of S-Π^1_1 RFN and PCA (i.e. each instance thereof) hold true of our inner model. We take up the schema of S-Π^1_1 RFN first. The argument used in the proof of Lemma 3.8.25 on page 104 appeals to arguably one of the most relevant results in constructibility theory.

DEFINITION 3.8.16. $\beth_\alpha$ is defined by transfinite recursion on α:

$$\begin{aligned} \beth_0 &:= \omega \\ \beth_{\alpha+1} &:= 2^{\beth_\alpha} \\ \beth_\lambda &:= \mathsf{sup}\{\beth_\alpha | \alpha < \lambda\}, \quad \text{for } \mathsf{Lim}(\lambda). \end{aligned}$$

DEFINITION 3.8.17. The *Continuum Hypothesis*, denoted by CH, is the assertion:

$$\omega_1 = \beth_1.$$

The *Generalized Continuum Hypothesis*, denoted by GCH, is the assertion:

$$\forall \alpha(\omega_\alpha = \beth_\alpha).$$

Here is an outline of the proof that V=L implies CH; similar reasoning establishes GCH. All the details of this argument are discussed at length in Gödel [10]. It is well-known that the constructible hierarchy grows at a much slower rate than the cumulative hierarchy. While $|V_{\omega+2}|$ is already as big as $2^{\beth_1}$,

$L_{\omega+2}$ is countable. In fact, $\wp(\omega) \in V_{\omega+2}$, but only denumerably many subsets of ω are in $L_{\omega+2}$. More, but not of all them, will appear at the next level, and so on. Hence we ask: how far along in the constructible hierarchy might we still be getting new subsets of ω? As shown by Gödel [10], in proving that V=L implies GCH, there is a bound to this gradual-growth process: $\wp(\omega) \subseteq L_{\omega_1}$, i.e. any subset of ω is constructed at some countable stage. Hence

$$|\wp(\omega)| = \beth_1 \leq |L_{\omega_1}| = \omega_1.$$

More generally, the following holds

THEOREM 3.8.18. *Let μ be a cardinal. If $x \in \wp(L_\alpha) \cap L$ for some $\alpha < \mu$, then $x \in L_\mu$.*

Obviously, if $\omega \geq \mu$, the result holds trivially, for $L_\mu = V_\mu$. This result tells us, that the cardinal levels of the constructible hierarchy are "super-transitive": closed under the constructible subsets of its elements. We will see the relevance of Thereom 3.8.18 in the proof of Lemma 3.8.22. We start by making a preliminary observation.

LEMMA 3.8.19.

$$\mathsf{sBL}_1 \vdash (\forall a \in L)(\wp(a) \cap L = \wp(a) \cap \mathsf{ac}(L)).$$

Proof. Let us argue informally within the theory sBL_1. Let $a \in L$ be given. By Lemma 3.8.11.(ii) we have

$$\wp(a) \cap L \subseteq \wp(a) \cap \mathsf{ac}(L).$$

Next we prove that

$$\wp(a) \cap \mathsf{ac}(L) \subseteq \wp(a) \cap L.$$

Let $x \in \wp(a) \cap \mathsf{ac}(L)$ be given. We must show that $x \in \wp(a) \cap L$. By making explicit the assumption "$x \in \wp(a) \cap \mathsf{ac}(L)$" we get

$$x \subseteq a \cap L \wedge \forall u \forall y (u \in L \wedge y = u \cap x \rightarrow y \in L).$$

Since $a \in L$, by transitivity of L, we have that $a \subseteq L$. Hence we obtain

$$x \subseteq a \wedge \forall u \forall y (u \in L \wedge y = u \cap x \rightarrow y \in L).$$

This last expression entails, in particular, that

$$(a \in L \wedge x = a \cap x \rightarrow x \in L).$$

At this stage we note that, by assumption, $a \in L$. And, since $x \subseteq a$, then $x = a \cap x$. Thus, $x \in L$. $\square$

COROLLARY 3.8.20.

$$\mathsf{sBL}_1 \vdash (\forall \alpha \in \mathbf{ON})(\wp(L_\alpha) \cap L = \wp(L_\alpha) \cap \mathsf{ac}(L)).$$

Proof. By Lemma 3.8.19, along with the observation that $L_\alpha \in L$. ☐

DEFINITION 3.8.21. For any $a \in L$ and for any set c, we define

$$\begin{aligned} \mathsf{amenable}(c,a) &:= c \subseteq a \wedge \forall u \forall y (u \in a \wedge y = u \cap c \rightarrow y \in a), \\ \mathsf{ac}(a) &:= \{\, c \mid \mathsf{amenable}(c,a) \,\}. \end{aligned}$$

Note that $\mathsf{ac}(a)$ is a set, for $\mathsf{ac}(a) \subseteq \wp(a)$.

LEMMA 3.8.22. *The following set-theoretic inclusion is provable in* sBL_1*:*
For any cardinal μ*,*

$$\wp(L_\mu) \cap \mathsf{ac}(L) \subseteq \mathsf{ac}(L_\mu).$$

Proof. By Corollary 3.8.20,

$$\wp(L_\mu) \cap L = \wp(L_\mu) \cap \mathsf{ac}(L).$$

We show that

$$\wp(L_\mu) \cap L \subseteq \mathsf{ac}(L_\mu).$$

Let $w \in \wp(L_\mu) \cap L$ be given. Obviously, $w \subseteq L_\mu$. We need to show that

$$\forall u \forall y (u \in L_\mu \wedge y = u \cap w \rightarrow y \in L_\mu).$$

Let $u \in L_\mu$ and $y = u \cap w$ be given. We distinguish between two cases:

$\underline{\mu \leq \omega}$ Then $L_\mu = V_\mu$. The result follows at once.

$\underline{\omega < \mu}$ Then μ is a limit ordinal. By the recursive definition of the constructible hierarchy, $u \in L_\alpha$, for some $\alpha < \mu$. Further $y \in L$, for y is the intersection of two constructible sets. By transitivity of L_α, we have that $u \subseteq L_\alpha$. Therefore, $y \subseteq u \subseteq L_\alpha$. It follows that $y \in \wp(L_\alpha) \cap L$ for some $\alpha < \mu$. By Theorem 3.8.18, $y \in L_\mu$. ☐

The set-theoretic inclusion established in Lemma 3.8.22 fails for ordinals which are not cardinals.

LEMMA 3.8.23.

$$\mathsf{sBL}_1 \vdash \wp(L_{\omega+1}) \cap L \nsubseteq \mathsf{ac}(L_{\omega+1}).$$

Proof. Let γ be the least ordinal greater than $\omega + 1$ for which

$$\wp(\omega) \cap (L_\gamma \setminus L_{\omega+1}) \neq \emptyset.$$

Such a γ must exist, for otherwise $\wp(\omega) \cap L \subseteq L_{\omega+1}$. But this then would violate the fact that $|L_{\omega+1}| = \aleph_0$ in L, for $|\wp(\omega) \cap L| > \aleph_0$ in L (The Cantor Theorem holds in L). Let x be an element of $\wp(\omega)$ such that $x \in (L_\gamma \setminus L_{\omega+1})$. Such an x must exist, for $\wp(\omega) \cap (L_\gamma \setminus L_{\omega+1}) \neq \emptyset$. Since $\wp(\omega) \subseteq \wp(L_{\omega+1})$, $x \in \wp(L_{\omega+1}) \cap L$. If $x \in \mathsf{ac}(L_{\omega+1})$, then

$$\forall u \forall y (u \in L_{\omega+1} \wedge y = u \cap w \rightarrow y \in L_{\omega+1}).$$

And this obviously entails $(\omega \in L_{\omega+1} \wedge x = \omega \cap x \rightarrow x \in L_{\omega+1})$. Note that $\omega \in L_{\omega+1}$ and, for $x \subseteq \omega$, $x = \omega \cap x$. Therefore, $x \in L_{\omega+1}$. But this contradicts the choice of x. □

For the proof of Lemma 3.8.25 on this page, the following result will be usefull too. See, for example, Chang and Keisler [6], p. 560.

LEMMA 3.8.24.

$$\forall \alpha (V_{\beth_\alpha} \cap L = L_{\beth_\alpha})$$

LEMMA 3.8.25.

$$\mathsf{sBL}_1 \vdash \Big(\text{S-}\Pi^1_1 \ \text{RFN}\Big)^{L,\mathsf{ac}(L)}.$$

Proof. We must show that if $\varphi(v_0, ..., v_n, C_0, ..., C_m)$ is any S-Π^1_1 formula of $\mathcal{L}_2$ in which z does not occur free and with no free variables besides the displayed ones and not necessarily all of them, then

$$\mathsf{sBL}_1 \vdash \Bigg(\forall v_0 ... \forall v_n \forall C_0 ... \forall C_m \Big(\varphi(v_0, ..., v_n, C_0, ..., C_m) \rightarrow$$

$$\rightarrow \exists z[\mathsf{Tran}(z) \wedge v_0, ..., v_n \in z \wedge \varphi^{(z)}(v_0, ..., v_n, C_0, ..., C_m)]\Big)\Bigg)^{L,\mathsf{ac}(L)}.$$

Let us argue informally within the theory sBL_1. Let $a_0, ..., a_n \in L$, $B_0, ..., B_m \in \mathsf{ac}(L)$ be given. We must check that

$$\Big(\varphi(a_0, ..., a_n, B_0, ..., B_m) \rightarrow$$

$$\rightarrow \exists z[\mathsf{Tran}(z) \wedge a_0, ..., a_n \in z \wedge \varphi^{(z)}(a_0, ..., a_n, B_0, ..., B_m)]\Big)^{L,\mathsf{ac}(L)}.$$

That is

$$\Big(\varphi(a_0, ..., a_n, B_0, ..., B_m)\Big)^{L,\mathsf{ac}(L)} \rightarrow$$

$$\rightarrow \Big(\exists z[\mathsf{Tran}(z) \wedge a_0, ..., a_n \in z \wedge \varphi^{(z)}(a_0, ..., a_n, B_0, ..., B_m)]\Big)^{L,\mathsf{ac}(L)}.$$

By definition of S-Π^1_1 formula we know that

$$\varphi(v_0, ..., v_n, C_0, ..., C_m) \equiv \forall W \psi(W, v_0, ..., v_n, C_0, ..., C_m),$$

where ψ has logical complexity Σ. Therefore under the assumption that

$$\forall W(\mathsf{amenable}(W, L) \rightarrow \psi^{(L)}(W, a_0, ..., a_n, B_0, ..., B_m)),$$

we must show, using Proposition 1.2.9, that

$$\Big(\exists z[\mathsf{Tran}(z) \wedge a_0, ..., a_n \in z \wedge$$
$$\wedge \forall W(W \subseteq z \to \psi^{(z)}(W, a_0, ..., a_n, B_0 \cap z, ..., B_m \cap z))]\Big)^{L,\mathsf{ac}(L)}.$$

This means that we seek a set $z \in L$ such that $\mathsf{Tran}(z)$ and $a_0, ..., a_n \in z$ and

$$\Big(\forall W(W \subseteq z \to \psi^{(z)}(W, a_0, ..., a_n, B_0 \cap z, ..., B_m \cap z))\Big)^{L,\mathsf{ac}(L)}.$$

That is

$$\forall W\Big(W \subseteq z \wedge \mathsf{amenable}(W, L) \to \psi^{(z\cap L)}(W, a_0, ..., a_n, B_0 \cap z, ..., B_m \cap z)\Big).$$

By making explicit the definition of "$\mathsf{amenable}(W, L)$" and in virtue of Proposition 2.1.3, then this formula is shown to be provably equivalent to

$$\forall w\Big(w \subseteq z \wedge w \subseteq L \wedge \forall u \forall y(u \in L \wedge y = w \cap u \to y \in L) \to$$
$$\to \psi^{(z\cap L)}(w, a_0, ..., a_n, B_0 \cap z, ..., B_m \cap z)\Big).$$

By transitivity of L, $z \subseteq L$. And for $z \subseteq L$, the formula above is shown to be equivalent to

$$\forall w\Big(w \subseteq z \wedge \forall u \forall y(u \in L \wedge y = w \cap u \to y \in L) \to$$
$$\to \psi^{(z)}(w, a_0, ..., a_n, B_0 \cap z, ..., B_m \cap z)\Big). \tag{1}$$

By Lemma 3.8.11.(iii), we clearly have that $B_i \cap z$, with $0 \le i \le m$, is an element of $\wp(z) \cap \mathsf{ac}(L)$. Therefore (1) is just another way of saying that

$$\langle\, z,\, \in^{[z]},\, \wp(z) \cap \mathsf{ac}(L),\, \in^{[z\times(\wp(z)\cap\mathsf{ac}(L))]}\,\rangle \models$$
$$\forall W \psi[W, a_0, ..., a_n, B_0 \cap z, ..., B_m \cap z].$$

Hence to sum up, for under the assumptions that $a_0, ..., a_n \in L$, $B_0, ..., B_m \in \mathsf{ac}(L)$ and

$$\forall W(\mathsf{amenable}(W, L) \to \psi^{(L)}(W, a_0, ..., a_n, B_0, ..., B_m)),$$

we must find a $z \in L$ such that $\mathsf{Tran}(z)$, $a_0, ..., a_n \in z$ and

$$\langle\, z,\, \in^{[z]},\, \wp(z) \cap \mathsf{ac}(L),\, \in^{[z\times(\wp(z)\cap\mathsf{ac}(L))]}\,\rangle \models$$
$$\forall W \psi[W, a_0, ..., a_n, B_0 \cap z, ..., B_m \cap z],$$

or equivalently, by Lemma 3.8.19,

$$\langle\, z,\, \in^{[z]},\, \wp(z) \cap L,\, \in^{[z \times (\wp(z) \cap L)]}\, \rangle \models$$

$$\forall W \psi[W, a_0, ..., a_n, B_0 \cap z, ..., B_m \cap z].$$

Before starting, let us recall the definition of "$\mathsf{amenable}(C, L)$":

$$\underbrace{\forall x(x \in C \to x \in L) \land \forall u \forall y((u \in L \land \forall x(x \in y \leftrightarrow x \in u \land x \in C)) \to y \in L)}_{\Pi^{\mathrm{C}}}.$$

Therefore

$$\underbrace{\forall W\big(\underbrace{\mathsf{amenable}(W, L)}_{\Pi^{\mathrm{C}}} \to \underbrace{\psi^{(L)}(W, a_0, ..., a_n, B_0, ..., B_m)}_{\Sigma^{\mathrm{C}}}\big)}_{\text{S-}\Pi^1_1}.$$

And we know this formula to hold true of $\mathbf{V}$. Therefore, by $\text{S-}\Pi^1_1$ Rfn itself, there exists a reflecting transitive set b such that $a_0, ..., a_n \in b$ and

$$\langle\, b,\, \in^{[b]},\, \wp(b),\, \in^{[b \times \wp(b)]}\, \rangle \models \forall W\big(\mathsf{amenable}(W, L \cap b) \to$$

$$\to \psi^{(L \cap b)}[W, a_0, ..., a_n, B_0 \cap b, ..., B_m \cap b]\big).$$

The reflecting transitive set b will appear in V_κ, for some ordinal κ. By transitivity of V_κ we then have that $b \subseteq V_\kappa$. At this stage, we consider the cardinal $\beth_\kappa$. Then, obviously, $b \subseteq V_\kappa \subseteq V_{\beth_\kappa}$. Hence

$$\langle\, b,\, \in^{[b]},\, \wp(b),\, \in^{[b \times \wp(b)]}\, \rangle \models \forall W\big(\mathsf{amenable}(W, L \cap (b \cap V_{\beth_\kappa})) \to$$

$$\to \psi^{(L \cap (b \cap V_{\beth_\kappa}))}[W, a_0, ..., a_n, B_0 \cap (b \cap V_{\beth_\kappa}), ..., B_m \cap (b \cap V_{\beth_\kappa})]\big).$$

That is

$$\langle\, b,\, \in^{[b]},\, \wp(b),\, \in^{[b \times \wp(b)]}\, \rangle \models \forall W\big(\mathsf{amenable}(W, (L \cap V_{\beth_\kappa}) \cap b) \to$$

$$\to \psi^{((L \cap V_{\beth_\kappa}) \cap b)}[W, a_0, ..., a_n, (B_0 \cap V_{\beth_\kappa}) \cap b, ..., (B_m \cap V_{\beth_\kappa}) \cap b]\big).$$

Thus, by Upward Persistency, we have

$$\langle\, V_{\beth_\kappa},\, \in^{[V_{\beth_\kappa}]},\, \wp(V_{\beth_\kappa}),\, \in^{[V_{\beth_\kappa} \times \wp(V_{\beth_\kappa})]}\, \rangle \models \forall W\big(\mathsf{amenable}(W, L \cap V_{\beth_\kappa}) \to$$

$$\to \psi^{(L \cap V_{\beth_\kappa})}[W, a_0, ..., a_n, B_0 \cap V_{\beth_\kappa}, ..., B_m \cap V_{\beth_\kappa}]\big).$$

By making explicit the definition of "$\mathsf{amenable}(W, L \cap V_{\beth_\kappa})$" we get

$$\langle\, V_{\beth_\kappa},\, \in^{[V_{\beth_\kappa}]},\, \wp(V_{\beth_\kappa}),\, \in^{[V_{\beth_\kappa} \times \wp(V_{\beth_\kappa})]}\, \rangle \models \forall W\Big(\forall v(v \in W \to v \in L \cap V_{\beth_\kappa}) \land$$

$$\land \forall u \forall y(u \in L \cap V_{\beth_\kappa} \land \forall x(x \in y \leftrightarrow x \in u \land x \in W) \to y \in L \cap V_{\beth_\kappa}) \to$$

$$\to \psi^{(L \cap V_{\beth_\kappa})}[W, a_0, ..., a_n, B_0 \cap V_{\beth_\kappa}, ..., B_m \cap V_{\beth_\kappa}]\Big).$$

By relativizing this formula to the set $V_{\beth_\kappa}$ we then obtain

$$\begin{aligned}
\forall w\Bigg(& w \subseteq V_{\beth_\kappa} \to \Bigg(\Bigg(\forall v(v \in w \to v \in L \cap V_{\beth_\kappa}) \wedge \\
& \wedge \forall u \forall y (u \in L \cap V_{\beth_\kappa} \wedge y \in V_{\beth_\kappa} \wedge \\
& \wedge \forall x (x \in y \leftrightarrow x \in u \wedge x \in w) \to y \in L \cap V_{\beth_\kappa})\Bigg) \to \\
& \to \psi^{(L \cap V_{\beth_\kappa})}(w, a_0, ..., a_n, B_0 \cap V_{\beth_\kappa}, ..., B_m \cap V_{\beth_\kappa})\Bigg)\Bigg).
\end{aligned}$$

Note that "$\mathsf{amenable}(B_i, L)$", with $0 \leq i \leq m$. And this entails, in particular, that $B_i \subseteq L$. Hence we can rewrite this last expression as

$$\begin{aligned}
\forall w\Bigg(& w \subseteq V_{\beth_\kappa} \to \Bigg(\Bigg(\forall v(v \in w \to v \in L \cap V_{\beth_\kappa}) \wedge \\
& \wedge \forall u \forall y (u \in L \cap V_{\beth_\kappa} \wedge y \in V_{\beth_\kappa} \wedge \\
& \wedge \forall x (x \in y \leftrightarrow x \in u \wedge x \in w) \to y \in L \cap V_{\beth_\kappa})\Bigg) \to \\
& \to \psi^{(L \cap V_{\beth_\kappa})}(w, a_0, ..., a_n, (B_0 \cap L) \cap V_{\beth_\kappa}, ..., (B_m \cap L) \cap V_{\beth_\kappa})\Bigg)\Bigg).
\end{aligned}$$

That is

$$\begin{aligned}
\forall w\Bigg(& w \subseteq V_{\beth_\kappa} \to \Bigg(\Bigg(\forall v(v \in w \to v \in L \cap V_{\beth_\kappa}) \wedge \\
& \wedge \forall u \forall y (u \in L \cap V_{\beth_\kappa} \wedge y \in V_{\beth_\kappa} \wedge \\
& \wedge \forall x (x \in y \leftrightarrow x \in u \wedge x \in w) \to y \in L \cap V_{\beth_\kappa})\Bigg) \to \\
& \to \psi^{(L \cap V_{\beth_\kappa})}(w, a_0, ..., a_n, B_0 \cap (L \cap V_{\beth_\kappa}), ..., B_m \cap (L \cap V_{\beth_\kappa})\Bigg)\Bigg).
\end{aligned}$$

By Lemma 3.8.24, we have that $V_{\beth_\kappa} \cap L = L_{\beth_\kappa}$. Thus

$$\begin{aligned}
\forall w\Bigg(\Bigg(& w \subseteq V_{\beth_\kappa} \wedge w \subseteq L_{\beth_\kappa} \wedge \\
& \wedge \forall u \forall y (u \in L_{\beth_\kappa} \wedge y \in V_{\beth_\kappa} \wedge y = u \cap w \to y \in L_{\beth_\kappa})\Bigg) \to \\
& \to \psi^{(L_{\beth_\kappa})}(w, a_0, ..., a_n, B_0 \cap L_{\beth_\kappa}, ..., B_m \cap L_{\beth_\kappa})\Bigg).
\end{aligned}$$

This last expression logically entails the following:

$$\forall w\Bigg(\Big(w \subseteq V_{\beth_\kappa} \cap L_{\beth_\kappa} \wedge \forall u \forall y(u \in L_{\beth_\kappa} \wedge y = u \cap w \rightarrow y \in L_{\beth_\kappa})\Big) \rightarrow$$

$$\rightarrow \psi^{(L_{\beth_\kappa})}(w, a_0, ..., a_n, B_0 \cap L_{\beth_\kappa}, ..., B_m \cap L_{\beth_\kappa})\Bigg).$$

By Lemma 3.8.7.(iii), we have that $L_{\beth_\kappa} \subseteq V_{\beth_\kappa}$. Therefore,

$$\forall w\Bigg(\Big(w \subseteq L_{\beth_\kappa} \wedge \forall u \forall y(u \in L_{\beth_\kappa} \wedge y = u \cap w \rightarrow y \in L_{\beth_\kappa})\Big) \rightarrow$$

$$\rightarrow \psi^{(L_{\beth_\kappa})}(w, a_0, ..., a_n, B_0 \cap L_{\beth_\kappa}, ..., B_m \cap L_{\beth_\kappa})\Bigg).$$

That is

$$\forall w(\mathsf{amenable}(w, L_{\beth_\kappa}) \rightarrow \psi^{(L_{\beth_\kappa})}(w, a_0, ..., a_n, B_0 \cap L_{\beth_\kappa}, ..., B_m \cap L_{\beth_\kappa})).$$

By Lemma 3.8.11.(iii), we clearly have that $B_i \cap L_{\beth_\kappa}$, with $0 \leq i \leq m$, is an element of $\wp(L_{\beth_\kappa}) \cap \mathsf{ac}(L)$. By lemma 3.8.22,

$$\wp(L_{\beth_\kappa}) \cap \mathsf{ac}(L) \subseteq \mathsf{ac}(L_{\beth_\kappa}).$$

Therefore,

$$\forall w(w \subseteq L_{\beth_\kappa} \wedge \mathsf{amenable}(w, L) \rightarrow \psi^{(L_{\beth_\kappa})}(w, a_0, ..., a_n, B_0 \cap L_{\beth_\kappa}, ..., B_m \cap L_{\beth_\kappa})).$$

That is,

$$\langle L_{\beth_\kappa}, \in^{[L_{\beth_\kappa}]}, \wp(L_{\beth_\kappa}) \cap \mathsf{ac}(L), \in^{[L_{\beth_\kappa} \times (\wp(L_{\beth_\kappa}) \cap \mathsf{ac}(L))]} \rangle \models$$

$$\forall W \psi[W, a_0, ..., a_n, B_0 \cap L_{\beth_\kappa}, ..., B_m \cap L_{\beth_\kappa}].$$

Which is, by Corollary 3.8.20, equivalent to

$$\langle L_{\beth_\kappa}, \in^{[L_{\beth_\kappa}]}, \wp(L_{\beth_\kappa}) \cap L, \in^{[L_{\beth_\kappa} \times (\wp(L_{\beth_\kappa}) \cap L)]} \rangle \models$$

$$\forall W \psi[W, a_0, ..., a_n, B_0 \cap L_{\beth_\kappa}, ..., B_m \cap L_{\beth_\kappa}].$$

Clearly $L_{\beth_\kappa}$ is as required. □

Next is the schema of Predicative Comprehension (PCA). In showing that each istance of PCA hold true of our inner model (see proof of Lemma 3.8.26 on the next page) we shall make use of the following well-known result:

The generalized reflection principle of VNB. *Let Z be a class and, for each $\alpha \in \mathbf{ON}$, Z_α is a transitive set. Suppose also*

- $\forall\alpha\forall\beta(\alpha \leq \beta \to Z_\alpha \subseteq Z_\beta)$,
- $\forall\lambda(\mathsf{lim}(\lambda) \to Z_\lambda = \bigcup_{\alpha<\lambda} Z_\alpha)$,
- $Z = \bigcup_{\alpha\in\mathbf{ON}} Z_\alpha$.

Let $\varphi(v_0, .., v_n, C_0, ..., C_m)$ *be a predicative formula of* $\mathcal{L}_2$ *with no variables besides the displayed ones free and not necessarily all of them. Then the following is a theorem of* VNB:

$$\forall\alpha\exists\beta\Big[\alpha < \beta \wedge \mathsf{lim}(\beta) \wedge \forall v_0...\forall v_n(v_0, .., v_n \in Z_\beta \to$$

$$\to (\varphi^{Z}(v_0, .., v_n, C_0, ..., C_m) \leftrightarrow \varphi^{Z_\beta}(v_0, .., v_n, C_0 \cap Z_\beta, ..., C_m \cap Z_\beta)))\Big].$$

The proof is just as for the Generalized Reflection Principle of ZF[1], with an additional base case, for the atomic formula "$v \in C$", thrown into the inductive argument. For a detailed argument the redear is referred, for example, to Gloede [9].

LEMMA 3.8.26.

$$\mathsf{sBL}_1 \vdash \Big(\mathrm{PCA}\Big)^{L,\mathsf{ac}(L)}.$$

Proof. Let $\varphi(v_0, ..., v_n, C_0, ..., C_m)$ be a predicative formula of $\mathcal{L}_2$ with no variables besides the displayed ones free and not necessarily all of them. We must check that

$$\mathsf{sBL}_1 \vdash \Big(\forall v_0...\forall v_n\forall C_0...\forall C_m\Big(\exists Y\forall x(x \in Y \leftrightarrow \varphi(x, v_0, ..., v_n, C_0, ..., C_m))\Big)\Big)^{L,\mathsf{ac}(L)}.$$

Let us argue informally within the theory sBL_1. Let $a_0, ..., a_n \in L$, $B_0, ..., B_m \in \mathsf{ac}(L)$ be given. We must show that

$$\exists Y\Big(\mathsf{amenable}(Y, L) \wedge \forall x(x \in L \to (x \in Y \leftrightarrow \varphi^{(L)}(x, a_0, ..., a_n, B_0, ..., B_m)))\Big).$$

By making explicit the definition of "$\mathsf{amenable}(Y, L)$", we have

$$\exists Y\Big(Y \subseteq L \wedge \forall u\forall y(u \in L \wedge y = u \cap Y \to y \in L) \wedge$$

$$\wedge \forall x(x \in L \to (x \in Y \leftrightarrow \varphi^{(L)}(x, a_0, ..., a_n, B_0, ..., B_m)))\Big).$$

That is

$$\exists Y\Big(Y \subseteq L \wedge \forall u\forall y(u \in L \wedge y = u \cap Y \to y \in L) \wedge$$

$$\wedge \forall x(x \in Y \wedge x \in L \leftrightarrow x \in L \wedge \varphi^{(L)}(x, a_0, ..., a_n, B_0, ..., B_m))\Big),$$

---

[1] Of course, by "class" in ZF, it is meant a definable class, i.e a class abstract $\mathbf{Z} = \{x|\varphi(x, v_0, .., v_n)\}$ where φ is a formula of $\mathcal{L}_\in$.

which is, in turn, equivalent to

$$\exists Y\Big(Y \subseteq L \wedge \forall u \forall y(u \in L \wedge y = u \cap Y \to y \in L) \wedge \\ \wedge \forall x(x \in Y \leftrightarrow x \in L \wedge \varphi^{(L)}(x, a_0, ..., a_n, B_0, ..., B_m))\Big). \tag{1}$$

By PCA itself we clearly have

$$\exists Y \forall x(x \in Y \leftrightarrow x \in L \wedge \varphi^{(L)}(x, a_0, ..., a_n, B_0, ..., B_m)).$$

And this, in turn, logically entails the following:

$$\exists Y\Big(Y \subseteq L \wedge \forall x(x \in Y \leftrightarrow x \in L \wedge \varphi^{(L)}(x, a_0, ..., a_n, B_0, ..., B_m))\Big).$$

Thus, having shown that

$$\exists Y\Big(Y \subseteq L \wedge Y = \{\, x \in L \,|\, \varphi^{(L)}(x, a_0, ..., a_n, B_0, ..., B_m) \,\}\Big), \tag{2}$$

next we prove that

$$\forall Y\Big(\Big(Y \subseteq L \wedge Y = \{\, x \in L \,|\, \varphi^{(L)}(x, a_0, ..., a_n, B_0, ..., B_m) \,\}\Big) \to \\ \to \forall u \forall y(u \in L \wedge y = u \cap Y \to y \in L)\Big). \tag{3}$$

Obviously, (2) and (3) logically entail (1). So, let

$$Y \subseteq L \wedge Y = \{\, x \in L \,|\, \varphi^{(L)}(x, a_0, ..., a_n, B_0, ..., B_m) \,\}$$

be given. We need to show that

$$\forall u \forall y(u \in L \wedge y = u \cap Y \to y \in L).$$

So let $u \in L$ and $y = u \cap Y$ be given. We need to show that $y \in L$. Note that

$$\begin{aligned} y &= u \cap Y \\ &= \{\, z \,|\, z \in Y \wedge z \in u \,\} \\ &= \{\, z \,|\, z \in \{\, x \in L \,|\, \varphi^{(L)}(x, a_0, ..., a_n, B_0, ..., B_m) \,\} \wedge z \in u \,\} \\ &= \{\, z \,|\, z \in L \wedge \varphi^{(L)}(z, a_0, ..., a_n, B_0, ..., B_m) \wedge z \in u \,\} \\ &= \{\, z \,|\, z \in L \cap u \wedge \varphi^{(L)}(z, a_0, ..., a_n, B_0, ..., B_m) \,\} \\ &= \{\, z \,|\, z \in u \wedge \varphi^{(L)}(z, a_0, ..., a_n, B_0, ..., B_m) \,\}. \end{aligned}$$

The last equality holds because, by assumption, we know that $u \in L$. And, by transitivity of L, $u \subseteq L$. At this point, it is worth pausing a moment to note the following. A schema of Predicative Comprehension, where class-parameters are not allowed to appear in the corresponding defining formulae is easily seen to hold true of our inner model, for

$$y = \{\, z \mid z \in u \wedge \varphi^{(L)}(z, a_0, ..., a_n) \,\}$$

is a constructibe set, by the corresponding instance of the schema of Separation of ZF in L (Lemma 3.8.7.(vii)). To overcome the difficulty given by the presence of class-parameters in the formula φ we use the Generalized Reflection Principle of VNB. Fix an α so that $u, a_0, ..., a_n \in L_\alpha$. By applying the Generalized Reflection Principle of VNB to the constructible hierarchy, we can find a $\beta > \alpha$ such that

$$\begin{aligned}(\forall z, a_0, ..., a_n, u \in L_\beta)((z \in u \wedge \varphi(z, a_0, ..., a_n, B_0, ..., B_m))^{(L)} \leftrightarrow \\ \leftrightarrow (z \in u \wedge \varphi(z, a_0, ..., a_n, B_0 \cap L_\beta, ..., B_m \cap L_\beta))^{(L_\beta)}).\end{aligned}$$

And this last expression, along with

$$y = \{\, z \mid z \in L_\beta \wedge (z \in u \wedge \varphi(z, a_0, ..., a_n, B_0, ..., B_m))^{(L)} \,\},$$

logically entails the following

$$\begin{aligned} y &= \{\, z \mid z \in L_\beta \wedge (z \in u \wedge \varphi(z, a_0, ..., a_n, B_0 \cap L_\beta, ..., B_m \cap L_\beta))^{(L_\beta)} \,\} \\ &= \{\, z \mid z \in L_\beta \wedge z \in u \wedge \varphi^{(L_\beta)}(z, a_0, ..., a_n, B_0 \cap L_\beta, ..., B_m \cap L_\beta) \,\} \\ &= \{\, z \mid z \in L_\beta \cap u \wedge \varphi^{(L_\beta)}(z, a_0, ..., a_n, B_0 \cap L_\beta, ..., B_m \cap L_\beta) \,\} \\ &= \{\, z \mid z \in u \wedge \varphi^{(L_\beta)}(z, a_0, ..., a_n, B_0 \cap L_\beta, ..., B_m \cap L_\beta) \,\}. \end{aligned}$$

At this stage we note that $B_i \cap L_\beta$, with $0 \leq i \leq m$, is an element of L, for the intersection of an amenable class with a constructible set is again a constructible set (Lemma 3.8.15). Hence $y \in L$, by the corresponding instance of the schema of Separation of ZF in L (Lemma 3.8.7.(vii)). □

REMARK 3.8.27. In our approach the theory of VNB has being cast with the schema of Predicative Comprehension. However it is well-known that VNB is finitely axiomatizable since such a schema can be replaced by a finite number of its instances (the two formulations of VNB are equivalent). This was indeed the approach undertaken by Gödel [10] in 1940, where it is also shown that the amenable classes are closed under these eight operations for generating classes. Lemma 3.8.26 obviously implies that any predicative class is also an amenable class. We do not know if it can be proved within the axiom system of VNB that any amenable class is also a predicative class.

THEOREM 3.8.28.

$$\mathsf{sBL}_1 \vdash \Big(\mathsf{sBL}_1\Big)^{L,\, \mathsf{ac}(L)}.$$

Proof. An immediate consequence of Lemmata 3.8.13, 3.8.14, 3.8.15, 3.8.25, 3.8.26. □

As shown by Gödel [10], $\mathsf{VNB} \vdash (\mathrm{V{=}L})^{L,\,\mathsf{ac}(L)}$. By Corollary 3.2.10, we know that VNB is a subsystem of sBL_1. Hence, $\mathsf{sBL}_1 \vdash (\mathrm{V{=}L})^{L,\,\mathsf{ac}(L)}$. Accordingly we also have

THEOREM 3.8.29.

$$\mathsf{sBL}_1 \vdash \Big(\mathsf{sBL}_1 \,+\, \mathrm{V{=}L}\Big)^{L,\,\mathsf{ac}(L)}.$$

As consequence of Theorem 3.8.29 we obtain that the theory sBL_1 is therefore consistent with Gödel's axiom of constructibility V=L. In other words, the consistency of the theory of $\mathsf{sBL}_1 + \mathrm{V{=}L}$ follows from the consistency of sBL_1: Given a proof of an inconsistency in $\mathsf{sBL}_1 + \mathrm{V{=}L}$ we can, in a higly effective way, produce from it a proof of an inconsistency in sBL_1. We begin by proving a more general theorem from which the above-mentioned equiconsistency result follows.

THEOREM 3.8.30. $\mathsf{sBL}_1 + \mathrm{V{=}L}$ *conservatively extends* sBL_1 *for set-theoretic* Σ_1 *sentences.*

Proof. Suppose that φ is a set-theoretic Σ_1 sentence derivable in $\mathsf{sBL}_1 + \mathrm{V{=}L}$. Let $\psi_0, ..., \psi_n$ be a formal proof of φ in the theory $\mathsf{sBL}_1 + \mathrm{V{=}L}$. Thus for each i $(0 \le i \le n)$, ψ_i is either an axiom of $\mathsf{sBL}_1 + \mathrm{V{=}L}$ or else follows from some of $\psi_0, ..., \psi_{i-1}$ by an application of a logical rule and ψ_n is the statement φ. Consider now the sequence $(\psi_0)^{L,\,\mathsf{ac}(L)}, ..., (\psi_n)^{L,\,\mathsf{ac}(L)}$. If ψ_i is an axiom of $\mathsf{sBL}_1 + \mathrm{V{=}L}$ then $\mathsf{sBL}_1 \vdash (\psi_i)^{L,\,\mathsf{ac}(L)}$. And if ψ_i follows from some of $\psi_0, ..., \psi_{i-1}$ by an application of a rule of logic, then $(\psi_i)^{L,\,\mathsf{ac}(L)}$ follows from the corresponding members of $(\psi_0)^{L,\,\mathsf{ac}(L)}, ..., (\psi_n)^{L,\,\mathsf{ac}(L)}$ by the same rule. Hence $(\psi_n)^{L,\,\mathsf{ac}(L)}$ is a theorem of sBL_1. That is $(\varphi)^{L,\,\mathsf{ac}(L)}$ is derivable in sBL_1. Hence, by persistency, φ is a theorem of sBL_1 too. ☐

COROLLARY 3.8.31. *If* sBL_1 *is a consistent theory then so too is* $\mathsf{sBL}_1 + \mathrm{V{=}L}$.

Proof. Suppose that $\mathsf{sBL}_1 + \mathrm{V{=}L}$ were not consistent. Then, in particular, the statement "$0 = 1$" would be derivable in $\mathsf{sBL}_1 + \mathrm{V{=}L}$. By Theorem 3.8.30 discussed above, such a contradictory statement would also be provable in sBL_1. Hence, sBL_1 would be inconsistent too. ☐

Analogous results hold here also for $\mathsf{sBL}_1 + \mathrm{AC}$ and $\mathsf{sBL}_1 + \mathrm{GCH}$.

Appendix A

The Operational Reflection Principle

A fruitful offshoot of the study of large cardinals has been the investigation of their various analogues in restricted contexts e.g., admissible set and recursion theory, constructive set theory and Explicit mathematics. The first substantive move in this direction was made in the early 1970's by Richter and Aczel [23] in the theory of inductive definitions. With the admissible ordinals playing the role of regular cardinals, analogues of Inaccessible, Mahlo and Indescribable cardinals were developed in this context.

To provide a general framework allowing an uniform treatment of these different analogues of such cardinals, Feferman proposed in [7], the *Operational Set Theory* (OST). The cardinal notions introduced there are for Inaccessible, Mahlo and Weakly Compact. A reflection principle entailing the existence of all these cardinals is also formulated in this context. The consistency strength of OST with this reflection principle adjoined, denoted by $\mathsf{OST} + \mathsf{Rfn}^{\forall}_{\mathsf{op}}$, has not been established yet. A partial result in this direction has however been achieved: we will show that the consistency of $\mathsf{OST} + \mathsf{Rfn}^{\forall}_{\mathsf{op}}$ is not provable in ZFC.

A.1 Operational Set Theory

Let $\mathcal{L}_{(\in,=)}$ denote the language of set theory given by countably many individual variables $a, b, c, x, y, z, ...$, the binary predicate symbols $\in, =$ and the logical operations $\neg, \wedge, \forall$. Assuming classical logic, $\vee, \rightarrow, \exists$ are defined as usual. *Formulae* of $\mathcal{L}_{(\in,=)}$ are built up from the *atomic formulae* $x \in y$, $x = y$ by closing under the logical operations as expected. As usual, ZF denotes Zermelo-Fraenkel set theory in $\mathcal{L}_{(\in,=)}$, ZFC that theory with axiom of choice adjoined.

The theory $\mathsf{OST} + \mathsf{Rfn}^{\forall}_{\mathsf{op}}$ starts off from the language $\mathbb{L}$ extending $\mathcal{L}_{(\in,=)}$ by a three-place relation symbol App, an infinite stock of operational variables

$f, g, h, \dots,$ the individual constants $\mathbf{k}, \mathbf{s}, \mathbf{t}, \mathbf{f}, \mathbf{el}, \mathbf{cnj}, \mathbf{neg}, \mathbf{all}$, the operational constants $\mathbf{S}, \mathbf{R}, \mathbf{C}$. *Basic terms* $(r, s, t, r_0, s_0, t_0, ...)$ are variables and constants of either sorts. *Atomic formulae* are then expanded to include $\mathsf{App}(r, s, t)$ for all terms r, s, t. *Formulae* $(\varphi, \psi, \theta, ...)$ are built up using the propositional operations and quantifiers applied both to individual $(\forall x, \exists x)$ and operational $(\forall f, \exists f)$ variables. By a $\forall$-op *formula*, we mean a formula in which all the quantifed occurrences of the operational variables are in positive $\forall$-form. The formula $\varphi^{(a)}$ is the result of relativizing all the unbounded individual quantifiers to a, that is replacing

$$\begin{aligned} &\exists x(...) \quad \text{by} \quad \exists x[x \in a \wedge (...)], \\ &\forall x(...) \quad \text{by} \quad \forall x[x \in a \rightarrow (...)]. \end{aligned}$$

Bounded quantification is abbreviated as usual:

$$\begin{aligned} (\exists x \in a)\varphi &:= \exists x[x \in a \wedge \varphi], \\ (\forall x \in a)\varphi &:= \forall x[x \in a \rightarrow \varphi]. \end{aligned}$$

The following abbreviations are adopted:

$$\begin{aligned} t \simeq x &:= t = x, \\ st \simeq z &:= \exists x \exists y[s \simeq x \wedge t \simeq y \wedge \mathsf{App}(x, y, z)], \\ s \simeq t &:= (s \downarrow \vee t \downarrow) \rightarrow (s = t), \\ t \downarrow &:= \exists x(x \simeq t), \\ s = t &:= \exists x \exists y[s \simeq x \wedge t \simeq y \wedge x = y], \\ t \in b &:= \exists x(t = x \wedge x \in b), \\ \varphi(t) &:= \exists x(t \simeq x \wedge \varphi(x)), \\ f : a \longrightarrow b &:= \forall x(x \in a \rightarrow fx \in b), \\ f : a^2 \longrightarrow b &:= \forall x \forall y(x \in a \wedge y \in a \rightarrow fxy \in b), \end{aligned}$$

We also adopt the convention of association to the left so that $s_1 s_2 ... s_n$ stands for $(...(s_1 s_2)...s_n)$. Additionally, we write $s(t_1, ..., t_n)$ for $st_1...t_n$. Note that

$$(f : a \longrightarrow b \wedge a' \subseteq a) \rightarrow f : a' \longrightarrow b.$$

The logical axioms of $\mathsf{OST} + \mathsf{Rfn}^{\forall}_{\mathsf{op}}$ comprise the usual axioms of classical first-order logic with equality. The non-logical axioms are divided into the following five groups.

APPLICATIVE AXIOMS

1. $xy \simeq z \wedge xy \simeq w \rightarrow z = w$,
2. $\mathbf{k}xy = x$,
3. $\mathbf{s}xyz \simeq (xz)(yz)$.

LOGICAL OPERATIONS

1. $\mathbf{t} \neq \mathbf{f}$,
2. $\mathbf{el} : V^2 \longrightarrow \{\mathbf{t}, \mathbf{f}\} \wedge \forall x \forall y[\mathbf{el}xy = \mathbf{t} \leftrightarrow x \in y]$,
3. $\mathbf{cnj} : \{\mathbf{t}, \mathbf{f}\}^2 \longrightarrow \{\mathbf{t}, \mathbf{f}\} \wedge \forall x \forall y[\mathbf{cnj}xy = \mathbf{t} \leftrightarrow x = \mathbf{t} \wedge y = \mathbf{t}]$,
4. $\mathbf{neg} : \{\mathbf{t}, \mathbf{f}\} \longrightarrow \{\mathbf{t}, \mathbf{f}\} \wedge \forall x[x = \mathbf{t} \vee y = \mathbf{f} \rightarrow \mathbf{neg}x \neq x]$,
5. $(f : a \longrightarrow \{\mathbf{t}, \mathbf{f}\}) \rightarrow \mathbf{all}fa \in \{\mathbf{t}, \mathbf{f}\} \wedge [\mathbf{all}fa = \mathbf{t} \leftrightarrow \forall x(x \in a \rightarrow fx = \mathbf{t})]$.

GENERAL SET AXIOMS

1. $\forall x(x \in a \leftrightarrow x \in b) \rightarrow a = b$,
2. $\exists y(y \in a) \rightarrow \exists y(y \in a \wedge \forall x(x \in y \rightarrow x \notin a))$.

SET EXISTENCE AXIOMS

1. $(f : a \longrightarrow \{\mathbf{t}, \mathbf{f}\}) \rightarrow \mathbf{S}fa \downarrow \wedge \forall x[x \in \mathbf{S}fa \leftrightarrow x \in a \wedge fx = \mathbf{t}]$,
2. $(f : a \longrightarrow V) \rightarrow \mathbf{R}fa \downarrow \wedge \forall y[y \in \mathbf{R}fa \leftrightarrow \exists x(x \in a \wedge fx = y)]$,
3. $\exists x(fx = \mathbf{t}) \rightarrow \mathbf{C}f \downarrow \wedge f(\mathbf{C}f) = \mathbf{t}$.

OPERATIONAL REFLECTION PRINCIPLE

For each $\forall$-op formula $\varphi(\underline{x}, \underline{f})$, we have

$$\varphi(\underline{x}, \underline{f}) \rightarrow \exists y[\mathsf{Tran}(y) \wedge \underline{x} \in y \wedge \varphi^{(y)}(\underline{x}, \underline{f})].$$

The Operational Reflection Principle is denoted by $\mathsf{Rfn}^{\forall}_{\mathsf{op}}$.

REMARK A.1.1. Note that in the process of relativization of a $\forall$-op formula, the oprational quantified variables remain unaffected. With respect to the original formulation of this axiom-system, as introduced by Feferman in [7], our theory $\mathsf{OST} + \mathsf{Rfn}^{\forall}_{\mathsf{op}}$ does not include the set-theoretic axioms of EMPTY SET, PAIRING, UNION and INFINITY, among the SET EXISTENCE AXIOMS. As we shall have occasion to see in the subsequent section, these axioms are all derivable in $\mathsf{OST} + \mathsf{Rfn}^{\forall}_{\mathsf{op}}$.

λ-abstraction and the fixed point theorem are well-known to be entailed by the APPLICATIVE AXIOMS.

THEOREM A.1.2 (λ-abstraction). *For each $\mathbb{L}$ term t and all variables x there exists an $\mathbb{L}$ term $\lambda x.t$, whose variables are those of t, excluding x, such that*

$$\mathsf{OST} + \mathsf{Rfn}^{\forall}_{\mathsf{op}} \vdash \lambda x.t \downarrow \wedge (\lambda x.t)y \simeq t[y/x].$$

THEOREM A.1.3 (Fixed point). *There exists a closed $\mathbb{L}$ term* **rec** *such that*

$$\mathsf{OST} + \mathsf{Rfn}^{\forall}_{\mathsf{op}} \vdash \mathbf{rec}f \downarrow \wedge [(\mathbf{rec}f = g) \rightarrow gx \simeq fgx].$$

A.2 ON THE STRENGTH OF $\mathsf{OST}+\mathsf{Rfn}^{\forall}_{\mathsf{op}}$

We are concerned with showing that the consistency of $\mathsf{OST}+\mathsf{Rfn}^{\forall}_{\mathsf{op}}$ is not provable in ZFC. We do this by showing that ZF is indeed a subsystem of $\mathsf{OST}+\mathsf{Rfn}^{\forall}_{\mathsf{op}}$. This, in turn, amounts to prove that PAIR, UNION, INFINITY, EXTENSIONALITY, Δ_0-$\mathsf{I}_{\in}$, POWER SET and any instance of the schemata of Separation and Replacement are all derivable in $\mathsf{OST}+\mathsf{Rfn}^{\forall}_{\mathsf{op}}$. Let us start by showing that any instance of the schemata of Separation and Replacement are derivable in $\mathsf{OST}+\mathsf{Rfn}^{\forall}_{\mathsf{op}}$.

LEMMA A.2.1. *Corresponding to each Δ_0 formula $\varphi(\underline{x})$ in the language of* ZF *there exists an associated closed term t_φ such that*

$$\mathsf{OST}+\mathsf{Rfn}^{\forall}_{\mathsf{op}} \vdash t_\varphi \downarrow \wedge (t_\varphi : V^n \longrightarrow \{\mathbf{t},\mathbf{f}\}) \wedge \forall \underline{x}[\varphi(\underline{x}) \leftrightarrow t_\varphi \underline{x} = \mathbf{t}].$$

Proof. See Feferman [7], Lemma 1.(i). □

Lemma A.2.1 allows the SET EXISTENCE AXIOMS 1 and 2 to take the place of the expected schemata for Δ_0 formulae. Hence

LEMMA A.2.2. *For all Δ_0 formulae in the language of* ZF*, the axiom schemata of Separation and Replacement are derivable in* $\mathsf{OST}+\mathsf{Rfn}^{\forall}_{\mathsf{op}}$.

We shall show that any instance of the schemata of Separation and Replacement are derivable in $\mathsf{OST}+\mathsf{Rfn}^{\forall}_{\mathsf{op}}$, by generalizing Lemma A.2.1 to arbitrary formulae in the language of ZF.

LEMMA A.2.3. *Let $\varphi(\underline{x})$ be any formula in the language of* ZF. *Then*

$$\mathsf{OST}+\mathsf{Rfn}^{\forall}_{\mathsf{op}} \vdash \exists f((f : V^n \longrightarrow \{\mathbf{t},\mathbf{f}\}) \wedge \forall \underline{x}[\varphi(\underline{x}) \leftrightarrow f\underline{x} = \mathbf{t}]).$$

Proof. This is accomplished by an adaptation of Specker's method concerning derivability of comprehension axiom schemata from reflection principles. We work informally within the theory $\mathsf{OST}+\mathsf{Rfn}^{\forall}_{\mathsf{op}}$. Assume without loss of generality that "φ" does not contain the variable "y" (this can be achieved by renaming, if necessary). Apply the operational reflection principle to the formula

$$\forall f((f : V^n \longrightarrow \{\mathbf{t},\mathbf{f}\}) \rightarrow \exists \underline{x}\neg[\varphi(\underline{x}) \leftrightarrow f\underline{x} = \mathbf{t}]),$$

which is indeed the negation of the formula we aim to prove. After eliminating all the abbreviations and relativizing, we infer

$$\begin{gathered}\forall f((f : V^n \longrightarrow \{\mathbf{t},\mathbf{f}\}) \rightarrow \exists \underline{x}\neg[\varphi(\underline{x}) \leftrightarrow f\underline{x} = \mathbf{t}]) \rightarrow \exists y[\mathsf{Tran}(y) \wedge \mathbf{t} \in y \wedge \mathbf{f} \in y \wedge \\ \wedge \forall f((\forall \underline{x}(\underline{x} \in y \rightarrow f\underline{x} \in \{\mathbf{t},\mathbf{f}\})) \rightarrow \exists \underline{x}(\underline{x} \in y \wedge \neg[\varphi^{(y)}(\underline{x}) \leftrightarrow f\underline{x} = \mathbf{t}]))]\end{gathered}$$

and from this

$$\begin{gathered}\forall f((f : V^n \longrightarrow \{\mathbf{t},\mathbf{f}\}) \rightarrow \exists \underline{x}\neg[\varphi(\underline{x}) \leftrightarrow f\underline{x} = \mathbf{t}]) \rightarrow \\ \rightarrow \exists y \forall f((\forall \underline{x}(f\underline{x} \in \{\mathbf{t},\mathbf{f}\})) \rightarrow \exists \underline{x}\neg[\varphi^{(y)}(\underline{x}) \leftrightarrow f\underline{x} = \mathbf{t}]).\end{gathered}$$

Therefore,

$$\forall f((f : V^n \longrightarrow \{\mathbf{t}, \mathbf{f}\}) \rightarrow \exists \underline{x} \neg[\varphi(\underline{x}) \leftrightarrow f\underline{x} = \mathbf{t}]) \rightarrow \tag{1}$$

$$\rightarrow \exists y \forall f((f : V^n \longrightarrow \{\mathbf{t}, \mathbf{f}\}) \rightarrow \exists \underline{x} \neg[\varphi^{(y)}(\underline{x}) \leftrightarrow f\underline{x} = \mathbf{t}]), \tag{2}$$

Here, "$\varphi^{(y)}(\underline{x})$" is a Δ_0 formula of the form $\psi(\underline{x}, y)$. By Lemma A.2.1, corresponding to the Δ_0 formula $\psi(\underline{c}, a)$ we have a closed term t_ψ such that

$$t_\psi \downarrow \wedge (t_\psi : V^{n+1} \longrightarrow \{\mathbf{t}, \mathbf{f}\}) \wedge \forall y \forall \underline{x}[\psi(\underline{x}, y) \leftrightarrow t_\psi \underline{x} y = \mathbf{t}],$$

as also

$$t_\psi \downarrow \wedge (t_\psi : V^{n+1} \longrightarrow \{\mathbf{t}, \mathbf{f}\}) \wedge \forall \underline{x}[\psi(\underline{x}, a) \leftrightarrow t_\psi \underline{x} a = \mathbf{t}].$$

By letting

$$t_{(a)} \equiv \lambda \underline{x}.t_\psi \underline{x} a,$$

we get

$$t_{(a)} \downarrow \wedge (t_{(a)} : V^n \longrightarrow \{\mathbf{t}, \mathbf{f}\}) \wedge \forall \underline{x}[\psi(\underline{x}, a) \leftrightarrow t_{(a)} \underline{x} = \mathbf{t}].$$

and

$$\exists f((f : V^n \longrightarrow \{\mathbf{t}, \mathbf{f}\}) \wedge \forall \underline{x}[\psi(\underline{x}, a) \leftrightarrow f\underline{x} = \mathbf{t}]).$$

By generalizing with respect to "a" we therefore infer

$$\forall y \exists f((f : V^n \longrightarrow \{\mathbf{t}, \mathbf{f}\}) \wedge \forall \underline{x}[\psi(\underline{x}, y) \leftrightarrow f\underline{x} = \mathbf{t}]).$$

and

$$\forall y \exists f((f : V^n \longrightarrow \{\mathbf{t}, \mathbf{f}\}) \wedge \forall \underline{x}[\varphi^{(y)}(\underline{x}) \leftrightarrow f\underline{x} = \mathbf{t}]).$$

which is the negation of (2). This implies, by MODUS TOLLENDO TOLLENS, the negation of (1), whence the result. □

LEMMA A.2.4. *For all formulae in the language of* ZF *the axiom schemata of Separation and Replacement are derivable in* $\mathsf{OST} + \mathsf{Rfn}^{\forall}_{\mathsf{op}}$.

LEMMA A.2.5. PAIR, UNION *and* INFINITY *are derivable in* $\mathsf{OST} + \mathsf{Rfn}^{\forall}_{\mathsf{op}}$.

Proof. For PAIRING, first apply $\mathsf{Rfn}^{\forall}_{\mathsf{op}}$ to the derivable formula $a = a \wedge b = b$ to obtain $\exists y(a \in y \wedge b \in y)$. Hence PAIR, by Separation. For UNION, first derive from $\mathsf{Rfn}^{\forall}_{\mathsf{op}}$ the axiom of TRANSITIVE HULL as in Proposition 1.3.3 and then use Separation as in Proposition 1.1.3. For INFINITY apply $\mathsf{Rfn}^{\forall}_{\mathsf{op}}$ to the derivable formula $\forall x \exists y \forall z(z \in y \leftrightarrow z = x)$. □

Since EXTENSIONALITY and Δ_0-$\mathsf{I}_\in$ are the GENERAL SET AXIOMS of the theory $\mathsf{OST} + \mathsf{Rfn}^{\forall}_{\mathsf{op}}$ all we are left with is to show that POWER SET is derivable in $\mathsf{OST} + \mathsf{Rfn}^{\forall}_{\mathsf{op}}$.

LEMMA A.2.6. POWER SET *is derivable in* $\mathsf{OST} + \mathsf{Rfn}^{\forall}_{\mathsf{op}}$

Proof. From the Set Existence Axiom 1,

$$(f : a \longrightarrow \{\mathbf{t}, \mathbf{f}\}) \rightarrow \mathbf{S}fa \downarrow \wedge \forall x(x \in \mathbf{S}fa \leftrightarrow x \in a \wedge fx = \mathbf{t}),$$

we readily infer its corresponding non-uniform version,

$$\forall f((f : a \longrightarrow \{\mathbf{t}, \mathbf{f}\}) \rightarrow \exists u \forall x(x \in u \leftrightarrow x \in a \wedge fx = \mathbf{t})),$$

which may briefly be denoted by "$\psi(a, \{\mathbf{t}, \mathbf{f}\})$". Taking for "$\varphi$" in the operational reflection principle the formula "$\psi(a, \{\mathbf{t}, \mathbf{f}\})$", we get through an application of the cut-rule

$$\exists y\, [\mathsf{Tran}(y) \wedge a \in y \wedge \{\mathbf{t}, \mathbf{f}\} \in y \wedge \psi^{(y)}(a, \{\mathbf{t}, \mathbf{f}\})\,].$$

Observe that

$$(\mathsf{Tran}(b) \wedge \{\mathbf{t}, \mathbf{f}\} \in b) \rightarrow \mathbf{t} \in b \wedge \mathbf{f} \in b.$$

We therefore obtain,

$$\exists y\, [\mathsf{Tran}(y) \wedge a \in y \wedge \mathbf{t} \in y \wedge \mathbf{f} \in y \wedge \psi^{(y)}(a, \{\mathbf{t}, \mathbf{f}\})\,].$$

As usual, before performing the relativization of "$\psi(a, \{\mathbf{t}, \mathbf{f}\})$" to the reflecting set "y" we have have to eliminate within "$\psi(a, \{\mathbf{t}, \mathbf{f}\})$" the abbreviations

$$(f : a \longrightarrow \{\mathbf{t}, \mathbf{f}\}) \qquad \text{and} \qquad fx = \mathbf{t}.$$

They will be reinstated afterwards. In place of "$\psi(a, \{\mathbf{t}, \mathbf{f}\})$" we thus obtain

$$\begin{aligned}&\forall f(\forall x(x \in a \rightarrow \exists z((\mathsf{App}[f, x, z] \wedge z = \mathbf{t}) \vee (\mathsf{App}[f, x, z] \wedge z = \mathbf{f}))\\ &\ \rightarrow \exists u \forall x(x \in u \leftrightarrow x \in a \wedge \exists z(\mathsf{App}[f, x, z] \wedge z = \mathbf{t}))).\end{aligned}$$

It is worth noticing that each operational variable is in functional position. By relativizing this last expression to the reflecting set "y", we therefore obtain

$$\begin{aligned}&\exists y\, [\mathsf{Tran}(y) \wedge a \in y \wedge \mathbf{t} \in y \wedge \mathbf{f} \in y \wedge\\ &\quad \forall f(\forall x(x \in a \rightarrow \exists z(z \in y \wedge ((\mathsf{App}[f, x, z] \wedge z = \mathbf{t}) \vee (\mathsf{App}[f, x, z] \wedge z = \mathbf{f})))\\ &\quad\ \rightarrow \exists u(u \in y \wedge \forall x(x \in u \leftrightarrow x \in a \wedge \exists z(z \in y \wedge \mathsf{App}[f, x, z] \wedge z = \mathbf{t}))))].\end{aligned}$$

Upon the conditions "$\mathbf{t} \in y$" and "$\mathbf{f} \in y$" and using the equality axiom we therefore infer

$$\begin{aligned}&\exists y\, [\mathsf{Tran}(y) \wedge a \in y \wedge \mathbf{t} \in y \wedge \mathbf{f} \in y \wedge\\ &\quad \forall f(\forall x(x \in a \rightarrow \exists z((\mathsf{App}[f, x, z] \wedge z = \mathbf{t}) \vee (\mathsf{App}[f, x, z] \wedge z = \mathbf{f})))\\ &\quad\ \rightarrow \exists u(u \in y \wedge \forall x(x \in u \leftrightarrow x \in a \wedge \exists z(\mathsf{App}[f, x, z] \wedge z = \mathbf{t}))))],\end{aligned}$$

and from this, in particular

$$\begin{aligned}&\exists y\, [\forall f(\forall x(x \in a \rightarrow \exists z((\mathsf{App}[f, x, z] \wedge z = \mathbf{t}) \vee (\mathsf{App}[f, x, z] \wedge z = \mathbf{f})))\\ &\quad\ \rightarrow \exists u(u \in y \wedge \forall x(x \in u \leftrightarrow x \in a \wedge \exists z(\mathsf{App}[f, x, z] \wedge z = \mathbf{t}))))].\end{aligned}$$

Reinstating the abbreviations, this last expression can be rewritten as

$$\exists y \forall f((f : a \longrightarrow \{\mathbf{t}, \mathbf{f}\}) \to \exists u(u \in y \wedge \forall x(x \in u \leftrightarrow x \in a \wedge fx = \mathbf{t}))). \tag{1}$$

Next we prove that

$$\forall w(w \subseteq a \to \exists f((f : a \longrightarrow \{\mathbf{t}, \mathbf{f}\}) \wedge \forall x(x \in w \leftrightarrow x \in a \wedge fx = \mathbf{t}))). \tag{2}$$

The proof of (2) goes as follows. Assume "$b \subseteq a$" for an arbitrary set "b". From the logical operation (ii), that is

$$(\mathbf{el} : V^2 \longrightarrow \{\mathbf{t}, \mathbf{f}\}) \wedge \forall x \forall y[x \in y \leftrightarrow \mathbf{el}xy = \mathbf{t}],$$

and letting

$$\lambda x.\mathbf{el}xb \equiv t_{(b)},$$

we get

$$(t_{(b)} : V \longrightarrow \{\mathbf{t}, \mathbf{f}\}) \wedge \forall x[x \in b \leftrightarrow t_{(b)}x = \mathbf{t}],$$

and from this

$$(t_{(b)} : a \longrightarrow \{\mathbf{t}, \mathbf{f}\}) \wedge \forall x(x \in a \to [x \in b \leftrightarrow t_{(b)}x = \mathbf{t}]),$$

and by means of propositional calculus we obtain

$$(t_{(b)} : a \longrightarrow \{\mathbf{t}, \mathbf{f}\}) \wedge \forall x(x \in a \wedge x \in b \leftrightarrow x \in a \wedge t_{(b)}x = \mathbf{t}).$$

Further, upon the assumption that "$b \subseteq a$" we get

$$b \subseteq a \to (t_{(b)} : a \longrightarrow \{\mathbf{t}, \mathbf{f}\}) \wedge \forall x(x \in b \leftrightarrow x \in a \wedge t_{(b)}x = \mathbf{t})),$$

and,

$$b \subseteq a \to \exists f((f : a \longrightarrow \{\mathbf{t}, \mathbf{f}\}) \wedge \forall x(x \in b \leftrightarrow x \in a \wedge fx = \mathbf{t})).$$

The assertion (2) is established by generalizing with respect to "b". At this stage we infer from (1) and (2) by making use of Extensionality

$$\exists y \forall w(w \subseteq a \to (u \in y \wedge w = u)).$$

From this, using the fact that

$$\forall y \forall w((u \in y \wedge w = u) \to w \in y)$$

we infer

$$\exists y \forall w(w \subseteq a \to w \in y).$$

This last expression asserts that each subset of the set "a" is an element of the set "y". The result is then obtained through an application of Separation,

$$c = \{w \in y \,|\, w \subseteq a\}.$$

It follows that Power set is derivable in $\mathsf{OST} + \mathsf{Rfn}^{\forall}_{\mathsf{op}}$. □

Theorem A.2.7. ZF *is a subsystem of* $\mathsf{OST} + \mathsf{Rfn}^{\forall}_{\mathsf{op}}$.

Theorem A.2.8. *The consistency of* $\mathsf{OST} + \mathsf{Rfn}^{\forall}_{\mathsf{op}}$ *is not provable in* ZFC.

Proof. If the consistency of $\mathsf{OST} + \mathsf{Rfn}^{\forall}_{\mathsf{op}}$ were to be derivable in ZFC, then by Theorem A.2.7, also the consistency of ZF would be derivable in ZFC. And this contradicts Gödel's equiconsistency result between ZF and ZFC. □

Appendix B

Open Problems

Here is a list of selected open problems.

- In Appendix A, we have proved that the consistency of $\mathsf{OST}+\mathsf{Rfn}^{\forall}_{\mathsf{op}}$ is not provable in ZFC. We are also confident that $\mathsf{OST}+\mathsf{Rfn}^{\forall}_{\mathsf{op}}$ entails the existence of all the "real" inaccessible and Mahlo cardinals and hence, in particular, the consistency of ZFC. On the other hand, it is not obvious whether the theory $\mathsf{OST}+\mathsf{Rfn}^{\forall}_{\mathsf{op}}$ is consistent. If so, then it would be reasonable to expect that this theory is as strong as BL_1.

- Friedman's conjecture: $\mathsf{sBL}_1 \vdash \mathrm{V{=}L} \rightarrow \Pi^1_1\ \textsc{Rfn}$. If so, then on the account of Theorem 3.8.30, we would have that BL_1 conservatively extends sBL_1 for set-theoretic Σ_1 sentences. In which case this result implies that for the consistency of the Π^1_1 reflection principle an external appeal to a weakly compact cardinal will be no longer necessary: the assumed consistency of sBL_1 would suffice. We are not far from a proof of this result, but still several technical points needed to be checked out. We are, however, confident of the soundness of our argument and we hope to present it in a future publication. The argument that we are actually carrying out has been suggested by Sy Friedman and it consists in a generalization of Barwise's Theorem VIII.9.7 [2] where instaed of the set-model H_κ with the standard interpretation of class variables as ranging over subsets of H_κ and where $\mathsf{cf}(\kappa) > \omega$, we use the proper-class L with classes interpreted as amenable classes. The main difficulty in this respect, is that by Tarski's argument of undefinability of truth, a uniform satisfaction relation for the proper-class L is formally undefinable in sBL_1. This limitation requires a reworking and adapatation to our context of the compactness argument used by Barwise. But the details do not look simple.

- What is the strength of $(\mathsf{strictBL}_1)^+$? We believe that this theory is a conservative extension of PA. Note that the existence of ω is not derivable in

$(\mathsf{strictBL}_1)^+$. The above-mentioned conservation result, could presumably be achieved using recursively saturated models.

- What is the strength of $(\mathsf{strictBL}_1)^{++}$? We conjecture that this theory has the same strength as $\mathsf{sKPu}^{\mathsf{r}}_2 + \textsc{Infinity}$. That $(\mathsf{strictBL}_1)^{++}$ is a subsystem of $\mathsf{sKPu}^{\mathsf{r}}_2 + \textsc{Infinity}$ is a triviality. For the converse direction, the only serious fact that needs to be verified is that every instance of Δ^{c}_1-CA is derivable in $(\mathsf{strictBL}_1)^{++}$. Note also that since we are working with a theory entailing the existence of ω the presence of urelements is superfluous in $\mathsf{sKPu}^{\mathsf{r}}_2 + \textsc{Infinity}$.

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LIST OF NOTATION

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