Dissertation submitted at the Faculty of Humanities at the University of Bern to obtain the degree of Doctor of Philosophy by

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THE SEA-BATTLE AND INTUITIONISM

(Printer)
Athens/Bern, 2008
Accepted by Prof. Dr. Gerhard Seel and Prof. Dr. Dirk van Dalen on behalf of the Faculty of Humanities
Bern, 23.3.2007, Dean: Prof. Dr. Karénina Kolmar-Paulenz

Revised Edition (September 2008)

Abstract

The Sea-Battle and Intuitionism - a new way out has a double purpose. First, and as the title suggests, it aims at presenting a new solution to the old Aristotelian puzzle about future contingents. Second, it aims at presenting a generalized world conception on the basis of intuitionistic logic and philosophy. More precisely, it generalizes and applies some old intuitionistic conceptions and techniques (which have so far been used for modeling the universe of intuitionistic mathematics), for modeling the universe *tout court*. One is here tempted to add, "...and from a temporal point of view", but strictly speaking this addition is quite unnecessary, since in Intuitionism time makes essential part of the underlying logic/semantics.

The two different goals of the book are in obvious interdependence. This "new way out" of the Sea-Battle problem is possible, exactly because the world is presented through that angle - i.e. evidence about the correctness of the perspective comes from the fact that by this perspective the problem does not even arise.

What is this "new perspective"?

To put it in a nutshell: it denies that Reality is complete - i.e. it denies that one element out of any pair of contradictory statements needs to be true. Consequently, no need remains for one out of, "There will be a Sea-Battle tomorrow", and, "There will be no Sea-Battle tomorrow" to be true. It might be the case that there is no corresponding fact of the matter with respect to any of the above sentences. However, and thanks to some devices of Beth semantics, as well as some techniques of my own that incorporate the metalanguage about models into the models themselves, the disjunction comes out true.

Acknowledgments

Ambition is a sly consultant. When I begun writing this book I was convinced that the Sea-Battle problem had done its historical circle, ending with Thomason's superevaluation solution. Back then I was aiming at a historical treatise. Its first version was historical all right. Later, when the idea came to me to use intuitionistic models, the historical aspect of the matter looked like a burden. I, now, wanted to focus attention on some attempt of mine to solve the problem by some new tools.

A book that was fantasizing to become a display of multifaceted scholastic knowledge is now fantasizing to state, in a self-contained manner, an elegant, new idea - whether or not this idea is either of the two.

This, *grosso modo*, was its journey.

Professor Gerhard Seel has been a constant source of inspiration, common sense, and -if I dare say- friendship. An earlier version of the book was defended as a PhD Thesis in Bern, under his guidance. Professor Dirk van Dalen, who was the other member of my committee, had the kindness and generosity to provide me with his constant support, and comment in *extesnso* the drafts I was giving him. Dr. Greg Restall was also very supportive. Besides, he was the first to read a manuscript about my intuitionistic models.

I also wish to thank Professor Paul Weingartner for giving me some advice on Appendix I. Professor Theodoros Scaltsas and Professor Christoff Rapp had the kindness to invite me to Edinburgh and Berlin to talk about my work. I am grateful to them both, as I am also grateful to the participants in the discussions that followed.

During the academic years 2004-07, I have been gratified by the seminar of Professor Michael Frede on *De Interpretatione*.

DK
Bern-Athens, 2007-2008

That the future is a faded song, a Royal Rose or a lavender spray
Of wistful regret for those who are not yet here to regret,
Pressed between yellow leaves of a book *that has never been opened*.

T. S. Eliot, *Four Quartets*

(Exquisite poetry, but rather dubious Time metaphysics)

Contents

Chapter 1

The Sea-Battle Affair

In part I, I briefly elaborate the reason why a "solution" to the Sea-Battle problem, that keeps intact the complete edifice of classical logic is no solution at all; I also present an informal *numerus clausus* hypothesis, aiming at delimiting all possible ways out of the problem. All but one of these correspond to the abandonment of a different classical dogma. In part II, I briefly comment on my methodological predispositions when approaching the problem.

I

Who has been a boy and has not dreamed of being king? And which boy has not feared -getting into bed on a cold wintry night- that the kingdom (*his* kingdom) has fallen into the hands of an evil tyrant?

The boy:

"The fields are now deserted, and endless night falls upon my land. My entire family is held captive by the traitor, and I, the rightful king, wander in the woods waiting and seeking, seeking and waiting, for a final opportunity. Will I make the right decision? Will I be able to recognize it, when it comes? For -an oracle once told me- the chance will be given to me, in times of great peril, to save the future of the land. I would only have to recognize it and act accordingly. Oh, if only I could know beforehand when this will be! And if only I could know the dilemma that I will have to face! For the oracle predicted that too! Will I be equal to the occasion?"

"But if," the boy now continues in subtle reverie, "if it is indeed the case that my kingdom is about to be threatened with ultimate extinction, and if, whether it is destroyed or not is the result of a future decision of mine, the sentence asserting that I will, when the time comes, make the right decision is either true or false. And, moreover, it has always been such. That is to say that, if it is now true that I will eventually make the wrong choice, nothing among the things that will take place between the present and that crucial (or isn't it so crucial after all?) moment can reverse this terrible development. And, if it is now true that I will take the right decision, what can possibly go wrong to overturn that fact? For fact it is!"

At this point, thoughts begin to weight heavily upon his young shoulders.

There is no escape from them.

"What a fine hero I shall be! And what an undeserved failure!" the boy finally whispers, bursting into tears. "Nothing depends on me. Whatever I do -whether right or wrong- it is (it has always been!) predetermined".

Now, what if, in order to stop being the puppet of sentences' truth-value, the boy assumes a sarcastic attitude and decides to surprise the future by doing exactly the contrary to what his fine (or is it not so fine after all?) nature suggests? If he decides -that is- not only not to fight the awful traitor, but, instead, to help him to power. Even then, his destiny will, once again, get the better of him. For, in that case too, it would have always have been the case that on that wintry night a potentially glorious king, would have turned into a sarcastic nihilist, after having entertained these specific thoughts about the relation obtaining between the future and the truth-value of sentences; and so on, and so forth.[1]

This is what *De Interpretatione* 9 says.

Not a causal, a purely logical/semantical determinism!

Luckily, the young do not meditate like this and, no doubt about it, that is a great relief to all teachers and parents. The interesting question, however, is whether the above state of moral devastation would have been philosophically justified, if the young did, actually, entertain such thoughts and come, actually, to such conclusions.

[1]Cf. the thought experiment of Anscombe, in G. E. ANSCOMBE, [1956], where (p. 13), someone is supposed daily to receive a letter, giving a full account of everything he has done between the moment the letter was posted, and the moment he opens it; and no matter what he does to surprise the sender, the sender is always one step ahead - i.e. the sender will anticipate all the bizarre things the receiver does to falsify the letter. Which means that, if he had done no bizarre thing at all, the sender would have been falsified. But this obtains only retrospectively. For, as the settings have it, if the receiver had actually, and on that basis, decided to do nothing strange for once, the envelope would have contained a letter with the words -e.g.- "at 2. pm the idea will come to you to take me aback by doing nothing strange for once"; and so on, and so forth.

Throughout this treatise, I will be upholding a series of positions the most central of which are the following.

First, I will maintain a position equivalent to saying that, if young boys did, actually, entertain thoughts like these, this state would be a totally justified state, if the logical/semantical framework of these meditations is classical. The reason why I want to emphasize the philosophical weight of the Sea-Battle argument is that (now and again), efforts have been made to diminish its importance and to present it as an easily refutable sophism. Those people who pronounce such judgments (and there are notorious scholars among them), tend to charge Aristotle with a plain logical fallacy. They assume that, in *De Interpretatione* 9, Aristotle has confused the claim: *it is necessary that, if the Sea-Battle will take place, then, it will take place*, with the claim: *if the Sea-Battle will take place, it will do so necessarily.*[2] And, hence, the fallacy.

I have argued elsewhere[3] against such an interpretation. More precisely, I have argued that the fatalistic conclusions of the first part of *De Interpretatione* 9[4] can be deduced out of the single, modal-free premise asserting that *there either will be a Sea-Battle tomorrow or not*, and, moreover, that the abovementioned fallacy lies nowhere within the steps leading to the fatalistic conclusions.

Now, I will also maintain that, although one cannot charge Aristotle with this fallacy, one could claim (as -e.g.- Quine has insightfully done)[5] that Aristotle has, when attempting to remove the fatalistic conclusions of the first part of *De Interpretatione* 9 arrived at a certain logical *bizarrerie*: the notorious Aristotelian "fantasy", which considers that a disjunction can be the case, without any of its disjuncts being the case. And, indeed, Aristotle does seem to assert: although it is the case that tomorrow there will (necessarily, but this is not important here) either be a Sea-Battle or not, neither, "There will be a Sea-Battle tomorrow" nor, "There will be no Sea-Battle tomorrow", is true. What this composite claim amounts to is that there can be true sentences of the general form $T(p \vee q)\&\neg T(p)\&\neg T(q)$, and this, of course, contradicts any classical intuition about disjunction. Moreover, when one considers more closely what the particular sentences p and q here are (i.e. if one considers that this particular q is the negation of p), one has to prepare oneself for the forthcoming $p \ \& \ \neg p$.[6]

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There are two general strategies, with which to confront the problem. The first denies that Aristotle claims any such thing as the fantasy in the first place

[2] Or, which is formally the same, that he has confused the claim that it is necessary that either the Sea-Battle will take place, or not, with the claim that it is either necessary that the Sea-Battle will take place, or it is necessary that the Sea-Battle will not take place.

[3] D. KAPANTAIS, [2005].

[4] The part ending at 19a6.

[5] W. O. QUINE, [1953b], p. 65.

[6] contradiction can be extracted anyway -i.e. even if q is not $\neg p$- by the De Morgan principles.

and considers that this exegetically erroneous conclusion comes from a certain ambiguity in the text. Aristotle -the same strategy goes on- *seems* to claim something like the fantasy, but, in reality, he has in mind something totally innocent. We are going to call this approach to the problem “conservative”.[7] The other general strategy accepts that the text says exactly what it *prima facie* seems to say, and tries to create a logical niche in which the fantasy no longer looks paradoxical; i.e. in which the fantasy is accepted as such, but no longer stings. Obviously, the logical framework constituting this niche has to be deviant. More precisely, the fantasy emerges, only when one accepts all four of the following principles:

a) The only truth-values are the True and the False.

b) All declarative sentences have a truth-value.

c) If a sentence is true, its negation is false, and *vice versa*.

d) The truth-value calculus cannot be reduced to the calculus of probabilities - i.e. these two are essentially different.

To deny any of these can (as we will see), provide an escape route out of the paradox, but it also makes one a logical renegade. It is in that sense that we will say that the second general strategy commits one to a certain relaxation of the standard logical apparatus; something which is well to be expected under the circumstances since to allow for $T(p \vee q) \& \neg T(p) \& \neg T(q)$ is formally forbidden in any non-deviant context, and therefore, in order to make it acceptable, one has to relax some of the principles of the same context. According to our *numerus clausus* hypothesis, one has, in order to create the appropriate niche, to abandon one of the above four principles.

But let us first begin with the conservative general strategy.

According to the conservative approach, the problem can be removed by implying that one is just *under the impression* of reading, in *De Interpretatione* 9, that “There will be a Sea-Battle tomorrow or not” is true, while neither of its disjuncts is such. As a matter of fact the above way of confronting the fantasy suggests that what Aristotle really means is that this disjunctive sentence has a certain modality of being true that none of its disjuncts has. More precisely, what this way out ultimately amounts to is the assumption that what is negated is not that one of the two disjuncts has to be true, but that it has to be true *determinately*. “Determinately true” meaning here that what the sentence asserts obtains in all possible futures, not only in the real future. As we said, what is actually claimed here is that Aristotle never formulated such a thing as the Quinean fantasy in the first place. Ammonius, under a certain interpretation

[7] I would prefer to call it “traditional”, but this could be misleading, since “traditional interpretation” of the Sea-Battle problem has, in the literature, the weight of a technical term referring to the contrary of what is, in general, supposed to be a “conservative” framework. The “traditional interpretation” of the problem usually refers to interpretations that attribute to Aristotle a denial of the Law of Bivalence.

of his Commentary, is the first to have given a semi-formal account of this.[8] The other general way to attack the fantasy is to confront it as it stands. If so, one thing is sure: the orthodox apparatus will lead us nowhere. Extracting a formal contradiction out of $T(p \vee q)\&\neg T(p)\&\neg T(q)$ is a matter of routine in any non-deviant context. Our *numerus clausus* hypothesis was that in order to endorse the fantasy into our logical universe we need to abandon some of the above-enumerated four dogmata.

Let us now see how this can be done in each case.

a) The first option consists in not accepting that there are only two truth-values. This path leads directly to the many-valued systems. In some of them, $T(p \vee q)\&\neg T(p)\&\neg T(q)$ becomes accepted as it stands,[9] while in others,[10] $\neg T(p)\&\neg T(q)$ is allowed, even when q is instantiated by $\neg p$, but the entire $T(p \vee q)\&\neg T(p)\&\neg T(q)$ is still unacceptable. This is because, in these latter systems, assuming $\neg T(p)\&\neg T(q)$ excludes the truth of "$p \vee q$" "; i.e. the solution endorses the contradictory (from the standpoint of bivalence): "p" is untrue and "$\neg p$" is untrue, but does not allow the truth of "$p \vee \neg p$", when they are both untrue. In fact, what this second many-valued way out accounts for is not the Quinean fantasy *per se*, but a certain deviant consequence the fantasy has if we take $\neg p$ for q. It allows for two contradictory sentences both being untrue, but still disallows that their disjunction might be true. It is still the case that if two sentences are not true neither is their disjunction. The reason that we did not classify this way out under the umbrella of the "conservative" solutions is that it accepts the possibility that both poles of a contradiction can be untrue. Otherwise, and similarly to what the conservative approach does, it renounces the right to read the fantasy into the Aristotelian text.[11]

b) The second option is to abandon the principle that all sentences have truth-values. Accepting that there can be truth-value gaps is gratifying for the fantasy on two accounts. First, it allows (just as the many-valued systems do), for "There will be a Sea-Battle tomorrow" and "There will be no Sea-Battle tomorrow" both to be untrue. And, second, it allows (thanks to some

[8]This account has been formalized in M. MIGNUCCI, [1996]. I have elsewhere argued why the Ammonian and, in general, the conservative way out cannot remove the fatalistic conclusions of *De Intepretatione* 9, up to 19a6, and, therefore, cannot be accepted as the way out Aristotle himself had in mind. See D. KAPANTAIS, [2005]. Note that Mignucci uses "definitely" for my "determinately" in rendering -literally- the Greek ΑΦΩΡΙΣΜΕΝΩΣ of the Commentary.

[9]e.g. Lukasiewicz's second system. J. LUKASIEWICZ , [1957].

[10]e.g. Lukasiewicz's first three-valued system: J. LUKASIEWICZ, [1920], and the intermediate system of 1930: J. LUKASIEWICZ, [1930].

[11]Another difference is that, while -e.g.- Ammonius (according to the interpretation of Mignucci), thinks that he remains faithful to the Aristotelian text, and, consequently, does not charge Aristotle with the fantasy, Lukasiewicz has altered his first many-valued system into that of 1957, exactly because he thought that the fantasy was present in the Aristotelian text and his first system was unable to capture that. Notice, here, that, according to some recent results by Restall (cf. G. RESTALL, [2005]), even the first three-valued system of Lukasiewicz can be made to respect the Excluded Middle.

supervaluation devices), for their disjunction to be true as well. The combination of these two gives justice to the entire $T(p \vee q)\&\neg T(p)\&\neg T(q)$, and this is made possible exactly because systems with gaps allow for compound sentences to have a value without necessitating that their sentential components have also a value. Which further implies that the calculation of the value for the compound sentence is no longer truth-functional. One of these systems is Thomason's.[12] Another solution along the same lines will be presented here for the first time and is based upon the intuitionistic Beth models.[13] The elaboration of this particular device is my main object in this book. The models appropriate to incorporate it will be introduced in chapter 4. They will be compared with those of Thomason in chapter 5. An intuitive approach to them will be attempted in chapter 3.

c) The third possible candidate to be the principle that must go is the dogma that no sentence can be false while its negation is also false. Challenging this principle is the most radical answer there is to the paradox and, since our goal can be achieved with much less drastic operations, we will not favor it here. Nonetheless, (c) is not without some considerable technical and philosophical interest of its own In fact, what it claims is that "There will be a Sea-Battle tomorrow" and "There will be no Sea-Battle tomorrow" are both false, upon the basis of their not having been proved as yet. Now, what is particularly interesting about this escape route is that it does *not* allow us to extract the contradiction "There neither will nor will not be a Sea-Battle tomorrow" out of the falsity of its components. The logical system fitting this solution is a kind of paraconsistent non-commutative logics.[14]

d) The forth alternative consists in replacing the calculus of truth for the calculus of probabilities. Once the substitution is brought home there will be no more reason for one to wonder how it can be that a disjunctive sentence has the probability-value 1, while none of its disjuncts has the same (probability) value. And so for the fantasy: the disjunctive sentence is a substitution instance of a theorem (i.e. the Excluded Middle), and therefore is assigned the (probability) value 1, while its disjuncts refer to not yet settled future events, such as tomorrow's Sea-Battle, and therefore acquire other values. Technically, this way out has striking similarities with the supervaluation methods of the second option above, yet, from the philosophical point of view, the two approaches differ to a considerable degree. The source of this difference is that in probabilities no extra argumentative labor is needed in order to convince someone that a disjunction might "superevaluate" its disjuncts. To put it another way, Quine's fantasy looks quite harmless when transcribed from the calculus of truth-values to the calculus of probabilities. Intuitively there seems to be nothing wrong with the assumption that a disjunctive sentence might have the probability-value 1, while none of its disjuncts has the same value. On the other hand, it looks *prima facie* as if everything has gone wrong in a logical system where the disjunction has the *truth*-value 1

[12] R. THOMASON, [1970]; [1984].

[13] If there is a logical framework which has some remote kinship with this one it is McCall's proposal in "Temporal Flux", (cf. S. McCALL, [1966]). Otherwise, among the non-formal proposals, the closest there is is that of Peter Aureoli (cf. PETER AUREOLI, [1952-1956]).

[14] I will present the technical details for that system in a forthcoming paper. It is based mostly on S. JASKOWSKI, [1969] and R. FAGIN - J. HALPERN , [1988].

and none of its disjuncts has the same *truth*-value.

These are *grosso modo* the alternatives by which the fantasy might be surpassed according to our *numerus clausus* classification. I have no *a priori* argument in favor of that particular number. It just seems to me that challenging the edifice of classical logic by the fantasy commits one either to renounce the fantasy and keep the entire edifice intact, or to proceed into one of these four relaxations. To the question of whether there are other ways to relax the classical apparatus so as to make sense of the fantasy, the answer is possibly affirmative but my guess is that they are even more radical. In general, Ockham's razor dictates that if, in order to avoid contradiction, one has to abandon some principles, one has to abandon the fewest and weakest possible - e.g. one could also try to drop the Law of Non Contradiction but this would have been (on this occasion) an unnecessary sacrifice.[15]

Before moving onto a brief comment concerning my general methods and goals, allow me a couple of further remarks on these two general strategies.

The first is closely related to the last paragraph. One might even wonder why one could ever prefer the second general strategy if the conservative strategy can both make the fantasy disappear and preserve the entire classical apparatus. "Isn't it always preferable", one might ask "to choose -in times of crisis- the escape route that keeps intact the greater part of the preexisting (*any* preexisting) edifice?" And it is indeed the case that the conservative solution assumes an all round classical framework. One sacrifices nothing there! On my own behalf, I claim that, although the framework used by the conservative solution is not deviant, the disappearance of the fantasy when achieved in that way, does not suffice to provide a successful way out of Aristotle's fatalistic conclusions up to 19a6. Which is to say, if it is the case that the premises of the fatalistic argument up to 19a6 are those that I think they are, the conservative solution constitutes no proper way out of it.[16] Now, if it was not for that, I would have no scruples whatsoever in adopting the conservative (according to some scholars, the Ammonian) solution. My working motto is that logic/semantics has to be as conservative as possible; not less, but not more either.

The second point: As we have seen, the two main strategies out of the fantasy are set apart by many aspects, the most flagrant of which is conservative respect for the classical apparatus and the deviant departures of all the rest. Now, as it often happens in such cases, this technical tension between the two strategies reflects a deeper philosophical tension. For consider it this way: for what reason does the technical apparatus of classical logic assume the dogmata, abandoned by the solutions (a) to (d)? It is none other than its theoretical approach as to what a proposition essentially is - i.e. according to it, a proposition is no secular item, but an atemporal entity, the very nature/essence of which dictates

[15] This, again, is dependent on one's intuitions. For example, G. Priest finds the Law of Excluded Middle more fundamental than the Law of Non Contradiction.

[16] See D. KAPANTAIS, [2005].

all the aspects denied by the relaxations effected by these four solutions. For example, the classical approach concerning the question of what is an (eternal) proposition dictates that it "always" has a truth-value, that it "never" changes the truth-value it has, that it can have no other value than the True and False etc. Now, among the deviant solutions, each one challenges at least one of these characteristics. The many-valued solution assumes more than two values and also accepts that propositions (better say "sentences" here) can change value as time goes by. The supervaluation techniques accept that a sentence might acquire a truth-value, when previously it had none. To cut a long story short, the remaining options, all belonging to the second general strategy, betray (in one way or another), the classical atemporal notion of propositions, and it is this betrayal which is, on each occasion, reflected by their deviant formalisms.

I want now to draw a terminological parallel between the pair of these two fundamental approaches to sentences and the two general strategies mentioned above. I will refer to the strategy that fits the conservative solution, as the "atemporal approach towards sentences" and to that which fits all the rest as the "non-atemporal approach towards sentences".[17] As far as my personal preferences are concerned it is not without some significance of its own that throughout the treatise I prefer to use the term "sentence" instead of the term "proposition".

II

Experience has shown that methodological questions about the way one should or should not approach ancient philosophical texts have rarely been fruitful. Nonetheless, and since we are going to approach the Sea-Battle problem from the angle of modern logic/semantics (an angle, which is nowadays heavily criticized by many "ancient philosophers"), some methodological remarks could be of help at this preliminary stage. These remarks would have been completely useless had my treatise aimed uniquely at a modern audience. That is to say, that I could have dropped completely all things having to do with Aristotle's *De Interpretatione* and its notorious Sea-Battle, and taken up the discussion of future contingents from the modern literature. Had I proceeded in this way, the question of whether or not I have stayed faithful to the Aristotelian intuition would not even have arisen.

The reason that I will not proceed that way is twofold.

First, it is because my formation is classical. I am -by formation- an "ancient philosopher" - whatever this funny[18] term has come to mean in the *milieu* of our profession. I have first approached the problem of future contingents by way's of Aristotle's *De Interpretatione* 9, and by the Commentaries of other -*stricto sensu* this time- ancient philosophers. Now, all would have been fine, if, after having concluded -as I certainly have- that modern logic can analyze

[17] These appellations will be further explained in the following chapter.

[18] In modern Greek, "ancient philosopher" means (unlike what it sometimes means in the Anglo-American philosophical jargon), exactly what it says: a philosopher living in ancient times.

and treat the problem in a much more efficient way than can any ancient tool, I had also abandoned the idea that my treatise also concerns my contemporary "ancient philosophers". Why I did not do this asks for an explanation. I keep referring to Aristotle because I still believe that my analysis might be of interest to anyone who reads and thinks about the Aristotelian text, no matter whether he is an "ancient philosopher" or not. To say that my analysis does not concern him would be equivalent to saying that it deals with modern topics which have nothing to do with Aristotle and *his* Sea-Battle. I am not willing to admit such a thing. Or, in order to see things from the other end of the telescope, to disdainfully deny any effort to approach ancient texts with modern tools is one thing, and to be right about upholding such an attitude, quite another.

I, then, *de facto* do not agree with that attitude and, consequently, I think that my text concerns (might be of interest to) any "ancient philosopher". I will not go on jingling on his doorbell and insisting upon how interesting this kind of analysis might be for him. It won't be me, though, who says that it isn't.

Especially now that the question is about *De Interpretatione 9*.

This leads me to the second reason, for which I uphold the view that my analysis concerns an *Aristotelian* matter.

De Interpretatione 9 represents a rare occasion in Aristotle and in ancient philosophy in general. For whereas problems raised in other fundamental texts of the Aristotelian corpus rejoice in a rather restricted autonomy with respect to the rest of the corpus, and, in general, with respect to their author and times, the degree of the Sea-Battle's autonomy approximates to that of a mathematical puzzle. What I mean by that is the following: Consider, for example, Goldbach's conjecture. A historian of mathematics could be deeply interested in the particular events that lead Goldbach to make such a conjecture, or even in what this particular conjecture meant for the mathematics of his era. The same historian might also be interested in the question of whether the sense and importance of the conjecture has altered throughout the centuries. All this is natural, if not necessary. Nonetheless, there is no question of the actual presence of working mathematicians whose sole interest is the conjecture *per se*. In other words, there are mathematicians who only know of Goldbach that he is the past master that stated the conjecture, on which they now work. These people are not interested in Goldbach's personality -not in the least- nor are they interested in what Goldbach had in mind when making the conjecture nor in the rest of his work, nor in what mathematics looked like then. They just want to test the conjecture. And yet it is the conjecture *of Goldbach* they work on, not an invention of the present.

The above analogy is a rude one. Philosophy is not mathematics, and genuine philosophical problems (if any), are much more dependant upon the philosopher who raises them and the times they are raised in, than are mathematical problems dependant on the mathematician and his times. Nonetheless -my claim is- some philosophical puzzles do rejoice in a relative autonomy with respect to the philosopher who formulated them or to his times.

The Sea-Battle is -in my judgment- such a puzzle.

Chapter 2

Terminological Clarifications

In part I, I present what I understand by "tensed ..." and what by "detensed ...", "eternal ...", etc., "... sentence". More precisely, I present a certain way of conceiving an eternalized sentence as a sentence that, though possibly having no time-dependant truth-value, contains no other verbal forms than tensed. In the second part, I clarify some points on the use I assume for causal terms. These clarifications are, for the most part, technical. They are not based on any background theory about causation, such that it could claim to settle the matter about determinism by its own. In part III, I clarify the sense, in which the model of my solution will be said to be "intuitionistic" but not "Kantian".

I

In the literature on "detensed", "eternalized", "atemporal", etc., sentences, there is a constantly re-occurring notional point of reference: the "atemporal" or "tenseless" or "timeless" present. More precisely, the grammatical present is often claimed to be semantically ambiguous, corresponding either to the tensed or to the atemporal semantical present. "There is (now) a Sea-Battle" is an occurrence of the former, and, "Seven plus five equals twelve", an occurrence of the latter.[1] For the semantics of this treatise, no such thing as the atemporal present will be necessitated, and, consequently, no such verbal form as the atemporal present will be defined. This -as it will subsequently be shown- comes as a straightforward consequence of our introducing the metalanguage about the models into the worlds of the models themselves. But let us take one step at a time and clarify first the sense we attribute to the term "eternalized sentence".

An "eternalized sentence" is usually called a sentence that has undergone the procedure of having been "eternalized" or "detensed". This particular procedure consists in replacing constants for all temporal indexicals and altering the tense

[1] See W. O. QUINE, [1960]; ch. 36. See, also, G. E. L. OWEN, [1986], p. 27: "Some statements couched in the present tense have no reference to time. They are, if you like, grammatically tensed but logically tenseless".

of the verbs to the atemporal present.[2] Of course, all this has to be done in such a way that the sentence continues to represent the same state of affairs, as it used to represent when it *was* tensed and *had* indexicals. Thus, “It rains” (the verb being present-tensed) will become, “It rains at n” (the verb now being atemporal and n denoting the moment that the event described by the former sentence is supposed to happen).[3] The *rationale* of such a procedure is to transform a context-depending sentence into a context-free sentence. The “It rains” above describes a different state if uttered at different times,[4] and if raining is no necessary condition in this world -as it isn't- “It rains” has different truth-values at different times. Exactly the opposite obtains for, “It rains at n”. Seen from this angle, the procedure of eternalizing a sentence consists in turning a sentence which when uttered at one moment, represents a certain state, and when uttered at another moment, represents another state, into a sentence that represents the same state no matter when the moment of its utterance happens to be.[5] Quine, for instance, insists upon the fact that an eternal sentence, unlike its tensed counterpart, is tenseless and its truth or falsity are no functions of any temporal context.[6] The thought implied here is that it is this very tenselessness of the verb together with the absence of indexicals that makes it possible that the truth and falsity of the sentence are no functions of the occasional context.

Let us here agree to follow a notational convention. Let us agree that, for every tensed sentence p, its eternalized counterpart will be noted by “$\{p\}$”. For our part, we are of the opinion that $\{p\}$'s truth-value depends indeed upon no context, not because it contains tenseless verbs, but because the claim it bears is constant for all moments. For consider it this way. “It rains” can have different truth-values in different temporal contexts, because it might be raining today, and not raining tomorrow. On the other hand, the fact that the sentence ‘It rains’[7] is, or was, or will be true,[8] if uttered at n (n denoting “now”), is

[2] What Quine does is to first eliminate all tenses other than the atemporal present, and then substitute the remaining indexicals (like -e.g.- “now”) by dates.

[3] In McTaggart's terms, the procedure consists in turning an A-series sentence into its correspondent B-series sentence.

[4] Not generically that is. The rain happening at n is the same general kind of weather condition as the rain happening at m. But one who utters “It rains” at n, asserts a different historic event from he who utters the same sentence at m.

[5] This is sometimes considered to be a necessary step on our way towards an objective and unambiguous science. For Quine a detensed language is the only language that can meet the standards of a relativistic science. For the possibility of a tensed relativistic language, see Appendix I.

[6] W. O. QUINE, [1960], p. 193: “In general, to specify a proposition without dependence on circumstances of utterance, we put for the ‘p’ of ‘[p]’ an eternal sentence: a sentence whose truth value stays fixed through time and from speaker to speaker.”; p. 227: “Insofar as a sentence can be said simply to be true, and not just true now or in this mouth is an eternal sentence”.

[7] In general, we will use double inverted commas for (i) quotation of passages, (ii) naming of sentences and terms and (iii) the rhetorical, “I do not mean it *stricto sensu*”. We will use single inverted commas for cases that we want to draw attention to the naming operation, such as here, or in cases of quotations/namings contained in one anot0her.

[8] Which one out of the three verbs has to be used is an obvious function of the moment that the claim is made. More precisely, the thus “eternalized” “It rains” is: (i) If now$< n$, then, ‘It rains’ will be true, if uttered at n, (ii) If now$= n$, then, ‘It rains’ is true, and (iii) If now$> n$, then, ‘It rains’ was true, if uttered at n. The whole disjunction, together with the tensed “now$< n$”, or “now$= n$”, or “now$> n$” that happens to be true now, picks up the consequent that is occasionally true.

constant for all moments. In that sense, one can understand {It rains} as the sentence bearing the claim that at all moments the following fact obtains: 'It rains' is/was/will be true, if uttered at n. Now, none among the verbs of this last sentence is atemporal/detensed. Nonetheless, the sentence does indeed bear a claim about the world, which (claim) does not vary according to the moment it is made.

Let us now agree to a further notational convention: When p is a non-eternalized sentence, let us agree to say that $[p]$ is its eternalized form according to the method of Quine, and $\epsilon(p)$ its "eternalized" form according to our proposal. One could say that $[p]$ is a *stricto sensu* detensed/atemporal sentence, while $\epsilon(p)$ does the same job that $[p]$ does, but is tensed. Differences in the truth-conditions of the two can emerge only in indeterministic Histories, and, more specifically, only if these Histories are evaluated in deviant logical frameworks.[9] In such Histories and frameworks, one can encounter indeterministic states that happen or fail to happen, while the sentence asserting that it is true to say that they happen/fail to happen has no constant value. For example: imagine that there will indeed be a Sea-Battle tomorrow, but that today this is a not yet settled future event. In such a case, 'There is a Sea-Battle at n'[10] is true no matter when uttered - today, of course, included. On the other hand, ϵ(There will be a Sea-Battle tomorrow)[11] is neither true nor false today, and it won't become such before the moment that the last missing condition for the advent of the Battle is satisfied. The truth-value behavior of "There will be a Sea-Battle tomorrow" and of "'There is a Sea-Battle' is/was/will be true, if uttered at n" goes hand in hand in these deviant contexts.[12] If one is truth-valueless, the other is too, and, so, $\epsilon(p)$, unlike $[p]$, has no constant truth-value. Apart from this discrepancy, the two methods generate sentences with identical truth-conditions. Moreover, there are good arguments for one to ignore the discrepancy in the first place, because, if the world is indeterministic and our logical framework is deviant, it is somehow out of context to adopt an *à la* Quine eternalization of sentences. Such an adoption would create semantical situations where -e.g.- 'There will be a Sea-Battle tomorrow' is neither true nor false, while 'There is a Sea-Battle at n' -the verb being detensed here- has a definite truth-value, even for now. (We will come back to these situations in chapter 5.)

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Generally, we see things thus. We believe that one who adopts the Quinean method of eternalizing sentences commits himself to the existence of what we call the "atemporal nature of things and of the world". More precisely, we think that one, in such a case, commits himself to the thesis that Time makes no essential part of the things and of the world. For how could there be any atemporal way to represent reality, if Time were essential to it? The various

[9] i.e. contexts that allow for gaps or more than two truth-values.

[10] n corresponding to "tomorrow", and "There is" being atemporal. This sentence is meant to be the [There will be a Sea-Battle tomorrow], see *supra*.

[11] i.e. "'There is (now) a Sea-Battle' is/was/will be true, if uttered at n".

[12] Why exactly this is so will be explained in chapter 5.

thought experiments involving atemporal beings that seize on the atemporal nature of things -like Kant's God- are to be classified in the opposite vein. For, by postulating atemporal observers who contemplate the same reality that we do, but from an atemporal angle, they *eo ipso* commit themselves to the thesis that things are not essentially temporal. If they were, they would also be temporal to the eyes of these atemporal observers. Moreover, by adopting the Quinean method of eternalizing sentences, one commits oneself to the view that human beings -though in Time- can grasp, or at least approximate, the cognitive status of the above atemporal observers, and can thereby formulate sentences fitting to their linguistic universes - i.e. linguistic universes having only dates and atemporal verbal forms. Numerous problems surround this supposed capacity of ours to assume the status of such creatures and meaningfully formulate sentences belonging to their linguistic universe. At the other end of the spectrum, similar problems surround the hypothesis that these beings can formulate and understand the tensed sentences of the linguistic universe of temporal beings.[13] But, in any event, these are assumptions one must undergo, if one attributes some sense to the -so called- "detensed verbal forms".

On the other hand, our preferred "non atemporal approach towards sentences" exploits the fact that the tensed language of any moment can be used as a metalanguage having as its object language the (tensed) language of other moments. It is in that sense that $\epsilon(p)$ has the same truth-conditions as p.[14] For $\epsilon(p)$ says that the (tensed) 'p' is/was/will be true, if uttered at a certain moment. The object language belongs to this latter moment and the metalanguage to the moment $\epsilon(p)$ occasionally belongs to.[15] Obviously, this approach does not attach any atemporal nature to things or to the world.

We, personally, prefer "eternalizing" sentences in that way rather than in the Quinean way.[16] This, however, does not mean that we definitely exclude the possibility that Time is not essential to things and to the world, and, so, (i) events can be contemplated from an atemporal angle, and (ii) we human beings can meaningfully formulate sentences that make a part of an atemporal language. We do not particularly like the idea, but we do not have any conclusive evidence against it either. Moreover, if the same idea is -despite what we think- correct, the Quinean way of eternalizing sentences should, in principle,

[13] Cf. the debate over whether or not an atemporal God could grasp the meaning of human utterances like "My time of need is now", or "It is twelve o'clock".

[14] There is an exception to this identification but it concerns contexts with truth-value gaps. See chapter 3. For the time being, we adopt a bivalent context without gaps.

[15] Which does not exclude that these two are one and the same.

[16] The tension between these two approaches is sometimes (e.g. S. HAACK, [1978]) identified as the tension separating the Quinean project of eliminating any tense in favor of atemporal verbal forms, and Prior's attitude in "taking tense seriously", and accepting as primitive only the present tense. The other tenses are formed, according to Prior, and according to the method that will be followed in this book (see chapter 4), by sentence-forming operators applicable to present tensed sentences and acquiring an interpretation by quantification over moments of Time. No atemporal verbal form is ever reached by that method. According to Prior, what is primitive and irreducible are the A-series, not the B-series. (A. PRIOR, [1957], [1967], [1968]). A sentential analysis that also insists upon the priority of the tensed over the detensed, and dismisses the idea that anything like the atemporal present is needed comes from the Swiss-Austrian philosopher Marty (see K. MULLIGAN, [1990], esp. the contribution of E. Morscher).

be preferred for it generates much shorter formulas than my $\epsilon(p)$, when fully analyzed. This is an easily anticipatable advantage of the Quinean technique because the same technique stipulates an additional verbal form: the atemporal present. Economizing with verbal forms condemns one to a more laborious procedure for the "eternalization" of tensed sentences and *vice versa*.[17] The only thing we definitely claim is that if one is to adopt the atemporal approach towards eternalized sentences one has to admit that an atemporal perspective has to exist. A timeless reality must be posited.

II[18]

We now pass on some remarks concerning the way we understand causal terms such as, "cause", "effect", "sufficient condition", "necessary condition" and "causal chain".

By paraphrasing Quine's apostrophe when facing the task of defining what sets are,[19] we could say that, *at the present stage, the notions of cause and effect are so fundamental that we cannot hope to define them in more fundamental terms*. Any story has to begin from somewhere, and one of the starting points of *this* story is the above notional pair. We can, if pressed upon the matter, say that "being a cause" is a state predicate that if true about a state implies that another state (or other states), which is (are) the "effect(s)" of the former, will be "brought about" or "generated" or "caused", be*cause* of the former.[20] "If this helps" -we paraphrase Quine again- "well and good!"[21] Now, the predicates of "being a cause" and "being an effect" are meant to apply to (have as range of significance for) a certain category of states, the scope of which can cover -in some theoretical frameworks- every kind of state. For instance, according to some models of relativity,[22] every state is causal[23]. Normally, however, not all states are considered such. A violent incident can be a cause for war, whereas $7 + 5 = 12$ is, usually, taken to be the cause of nothing. About causes and effects, we will assume that any item that is either a cause or an effect has a cause of its own and that it has at least one effect of its own.[24]

[17]This is particularly evident for mathematical sentences, where the non-atemporal approach cannot afford the luxury of the atemporal -e.g.- "Seven plus five equals twelve". The above sentence has to be taken -according to the non atemporal approach- as true at a moment, even if it is true at all moments, and even if the fact that it is true at all moments obtains at all moments.

[18]The lengthy footnotes of this part are not indispensable for the presentation. They focus on the various ways, in which the terms, as defined here, can be made more specific, and on some alternative interpretations of them. There is a tableau at the end of the chapter resuming the main notions defined here.

[19]W. O. QUINE, [1969], p. 1.

[20]This state predicate applies not only to state of affairs that are the case, but also to state of affairs that are not - e.g. Hitler's entering Poland being the cause of World War II, and Hitler's deciding not to enter Poland being the cause of World's War II avoidance.

[21]Cf. J. LUKASIEWICZ, [1922], where the notions of cause and effect as treated as quasi-primitives, although in different forms than here; e.g. the cause relation is taken to be transitive.

[22]Cf. Appendix I.

[23]For what exactly we understand by "causal state" see (iv) at the end of this chapter.

[24]This is independent of whether the universe is deterministic or not (see *infra*).

The relation binding together a cause and its effects is a relation apart from the more general relation between sufficient conditions and the state they are sufficient conditions for. For example: if p is the cause of q, any true conjunctive sentence having p as one of its conjuncts will represent a sufficient condition for q, but it will not be *the* cause of q. Here we resist the temptation of introducing the notion of cause by another notion; on this occasion, by the notion of sufficient condition. The temptation we resist will become clearer from the following example. One might suppose that, if there are several states before tomorrow's Sea-Battle, which all constitute sufficient conditions for the event,[25] the cause of the Battle will be their intersection. This will be because -the same attempt goes on- all these states must share a definite sub-state in common: the cause of the Battle. The effect could not have occurred without its cause. Hence, the necessity for it to be (or at least to belong to) the intersection of these states. The hypothesis seems to be making sense, but falls short, for the following reason. A state that represents a sufficient condition for the Battle could be such that it does not contain the cause of the Battle - e.g. consider what happens if this state instead of containing the cause of the Battle contains the cause of the cause of Battle. In such a case, the state would have represented some sufficient conditions for the Sea-Battle although the cause of it would not make up a part of its sub-states. Which means that a sufficient condition for p not only can take the form of a conjunctive state containing the cause of p, but also of a state that forces the cause of p to come about. The presence of the cause of p implies the satisfaction of some sufficient conditions for p, but the inverse does not obtain. This is particularly evident as for deterministic universes, where there are always sufficient conditions for everything, though it is not the case that the cause of every effect is always there. The cause of the Battle could be a violent incident of this year, although this incident could have been necessary from the beginning of Time. And, so, we will say -on this occasion- that although the cause of the Battle could not have happened before this year there were always sufficient conditions for it.[26] For the same reason,

[25] For the distinction between state of affairs and events see the tableau at the end.

[26] We have to admit that this distinction is debatable, and one could alternatively drop the idea that there is any intrinsic difference separating the sufficient condition for an event from the cause of the event. Which is to say that, alternatively, and if we opt for a more economic causal apparatus we can completely drop the primitiveness of cause, and focus on sufficient conditions. We will not insist upon the evaluation/comparison of these two options here. In general, we are reluctant to dismiss the idea that there are some states, which are essentially causes; we prefer to consider "being a cause" as an irreducible state-predicate binding together a causal state with another causal state: its effect. A reason for this preference of ours is relativity (see Appendix I). Another reason is more intuitive. Say, as above, that we inhabit a deterministic universe, where there are sufficient conditions for everything at any time. If we identify causes with the presence of sufficient conditions, we end-up with a single cause for everything. Now, my impression is that this notion is insufficient to capture the relation between a cause and *its* effect(-s). Figuratively speaking, I believe that in the final exams of a scientific Academy constituted by Laplacean demons, and to the question "What was the cause of World War II" the right answer would not be "The Big Bang", even if, in deterministic universes, and as any Laplacean demon knows, there *were* sufficient conditions for it at that time. The right answer might possibly not have been "the invasion of Poland" either, but something more structural or thoroughgoing or whatever; in any case, something that is intrinsically related to this war, or to this war and some other states, but not to every state there is, as is the Big Bang. On the other hand this insistence of ours on the non-definable character of causes could reveal a kind of philosophical superstition. Therefore, it is essential to note here that for the formal part (i.e. chapter 4) no problem will emerge if one makes no discrimination between "cause of p" and "sufficient conditions for p". The

we resist the temptation of identifying the cause of an event with the set of all the necessary conditions for the event.[27]

We postulate here that no effect can temporarily precede its cause,[28] unless travel backwards in Time is allowed. (We won't be allowing any of that.)

We also pose that, in deterministic universes, causes always differ from their effects, and that each effect can be caused by one and one cause only.[29] Take the above double condition to be the definition of what a deterministic universe is.

To spell it out: we presently pose that a universe will be called "deterministic", if, and only if,

(i) all causes differ from their effects, and

(ii) each effect can be caused by one, and one cause only.[30]

Indeterministic universes differ from deterministic universes in so far as either (i) some events are causes of themselves (these events will be called "self-caused"),[31] or (ii), although each cause has a single effect, this latter is not

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formalism stays indifferent to the interpretation one provides of the cause-effect relation.

[27] As for that, there is an extra reason too. We will see later that in indeterministic universes with convergent futures, one encounters situations where there are sufficient conditions for an event which are not also necessary.

[28] Though in non-relativistic universes where action at a distance is theoretically possible, causes can be simultaneous to their effects.

[29] Observe that this allows causes to have more than one effect.

[30] We note *in passim* the possibility of two further specifications on this definition:
The first is necessary, if one wants to avoid endless causal chains "imprisoned" in one moment. The clause allowing causes to be simultaneous with their effects in deterministic universes opens the possibility for the following two situations, which, though formally innocent, look puzzling. The first is the possibility of an endless series of causes and effects (all different from one another) propagating in space "during" a single moment, and having no causal relation to any other moment (prior or posterior). The other is causal chains which endlessly double back upon themselves "during" one single moment - i.e. the first link of the chain is caused by the last effect. (We mean by "link" what is usually meant by "linked element"; see note 37). It is not difficult to take formal action against such cases, but, since they are formally innocent, we will just ignore them.
The second possible specification does justice to a general intuition, according to which the same state can have a certain effect in this world, but fail to have the same effect in another. If so, we could take causes to be intensional objects - i.e. the "cause of p" would, then, be a function from possible worlds to states: to each world, the corresponding state would be the cause of p as for that world.

[31] This appellation is non standard and might sound awkward, because one would intuitively expect that a cause has to cause something different from itself. The reasons motivating this particular terminological choice will become clearer when we introduce causal chains, for a causal chain is, according to a possible definition (see *infra*), a sequence of causes and effects. Obviously, the intermediate links of the chain are effects of some cause (i.e. of the previous link), and causes of other effects (i.e. of the following link). Chains beginning by indeterministic states will have, on top of them, a state that causes something else (i.e. its successor) but is caused by nothing (else). Now one has two options: in the first place, one can assume that this state has no cause. Otherwise, one can consider it as self-caused. For a similar terminological choice see SPINOZA, [1925], § 92: "*causa sui*" (although in a different sense than the one implied here), and I. KANT, [1956], A/488-B/516: "[...] von selbst gewirkte Begebenheiten". It has to be noticed also that, in the way they are defined

always singularizing.[32] This says that in such universes one and the same effect can be brought about by different causes.

As will subsequently be shown, our allowing for self-caused events allows the future to diverge (fork) at some points, and our allowing for effects that have more than one possible cause allows the future to converge. For the moment, we will provide an example for each of these two cases of indeterministic universes. If the state (p at n) is self-caused, this means that there can be no (causal) explanation of that state before n.[33] Therefore, the cause of (p at n) cannot precede n; (p at n) will be considered as causing itself. On the other hand, if the future converges, two different possible Histories will have to merge into one, after a certain moment. If we allow for effects having two possible (mutually exclusive) causes we have a nice model for convergence. More precisely, take $w(1)$ and $w(2)$ to be the last two universal states, before the convergence. If so, the universal state that comes afterwards has to be the same for both. Now, $w(1)$ and $w(2)$ have to differ for at least one state - say that p belongs to $w(1)$ and $\neg p$ to $w(2)$. A reasonable explanation of the convergence is that p and $\neg p$ are both causes but they bring about the same effect, hence the identity of the successors of $w(1)$ and $w(2)$.[34]

Formally, the relation between cause and effect will be noted by "$\ldots \rightarrow_c$—", and read: "…causes—". This connective will be called "causal implication", and the whole conditional, "causal conditional".[35] It will be taken to imply the simple implication but not to be implied by it. (I will explain in the next chapters why I call this latter "simple", instead of "material".)

here, self-caused states might generate endless causal sequences constituted out of a one and only state: the self-caused event. This is unwelcome, and we will formally forbid it, when introducing causal chains.

[32] We will say that "q is a singularizing effect of p", if, and only if, by the information that q is the case, one can conclude that p has been/is the case, and that p is the cause of q. What we mean is that, ideally, if A has the information that q is the case at n, A can validly conclude that p has been/is the case at some $m \leq n$, and that it is the cause of q. (The "…/is" clause concerns the self-caused case, i.e. $n = m$). "Non singularizing effect", like "self-caused state" can sound awkward. The reason is similar. An effect which is not singularizing could arguably be not an effect at all. See the previous note.

[33] A clarification about hybrid states that are partially deterministic and partially not, is necessary at this point. Say p stands for a compound causal state having some elements that have causes other than themselves and some elements that are self-caused: p will also be self-caused. And as for convergence: say that p is a compound causal state having some elements that are singularizing effects of some states and others that are effects, but not singularizing: p will not be considered as a singularizing effect of anything.

[34] Future convergence can be modeled in other ways as well. For example, instead of stipulating that p and $\neg p$ above have the same effect, one could stipulate that they have no effect at all. The reason we prefer to say that they have an effect, but that this effect is not singularizing, instead of saying that they have no effect, is parallel to the reason why we prefer to say that there are self-caused events, instead of saying that there are events having no cause.

[35] Obviously, $p \rightarrow_c p$ obtains, if, and only if, p is self-caused; $(p \rightarrow_c q)\&(r \rightarrow_c q)\&(p \neq r)$ obtains, if, and only if, q is no singularizing effect of either p or r.

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"Sufficient condition for..." is to be taken as equivalent with "state, making impossible the non-advent of...", and "necessary condition for..." as equivalent with "state necessitated for the advent of..." It is important to insist here upon the fact that we consider all these conditions diachronically - i.e. by "set of necessary conditions for p", we do not mean the set of necessary conditions for p that have time indexes smaller or equal to now, but the set of *all* necessary conditions for p, no matter whether they refer to the present and the past or to the future.[36] This specification is important, because it is only then that one can claim, in general, that the set of all necessary conditions for p is always a sufficient condition for p. Note that the converse does not obtain in universes with convergent futures, because, in such universes, one encounters states for which there can be many (incompatible with one another), sufficient conditions and, consequently, the set of necessary conditions has to include disjunctive sentences containing all of them - e.g. if p is a sufficient condition for q, and so is r, and neither p implies r, nor r implies p, the set of necessary conditions for q would have to contain a $p \vee r$ condition. Neither p, nor r contains such a condition, and, therefore, they are not necessary. So, in deterministic universes, if there are sufficient conditions for p, all the necessary conditions for p are satisfied and *vice versa*. In indeterministic universes, and more precisely, in universes with convergent futures, there are cases of conditions that are sufficient, but not necessary. Since in such universes it is possible to have two different and mutually exclusive causes of the same effect, neither of them would be a necessary condition for it though both of them would be sufficient.

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The notion of "causal chain", although intuitively clear, presents several formal complications that call for a more detailed presentation.

We will provide three alternative (non exclusive) definitions, which will be used, thereafter, according to the occasional context.

(A) concerns causal chains of causes and effects,

(B) concerns what we shall call "weak causal chains" and

(C) what we shall call "strong causal chains".

[36] e.g. consider that we are one day away from the Sea-Battle and the positive deliberation of the admiral is scheduled for sometime during the following morning and that it is an indispensable parameter for the advent of the Battle. We call this "a necessary condition for the Battle", even today. Consequently, when we will say that "all the necessary conditions for a future event are now satisfied", we will include the ones referring to the future (e.g. admiral's deliberation). For such cases, "satisfied" would be equivalent with "impossible not to happen". More generally, if p is a causal condition (necessary or sufficient), we will say that p is satisfied at n, if and only if we have at n enough evidence that p was, is or will be the case.

A) The first problem that one encounters when trying to define what causal chains are concerns a certain misfit between the intuitive understanding of what constitutes a succession of interlinked causes and effects and what this succession looks like when formalized. For when in everyday language we speak of a series of events that have led to a certain situation we sometimes have in mind that the event at the top of the series causes its successor, that this latter causes its own successor, and so on, until one reaches the final situation. This is indeed what one usually means by locutions of the sort: "and one thing has led to another, and here we are now in the state we are in". The links of the chain represent states causally related to one another and leading to a certain final result (the chain's final link).[37] Now, the first problem. One can always be certain that the final link of the chain represents a state that will indeed be actualized (i.e. satisfied, be the case), if one knows that its first link is unavoidable. This is because there are within the state at the top of the chain sufficient conditions for the state at the other end. For example, assume the chain $\langle p(1), p(2), \ldots, p(n)\rangle$. It is obvious that if $p(1)$ is the cause of $p(2)$, $p(2)$ is the cause of $p(3)$ [...] $p(n-1)$ is the cause of $p(n)$, then $p(1)$ represents sufficient conditions for $p(n)$. For, if $p(1)$ cannot but generate $p(2)$, and $p(2)$ cannot but generate $p(3)$ [...] $p(n-1)$ cannot but generate $p(n)$, by the time $p(1)$ has become necessary, $p(n)$ cannot but be destined to happen.

B) According to (A), causal chains, of which the first link is satisfied, cannot represent successions of interlinked states that could, but did not, finally arrive at their last link. There are occasions, however, when what one means by "series of events leading to a certain result" does allow for that. For when one expresses himself in the above way one has possibly in mind that something occurs and helps another thing to happen and this thing does indeed happen and then, combined with other things, generates a third, and this third thing, combined with still other parameters, has a forth consequence, and so on until the end of the chain. And in contexts where this is indeed what one intuitively understands by "causal chain" a causal chain does not necessarily run all the way to its final link. This means that, in such cases, one is not in a position to deduce that the final link will be reached solely out of the information that the beginning is necessary. On these occasions, within the expression "one thing leads to another", this "one thing" is from the formal point of view a *necessary*, not a *sufficient* condition; a necessary condition that when added to the rest of the necessary conditions makes the whole sufficient. Let us now try to formalize this variation. The ordered n-tuple $\langle p(1), p(2), \ldots, p(n)\rangle$ will be said to be a causal chain, if, and only

[37] More formally, "links" and "chains" are here understood as follows: "Links" are the elements of ordered n-tuples $\langle p(1), p(2), \ldots, p(n)\rangle$, where, for every $p(m)$, $p(m)$ is a state causing $p(m+1)$ and being caused by $p(m-1)$. "Chains" are the ordered n-tuples themselves. This terminology is somewhat different from the standard. Usually one calls a relation -e.g. "R" of aRb- a "link" and a and b the "linked elements". In our vocabulary the link and its elements are one and the same. Cf. a remark of Wittgenstein's, according to which what holds the chain together are the links themselves. There are no extra things uniting the linked elements so as to turn them into a chain. L. WITTGENSTEIN, [1973], p. 23: "There isn't anything third that connects the links but the links themselves make the connection with one another". Notice also that, as for the self-caused event p, $\langle p, p\rangle$ is a causal chain under this definition. In order to avoid chains constituted out of endless loops (i.e. $\langle p, p, p, \ldots\rangle$), we have to impose the stricture that no two occurrences of the same link are allowed within a single chain.

if, $p(1)$ is a necessary condition for $p(2)$, $p(2)$ is a necessary condition for $p(3)$ [...], $p(n-1)$ is a necessary condition for $p(n)$. This, we call a "weak causal chain". One of its characteristics is that it can "break down" at any link. For this to happen it suffices that while some link is effectively reached some, among the rest of the necessary conditions for the successor of that link, will not be satisfied at the end. This successor will be said to be the link where the chain "breaks down".[38]

C) We now go back to the intuition of (A), aiming to define causal chains, the first links of which causally imply all the rest - i.e. we try anew to formalize everyday language locutions, where one speaks of "things leading to other things", and meaning *effectively* leading to them. One way to formalize these is (A) itself. Another is to employ chains of sufficient conditions. These suffer from the same rigidness that (A) suffers from.[39] We will now follow a more flexible device based on sets of necessary conditions. The following notational facility will help:

$\langle p, q\rangle' =_{\text{df.}}$ The set containing all the necessary conditions for q *minus* p.[40]

It is obvious that, if $\langle p(1), p(2), \ldots, p(n)\rangle$ is a weak causal chain, by the above standards then the following must obtain: $p(1) \cup \langle p(1), p(2)\rangle'$ is a sufficient condition for $p(2)$ & $p(2) \cup \langle p(2), p(3)\rangle'$ is a sufficient condition for $p(3)$ & [...] & $p(n-1) \cup \langle p(n-1), p(n)\rangle'$ is a sufficient condition for $p(n)$. We proceed with the definition. We will say that the ordered n-tuple $\langle p(1), p(2), \ldots, p(n)\rangle$ is a "strong causal chain", if, and only if, $p(1)$ is a necessary condition for $p(2)$, $p(2)$ is a necessary condition for $p(3)$, [...] , $p(n-1)$ is a necessary condition for $p(n)$,[41] and every element of

$$\{\langle p(1), p(2)\rangle', \langle p(2), p(3)\rangle', \ldots, \langle p(n-1), p(n)\rangle'\}$$

is satisfied.[42] We will call the chain "strong", because, with the above extra condition breakings are no longer possible - i.e. if $p(1)$ is satisfied, and

[38] More formally, we will say that a causal chain "breaks down at the link n", if, and only if, all the states corresponding to the links coming before n are reached, but n will never be. Symmetrically, we will say that a weak causal chain "is fortified (or strengthened) at a link", if, and only if, the chain reaches that link, and by the time it reaches it all the rest of the necessary conditions for the rest of the links have also become satisfied, while previously they were not.

[39] Chains of sufficient conditions would have been even less useful than chains of causes and effects. For example, in deterministic universes that are non-endless towards the past, all causal chains of sufficient conditions with more than two links could be reduced in two link chains. They all could be of the form $\langle$Big Bang, $p\rangle$, p being any historical state there is. This is because "being a sufficient condition" -unlike "being a cause"- is transitive. (The non-transitivity of our cause to effect relation can be debated as well, but we will not pursue the relevant discussion here).

[40] Notice that according to this definition p need not be a necessary condition for q. In such a case $\langle p, q\rangle'$ contains all the necessary conditions for q.

[41] i.e. $\langle p(1), p(2), \ldots, p(n)\rangle$ is a weak causal chain.

[42] The second proviso of the definition obliges "strong causal chains" to be of a partly set-theoretical and partly sentential nature. On the one hand they are ordered n-tuples of conditions and on the other they are sentences - i.e. to say that $\langle p(1), p(2), \ldots, p(n)\rangle$ is a strong causal chain says that $p(1), p(2), \ldots, p(n)$ are elements ordered in a certain way, but also asserts a conjunctive sentence about the satisfaction of some conditions (because of the second part of the definition). This ambiguity is not present in the notion of "weak causal chain".

the chain is strong in that sense, $p(n)$ is destined to happen. The present definition of a (strong) causal chain is more flexible than (A) for the following reason: there can be more than one necessary condition for the same state and consequently there can be more than one strong causal chain leading to the same result - e.g. assume that p and q are the two necessary conditions for r; then, $\langle p, r\rangle$ and $\langle q, r\rangle$ could be different strong causal chains leading to r. The difference between these two and the corresponding weak chains is that by knowing that p or q (respectively) obtains one can deduce r. Notice, also, that, if we disregard cases of future convergence each cause will also be a sufficient *and* necessary condition for its effect and therefore if there is no convergence and $p \rightarrow_c q$ obtains, $\langle p, q\rangle$ is both a strong causal chain and a causal chain according to (A). Matters are more complicated in future convergence cases. In the first place, and in such cases, one can encounter situations where $p \rightarrow_c q$ is a causal conditional, p does not obtain but there is a strong causal chain leading effectively to q. This will be, if -e.g.- $r \rightarrow_c q$ is also a causal conditional, and r is necessary. Besides, there could be cases where there is a strong causal chain leading to a certain state but no corresponding "actualized" chain[43] belonging to the set of chains satisfying (A). Consider, for example, the case where both $p \rightarrow_c q$ and $r \rightarrow_c q$ obtain but neither p nor r is satisfied though $p \vee r$ is necessary.[44] The strong causal chain that obtains is $\langle p \vee r, q\rangle$, but there is no actualized causal chain, in the sense (A).

The above distinction between strong and weak chains enables us to give a formal explanation by disambiguation of the possible co-truth of "There is now no causal chain leading to p" and "A causal chain leading to p is now actualized". The first sentence could be considered as true on the basis of some necessary conditions for p that are now missing while the latter, on the basis of some necessary conditions for p that are now fulfilled. For example: consider, again, the Sea-Battle. According to the above definitions it is possible that there is today a weak causal chain leading to the Battle even when an indispensable parameter for its occurrence is not yet satisfied - e.g. the admiral's deliberation in favor of it. It just suffices that another necessary parameter for the event (e.g. a violent incident of today), is. In such case and if $p(n)$ stands for the Sea-Battle and $p(m)$ for the violent incident, $\langle p(m), p(n)\rangle$ is a weak causal chain, which is now actualized. Nonetheless, within $\langle p(m), p(n)\rangle'$ there is a future indeterministic event (i.e. admiral's deliberation) and so there is now no actualized strong causal chain leading to the Battle. In other words, saying that "a strong causal chain leading to $p(n)$ is now actualized" has to be co-extensive with: there are now sufficient conditions for $p(n)$ - i.e. not only $p(m)$ but also $\langle p(m), p(n)\rangle'$ is satisfied. This fits well with the intuition that the admiral's (free) deliberation about the Battle will generate a new causal chain. It will generate a new *strong* causal chain because unless all the necessary conditions for an event are met there can be no actualized strong causal chain leading to it and since the

[43] "Actualized" here means that one of its links concerns the present. See also points (ii) and (iii) at the end of this chapter.

[44] Notice here that the disjunction could be such as to cover the entire logical space - e.g. take $\neg p$ for r. In such cases one could deduce q upon zero premises - i.e. if both p and $\neg p$ are causes for q, one gets $\emptyset \vdash q$.

admiral's deliberation concerns the future and is a self-caused event one can conclude that there can be no actualized strong causal chain leading to the Battle at present though many weak ones are there already.

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A final four terminological remarks about causal chains.

i) We will say that a weak causal chain "stops dead at the link n" if, and only if, the state standing for that link has no singularizing effect of its own.[45] On that basis it is interesting to see some among the interrelations between weak and strong causal chains. Weak causal chains leading to events can be inaugurated before the corresponding strong ones are inaugurated. This is trivial. There can be some necessary conditions for an event before the corresponding sufficient conditions. On the other hand, once a weak causal chain stops dead there can no longer be any actualized strong causal chain containing as a segment of it the weak chain that has stopped dead. Being a weak causal chain is by definition[46] a *sine qua non* for strength.

ii) We will say that a chain "traverses", "passes through" etc., moment n, if and only if, a state represented by a link of that chain concerns moment n. This way of speaking is intuitively clear, I hope, but intolerably vague from the formal point of view. Unfortunately, it will remain so until chapter 4, where we will introduce the formation rules for temporal indexes. From chapter 4 onwards to say that "a causal chain traverses (passes through) moment n" will be equivalent to saying that it has a link of the general form: $p(n)$.

iii) To say "a chain is *now* actualized" or "is *now* on" will be equivalent to saying that the chain passes through the occasional present moment. Notice, then, that as for past events one can guarantee that they have happened only when they are upon a weak causal chain that is *now* actualized, but for future events one can guarantee that they will happen only when they are upon a strong chain that is *now* actualized.

iv) We will call a state "causal" if, and only if, it appears upon a causal chain.[47]

[45] Notice that it is another thing to say that $p(n)$ is the final link of a chain, and another to say that the chain stops dead at $p(n)$. If we write down the referent of a chain ending at $p(n)$ this will look like: $\langle p(1), p(2), \ldots, p(n)\rangle$; $p(n)$ will be the final link of the chain. Nonetheless, if $p(n)$ has any singularizing effect of its own, we won't say that the chain "stops dead in $p(n)$". However these two become identical if we "maximize" the chain. A chain $\langle p(1), p(2), \ldots, p(n)\rangle$, will be said to be "maximal", if, and only if, there is no $q(1)$ and there is no $q(2)$, such that $\langle q(1), p(1), p(2), \ldots, p(n)\rangle$ or $\langle p(1), p(2), \ldots, p(n), q(2)\rangle$ is a causal chain. Maximal chains are not to be confused with maximal paths (see chapters 3 and 4). These latter have not only maximal length, but density as well. One can also define a "dense" causal chain by some further conditions but this will not be necessary here.

[46] Cf. (C).

[47] Notice that this is wider than "being either a cause or an effect", because it comprises necessary and sufficient conditions.

III

As I have said, the solution, in favor of which I am going to argue is "intuitionistic". In a sense this is not an entirely accurate description of my device.

It is accurate, in so far as the models that will generate the present way out of the Sea-Battle problem are generalizations of the models that the Dutch logician, E. W. Beth presented in 1956, in order to give a semantic foundation to the intuitionistic calculus. In fact the models that will incorporate my answer to the Sea-Battle problem are conservative extensions of these Beth models. Now to some people the characterization, "intuitionistic" is equivalent with "Kantian" or, more generally, "conscious-centered". From that angle it is wrong to say that my models are "intuitionistic". For, as I said in part I, I do not believe that Time makes no essential part of reality. In contrast, any intuitionistic approach that stays obedient to a rigidly Kantian *Weltanschauung* would be formalizing a subject-centered solution of a temporal observer. This, on the occasion, says that since Kant thinks that things are not essentially temporal the (temporal) Beth models are not representations of the world *per se*, but representations of how the world appears to the apprehending (human, temporal) individual. It is essential for my presentation to clarify this beforehand, because a certain tension will emerge between chapters 3 and 4. In chapter 3, I will elaborate the "intuitionistic" solution by means of a thought experiment involving an idealized temporal scientist. This might tempt one to assume that my solution is Kantian, in the sense that it formalizes not "tomorrow's Sea-Battle" *per se*, but the way human individuals are predestined to approach the event. Or, more precisely, that it formalizes the *a priori* epistemic shortcomings of the same beings towards the event. Consider the idealized "outside of Time" observer of part I. For him, and in the atemporal language he uses, the Sea-Battle problem does not even arise. The truth-value assignments that his atemporal standpoint provides to the atemporal sentences of his linguistic universe are classical throughout. In fact, if we assent to the reality of this linguistic universe the appropriate "solution" to the fantasy is that codified in chapter 1 as the "conservative solution". It says that $T(p \vee q)$ & $\neg T(p)$ & $\neg T(q)$ obtains for future contingents because there is a tacit operator between the two negation symbols and the sentences they govern - i.e. the operator standing for "determinately". The distinction between determinately and non-determinately being a distinction concerning the limitations that our temporal perspective imposes on our grasping the (atemporal) truth-value of sentences. For atemporal beings occupying the outside of Time standpoint everything is known - even the indeterminate states of our temporal point of view. Once this understood, the "objective problem" vanishes. And, in that sense, my models, if Kantian, would be depicting the epistemic shortcomings of human beings. In chapter 4, however, it will become clear that my models are subject-free, and, as such, can be put in agreement with the general view that Time makes essential part of reality.[48]

Mutatis mutandis, the same obtains as for causes and effects, and, in general, as for the causal apparatus of my models. A rigidly Kantian approach -i.e. an

[48] My attachment to the tensed apparatus testifies just that. For -my claim is- there are no *stricto sensu* detensed sentences; my $\epsilon(p)$ sentences are no *stricto sensu* detensed sentences.

approach that takes causes and effects not to belong to the things themselves but to the Categories- would be making of my models models that depict the cognitive shortcomings of the apprehending (human) individual. However, the conscious-free presentation of chapter 4 will make it clear that my models are compatible with the view that the causal apparatus makes part of the world *per se* and not of the way humans are cognitively “condemned” to approach it.

The previous observations will, in my models, find the following formal account. “Time”, “causes” and “effects” will be taken to be features of the world *per se*, and all these characteristics, although possibly analyzable into more primitive ones, will not concern the apprehending individual. Evidence for that is that I allow for consciousless universes by allowing for models for lifeless Histories. Thus I will be considering the possibility that indeterministic events take place in Histories containing no living beings. The Sea-Battle will no longer do -too many people are involved there- but the “chronicles of an electron” would. For example, we will say that the sentence, ‘The electron will move in the upward direction’ is truth-valueless as long as this is (objectively) undetermined, and even if the electron inhabits a lifeless world.[49] This, transposed to the Sea-Battle, says that we are at present unable to assert either, “There will be a Sea-Battle tomorrow” or, “There will be no Sea-Battle tomorrow” because there is at present no fact of the matter corresponding to any of the above sentences; not because there is one but we are cognitively closed with respect to it. The gaps (seen thus) are not cognitive they are real and concern the world itself, not the people living in it. In other worlds, if an ideal scientist living in Time ignores that p is the case, it can only be because p is-not the case;[50] not because p is the case, but the ideal scientist, since temporal, is unable to reach that knowledge.

My models are intuitionistic. To what extent they are Kantian, or conscious-centered is now to a certain extent, clarified. They are Kantian in as much as they take Time and causality to be intrinsic parts of the underlying logic, and they are not Kantian in as much as they do not do so *faute de mieux* - i.e. not because of the cognitive limitations human (and in general temporal) beings suffer from.

[49] This, of course, does not mean that there can be any linguistic model of these consciousless Histories in worlds where there are no cognitive beings. But I do not go from the assumption that to create a linguistic model for a consciousless universe necessitates a universe inhabited by speaking individuals to the conclusion that to assume a consciousless (or even lifeless), universe is self-contradictory.

[50] Which, however, does not imply that not-p is the case. See the next two chapters.

Synopsis of the main causal notions

cause/effect	Non-defined primitive notions. State predicates characterized by a relation connecting the one (cause) to the other (effect). We pose that no effect occurs without a cause and that no cause can have more than one effect. (This does not imply that the universe is deterministic; cf. *infra.*) Moreover we pose that no effect can temporally precede its cause. The predicate is not restricted to states that obtain. Formally, causes are bound to effects by the "causal conditional": $\ldots \rightarrow_c$— The relation is called "causal implication" but we will sometimes use the former term for both.
singularizing effect	An effect that can be caused by one and one cause only - i.e. if y is an effect, we say that y is a singularizing effect, if, and only if, there exists a state x, such that by the information that y obtains, one can conclude that x is its cause; in such case y is called the "singularizing effect of x".
deterministic universe	A universe in which (i) all causes differ from their effects and (ii) each effect can be caused by one cause only - i.e. all effects are singularizing.
indeterministic universe	A universe that is not deterministic - i.e. a universe that either (i) some effects are the same with their causes or/and (ii) some effects are caused by more than one causes. Universes falling under (i) have divergent futures, and universes falling under (ii) have convergent futures.
event	Kind of state obedient to the condition: if x is a state, x is an event at t, if, and only if, x obtains at t, but does not obtain at *t-1*. (This definition applies to Time discrete).
self-caused event	An event that is an effect having itself as its cause - cf. indeterministic universe (i).
indeterministic event	An event that either (i) is self-caused or (ii) is an effect that is not singularizing.
sufficient condition	State predicate characterized by a relation binding together the sufficient condition with a state (states) it is sufficient condition of. If x is a state and is sufficient condition for y, then, if x obtains, y is unavoidable.
necessary condition	State predicate characterized by a relation connecting the necessary condition with a state (states) it is necessary condition of. If x is a state and is necessary condition for y, then, y cannot obtain, unless x obtains.
satisfaction/actualization (for conditions)	A condition is satisfied/actualized, if it obtains. The predicate is applied diachronically - i.e. if (x, at t) is a condition for (y, at t'), then, if x obtains at t, we say that the condition is satisfied at any t'', provided that "x, at t" is true at t''.
causal chain (sense A)	A sequence of causes and effects.
causal chain (sense B: *weak* causal chain)	A sequence of necessary conditions and the states of which they are necessary conditions.

causal chain (sense C: *strong* causal chain)	A sequence of necessary conditions and the states of which they are necessary conditions. It differs from the weak causal chain, in as much as, by the information that a state belonging to the chain obtains, we get the additional information that the rest of necessary conditions for its successor are also satisfied.
link (in a causal chain)	An element of the chain. Links and elements are extensionally the same.
strengthening/fortification (of a weak causal chain)	A weak causal chain is strengthened/fortified at its n^{th} element, if, and only if, it becomes strong at that element.
breaking down of a causal chain	A causal chain breaks down at its n^{th} element, if, and only if, this element is satisfied, but the element *n+1* cannot possibly be satisfied.
stopping dead (of a weak causal chain)	A weak causal chain stops dead at its n^{th} element, if, and only if, the state standing for that element has no singularazing effect.
passing through/traversing (for causal chains)	A causal chain passes through the moment t, if, and only if, an element of the chain "concerns" moment t. For a more formal elaboration, without the vague "concerns", see chapter 4: the element of the chain is of the general form, $p(t)$.
actualization (for causal chains)	A causal chain is actualized at t, if, and only if, an element of the chain traverses t, and the element corresponding to t is actualized.
causal state	A state that belongs to a causal chain (A, B or C).

Chapter 3

The Intuitionistic Solution - Intuitive Approach

In part I, I try to clarify the sense in which my solution belongs to the tradition of "logical atomism" and the necessity motivating such a classification. Part II gives a narrow picture of the model in which the solution emerges and, in part III, I present the solution through a thought experiment based on an idealized scientist who much resembles the *Creating Subject* of intuitionists, save that he establishes truths about History rather than mathematics. The general style of the presentation is informal: the chapter is intended (especially parts II and III) to be the informal counterpart of chapter 4.

I[1]

There is an isomorphic correspondence between the world and language. To every state of affairs there corresponds a declarative sentence representing/describing it. The world is constituted out of a number of simple objects while in language there is an equal number of names that denote these simple objects. Atomic states stand for the ways these objects are combined and atomic sentences are the corresponding configurations of the names denoting these objects. Compound states are combinations of atomic states. The whole world is such a compound state. Therefore, the whole world can be reduced to the atomic states that obtain in it. On the linguistic level compound sentences are well-formed combinations (*via* the primitive connectives) of atomic sentences. The truth-values of compound sentences are reachable by truth tables. There is a (compound) sentence that describes the universal state.

The truth-value of a compound sentence is always a function of the truth-values of the sentences it is composed of. These latter might be atomic, as they might be not atomic. But at the end each sentence is revealed to be a well formed string of atomic sentences and primitive connectives (if it is not atomic).[2]

[1] Some acquaintance with the literature of logical atomism is presupposed for parts I and II. In order not to overcharge the exposition I have systematically suppressed references. Logical atomism can be seen as a background theory.

[2] As we will see, this is a controversial issue even among atomists. For some (e.g. Russell) there are such things as primitive connectives. For others (e.g. Wittgenstein) the very idea of

The promethean atomistic enterprise of analyzing/correcting natural and pseudoscientific language consisted in the following motto. One should constantly analyze away (eliminate) any sentential part which is not primitive. Thus, any -e.g.- apparent name (e.g. a definite description) would be eliminated in favor of more primitive expressions, until one reaches the names of simple objects. The fully analyzed sentence will be revealed to be such a sequence of simple names and, if it not atomic, of atomic sentences and primitive connectives. Together with the elimination of the apparent names, the misleading ontological presuppositions surrounding these latter will also disappear. Ideally, when reaching the ultimate names one will be able to grasp the fact that they denote something instead of nothing by the -so to speak- "way they sound". And one will also be able to recognize whether the sentences formed by combinations of simple names are true or not by one's being acquainted, or by one's not being acquainted with the atomic states these combinations describe, or fail to describe. Then one, ideally, will be able to reconstruct the complete linguistic apparatus, by introducing abbreviations, incomplete symbols, non primitive connectives, etc., but one will no longer be under the illusion that these are primitive. For by that time, one will already have been delivered from all the misunderstandings one's previous lack of knowledge about language and (correspondingly) about the world had been causing.

The previous two paragraphs, we have to confess, are not the quintessence of either eloquence or clarity. This is partly our fault. Logical atomism is a horribly complicated matter. Nonetheless, we have to add, the same lack of clarity is not entirely our fault.

For example, let us assume that atomic sentences are sequences of names. But then what kind of things do they name? Are simple objects individuals only, or do they also contain logical forms and qualities? Correspondingly, are simple names only the names of individuals or are they also simple names of, e.g., connectives, predicates, etc.? Besides, how do we know that the analysis ever comes to an end? How do we *a priori* know that some simple names will emerge at some final terminus and that the analysis does not go on *ad infinitum*?[3] And what about existence? Is existence a derived notion? Will we no longer be allowed to say that a simple object exists, when, having reached the final stage of analysis? If so, in what sense do we now assert that the fundamental fact about the world is the existence (persistence) of simple objects? What does this existence amount to when put in terms of the aimed final stage of analysis? And, if operators, predicates, relations, quantifiers, etc., will ultimately be revealed to be non-primitive, how will a sequence like "Paul, John, Peter, Mary," (for these are proper names aren't they?) constitute a meaningful sentence, that describes an elementary feature of this world? Of course we have Wittgenstein's cautious remark that what language will look like after the final stage need not have anything to do with what language (even the most formal language) looks like now.[4] So, possibly, simple names will be revealed to

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a connective which is primitive is contradictory. Here we will adopt the simpler (Russellian) version, which gave rise to modern model theory (see *infra*).

[3]There is, of course the notorious "argument for substance" in the *Tractatus*, but scholars do not agree on what exactly this argument is - let alone whether it is sound.

[4]Cf. F. WAISMANN, [1967], p. 42: "The logical structure of elementary propositions

have completely different "looks" from those linguistic items that we, at present, call "names". But then again, one might ask, doesn't logical atomism use the present unaccomplished state of the analysis and the awareness of the fact that it is unaccomplished as a cover (i.e. pretext) for reassuring itself that, in the end, everything will be revealed to be exactly as promised? Are we not hiding behind the fact that we are still (and God knows for how long) underway? That is to say, are we not using our *de facto* present incapacity to name a single simple object in order to feed our optimism about the whole enterprise, instead of taking the same incapacity as evidence against the project itself? For it looks much as if we have set off with a relatively clear and simple notion: the idea that the world is a configuration of simple objects we then, have posited that primitive language reflects this configuration and, finally, have posited that natural language will be revealed -when fully analyzed/corrected- to be the very same primitive language that we are now dreaming of. It looks much as if we have begun with a project that seems to be a reasonable one, have then entered (eyes wide shut) into an abyss and we now hope that when we come out of the abyss again, we will reopen our eyes only to realize that we are holding a perfect language in our hands and inhabit the promised land of a perfectly transparent logical world. At that point -i.e. at our exit out of the abyss- the fact that we possess a perfect language will retrospectively testify in favor of the clarity of the notions we now possess. On the other hand, the fact that we really need to go through the abyss and that we, at present, haven't got through it as yet speaks in favor of our present optimism. "This is how things are going to be revealed to be. We cannot show it yet, because of the abyss, but when we are in a position to undertake the whole journey we will be ultimately justified!"

Very few things in life are less graceful than forcing our way through open doors. Since Wittgenstein's notorious turn against his own early Picture Theory of Language (and even earlier than that) logical atomism has been heavily criticized for these and similar points.[5] I am certainly not writing this treatise to add yet another clue to the tormented body of this doctrine, and, far more than that, I do not compliment myself on the thought of having found pitfalls in it.

Now to the obvious question, why is it that, despite all this, I want to state my intuitionistic way out of the Sea-Battle in terms of logical atomism, the answer is twofold.

Firstly, it is because, as I was saying in the previous chapter, I want to dissociate the intuitionistic way out from the Kantian (and, more generally, from the conscious-centered) connotations the appellation "intuitionistic" might suggest.

need not have the slightest similarity with the logical structure of propositions". The above quotation is a remark made to Waismann by Wittgenstein.

[5] I have here mentioned the more general points of the critique. Other more technical and specific points are left aside - e.g. the problem of color-exclusion, strikes me as less central because it is based upon the assumption that a sentence like, "This point is red" is (or is very close to) an atomic sentence. Otherwise, the critique doesn't get through, the atomist can still challenge this very assumption. In my mind, the real problem an atomist faces is rather that at the present stage there can be no effective counterexamples to the atomistic project, exactly because the project is unaccomplished, hence, the dubious strategy I have been talking about.

The intuitionistic setting will be thus "objectified" in a way. Since we accept of no atemporal observer, it no longer makes any sense to speak of the things in themselves *qua* the things known to the same observer. Therefore, the impossibility to grasp some of the features of our world is not accountable by our cognitive limitations. It is accounted by the fact that these same features are genuinely underdetermined. "Genuinely" meaning here: not because we cannot do better, while others (e.g. an atemporal God) can.Now, since the fantasy will find room in such a rigid context of analysis, it, *a fortiori*, has room enough, in other less rigid contexts, such as the epistemic.

Secondly, I will refer to the atomistic machinery because despite the fact that nowadays very few people support logical atomism the main part of the literature about models uses atomistic notions, without even recognizing them as such. For example, all handbooks on model theory describe model formation in terms of what was once thought to be the final analysis of language. They assume a vocabulary of sentential variables, primitive connectives, individual variables, constants for individuals, predicate letters (sometimes predicate variables), function letters (sometimes function variables) and quantifiers. And they also assume that all well-formed sentences can be reached out of these.

Moreover, they use the notion of "atomic sentence true in a model (interpretation)" and then go on to construct whatever else is true in the model by the truth-value that these "atomic sentences" have in it, and the definitions of the connectives. Now if this technique has no kinship with logical atomism, I cannot think of what else logical atomism is kindred with. In fact, the general schema resembles (if not is "identical with") a certain conception of logical atomism Russell once had.[6] For in the early stages of the theory Russell thought that the world is constituted out of individuals which have some properties and are related with one another in certain ways as he also thought that an exhaustive (final) analysis of the world will be attained only when all these properties and relations are depicted within a language that has only names for the individuals, names for the properties they have (i.e. one place predicates), names for the relations they are governed by (i.e. more that one place predicates), quantifiers (Russell thought that existence is primitive), and the rest of the primitive connectives. According to this early Russellian notion, each sentence describes a feature of the world and atomic sentences are well-formed sequences constituted out of n names of individuals and n-adic predicate letters. The truth-value of atomic sentences is knowable by acquaintance, and all other sentences are constructible from atomic sentences and logical forms (represented in language by the connectives). Now substitute the idea that the truth-values of atomic sentences are known by acquaintance, to the idea that they are defined arbitrarily for each model/interpretation and the degree of kinship between contemporary model theory and Russell's early atomistic conception will jump into the eye. Wittgenstein's critique of this early Russellian conception has, perhaps, been fatal for it. To take an example, logical forms -as Wittgenstein has claimed- might indeed be impossible to be primitive and consequently primitive connectives might be a self-defeating notion. Besides, Russell himself conceded much

[6] This is around 1910. See esp. B. RUSSELL, [1984]; A. N. WHITEHEAD - B. RUSSELL, [1910-1913]; Introduction.

of Wittgenstein's critique and abandoned his own early version of the theory. However, although the same "defects" underlie all literature on model theory, no one seems to be worried enough about them to apply the same criticism against model theory. And yet, it is the very same fundamental conception of the world that inspires them both. Their difference is that logical atomism was aiming to become an all-embracing exegetical scheme, while model theory was founded in contexts which had to do with truth, consistency and completeness - i.e. it was of a somehow narrower inspiration and scope.[7] This, however, does not mean that it is not open to the same general line of critique.

The third, and final, reason why I want to incorporate my solution to an atomistic background is that despite the above mentioned "defects" in the whole conception and the possible need to restate some of its fundamental principles, my analysis will prove the atomistic conception -either in its historical or its reevaluated form- to be incompatible with any indeterminacy and, consequently, with historical indeterminacy as well. This will be an essential discovery, since it will prove that the atomistic *Weltanschauung* is incompatible with indeterminism and therefore inappropriate for the analysis of events such as the Sea-Battle. Following this proof I will present a way to further relax the project so as to make it compatible with such events and make it capable of endorsing claims such as the "fantasy", or future openness.

To these ends, we are now going to assume the following:

We will agree that this early Russellian conception of language, which is taken up by model theory, does not represent the final stage of analysis -if there is any such stage- but rather that it corresponds to a stage that one will necessarily have to pass through on his way down to primitiveness. That is to say, we will assume that one will never reach the "final stage" without passing through the stage that this early Russellian version stands for.

The above commitment says that the worlds of our models are (or are isomorphic to) exhaustive descriptions of all the individuals inhabiting them by means of the Russellian primitive vocabulary. It also says that at a future stage this description will possibly give way to a more fundamental, and this latter might contain no names for -e.g.- connectives. What, however, this latter further analysis will reveal is not relevant to our present discussion. That is to say that we postpone any judgment about whether a deeper analysis will reveal that there are only names for individuals, or concepts, or qualities, or events, or whatever. We only assume that the actual stage of analysis -though not the final one- is properly constructed upon the latter; i.e. we assume that it does not mislead us in any way unless we mistake it for the final stage.[8]

[7]If ever a scope including "truth", "consistency" and "completeness" might be called "narrow". But everything is a matter of context, and in comparison to the scope of logical atomism it is narrow indeed.

[8]This optimism is, of course, open to the same kind of criticism we have mentioned before, when targeting logical atomism in general. One could say that we are hiding behind the actual non-conclusive state of the project and that we call for the possibility that it is not excluded that things might finally be revealed to be exactly as the project says they are.

II[9]

(i) The world is in isomorphic correspondence with language.

(ii) To every state of affairs there corresponds a well-formed sentence that describes (represents) it.

(iii) All states of affairs can be reduced to combinations of atomic states.[10]

(iv) Sentences are either atomic or generated by atomic sentences and the formation rules.

(v) No atomic sentence's truth-value depends upon the value of another atomic sentence.[11]

(ii) can be considered as redundant on the basis of (i) provided that we include in the set of states of affairs states that are not the case. (iv) is redundant on the basis of (iii) and (i). We assume that there is a set of sentences that contain only simple names and simple n-adic predicates. These represent all the possible features of simple objects and constitute the set of atomic sentences. Everything else is built upon these and the formation rules. (v) makes it clear that negated sentences represent no atomic states. Otherwise there would have been atomic sentences which would be truth-functions of other (atomic) sentences since there exists a table for negation.

The set of atomic sentences belongs to the predicate calculus but we will limit ourselves here to the sentential calculus - i.e. if p is said to be atomic the reader will have to have in mind that it stands for a sentence of the form $R^n(a_1, a_2, \cdots, a_n)$, where "$R$" stands for an n-adic predicate and "a_1", "a_2",$\cdots$, "a_n" are n in number simple names.

Let us call the set of atomic sentences AS. By (v), it can easily be shown that any subset of AS is logically non-contradictory. Since no atomic sentence's truth-value depends upon the value of any other atomic sentence, any distribution of truth-values within AS represents a logically consistent world. Saying that any subset of AS is non-contradictory, states just that - i.e. assume a distribution that assigns truth to AS'; AS' being a random subset of AS. AS', if taken to be the set of atomic sentences that are true, represents a non-contradictory state. The truth-value of sentences belonging to the complement of AS' with respect to AS is the False, and the negations of all these (which, when negated are no longer atomic) are true. On the basis of the fact that all other sentences can be reduced to atomic sentences and primitive connectives, we will say that any subset of AS represents a non-contradictory *universal* state. And on the basis of the same fact, we will also say that the power-set of AS represents *all*

[9]My intention in this part is not to introduce the philosophical basis of any new *Weltanschauung*. I take up the atomistic *Weltanschauung* as such, in its criticized proto-Russellian form and for the reasons mentioned above.

[10]I will not touch the question of whether atomic sentences are finite or infinite in number.

[11]According to the formulation of the *Tractatus*, "5: "[$\cdots$] (An elementary proposition is a truth-function of itself)". (Ogden's translation).

non-contradictory universal states. For out of any subset of AS one can generate only one maximal and consistent set of sentences.[12] Finally, on the basis of the fact that the power-set of AS represents all non-contradictory universal states, we will also say that the same set represents all the (logically) possible universal states, and so all (logically) possible worlds.

Saying that the truth-value of any sentence of the sentential calculus can be computed by the truth tables and that all sentences can be analyzed into well-formed sequences of atomic sentences and connectives amounts to saying that the subset of AS that happens to be true determines completely which is the set of sentences that are occasionally true. As Russell-Whitehead put it: "Given all true atomic propositions, together with the fact that they are all, every other proposition can theoretically be deduced by logical methods".[13] If we have a subset of AS that is constituted out of true sentences and we know that there is no other member of AS that is true we know everything. By that subset and by the truth tables defining the connectives we can calculate the truth-value of any other sentence.[14] Thus, "every other <sc. true> proposition can theoretically be deduced".

Now, starting from the fact that the power-set of the set of all atomic sentences is the set of all possible worlds,[15] and from the fact that all true sentences can be deduced from the set of atomic sentences that are occasionally true, we can ask ourselves what kind of sentences these "all true sentences" are. The answer is trivial. They are all kinds of sentences, since the deduced set stands for a universal state. Therefore, they are sentences about logical and mathematical truths, such as the Law of Non Contradiction and 7+5=12, they are sentences representing moral imperatives (if there are such things), they are sentences representing fundamental physical laws, they are also sentences about events. (In brief, the deduced set will contain every true sentence and, consequently, sentences of all kinds).[16] Now, because the set of true atomic sentences about

[12] A set of sentences is "consistent" when no two sentences of the form p and $\neg p$ belong to it and is "maximal" when one element out of any pair of sentences p and $\neg p$ belongs to it. Note that up to this point we have been using the expression "generated" with respect to sentences in a purely syntactical sense - e.g. we have said that out of some terms, connectives, formation rules, etc., one can *generate* all well-formed sentences. From this point onwards, we will also use it in a semantical sense. Which is to say that we will also say that a sentence q is generated out of the set of sentences A, if, and only if, $q \in \{p : A \vdash p\}$. Obviously, by saying that q can be generated *simpliciter*, we will mean: $\emptyset \vdash q$.

[13] A. N. WHITEHEAD - B. RUSSELL, [1910-1913], Introduction, p. xv.

[14] This general scheme might be open to circularity objections. For, if the truth-value of every compound sentence follows from the truth-values of its atomic subsentences and the tables, then the rules that one applies, when calculating the truth-value of a compound sentence, need to be state-able in a rule, the validity of which also depends upon atomic sentences. (Everything depends on them). This rule concerns the way the tables should be applied. Which says that, in order to calculate the truth-value of compound sentences, we already have to assume a portion of these. Hence the circle. This bizarreness can be somehow smoothened by the usual aggressive atomistic retreat mentioned earlier; it can be smoothened by the claim that the present stage of analysis is not conclusive. This suggests that the circle is there, because the sentences that we now take to be atomic are not the *stricto sensu* atomic sentences, and so the compound sentences are not constructed upon *stricto sensu* atomic sentences.

[15] From now on we will sometimes speak as if worlds are sets of sentences and not what is represented by them. We can safely do that thanks to the isomorphism stated in (i).

[16] No matter how childish the optimism of such a project might look today, one should not forget that this was a project that the founders of modern logic took very seriously. In the

this world is the one it is and no other, the set of all true sentences about this world is the one it is and no other. The question is, of course, what happens as for other possible worlds. The idea is, again, that every possible world differs from all others with respect to one aspect at least. This is trivial since such an aspect can be immediately identified with the state represented by the conjunction of all atomic sentences true about that world. What is more interesting is to ask ourselves what kind of sentences could the sentences differencing worlds from one another be? From the examples mentioned above, the first that come to mind are sentences about events. Had the configuration of atomic states been different we would perhaps not be asserting today that -e.g.- Constantinople fell in 1453. Physical laws are harder nuts to crack, but some people believe that -e.g.- relativity (and any other theory about physics), does not state tautologies. In that sense physical laws could also vary from world to world. Now as for logical and mathematical truths the situation is more delicate. One could try arguing that even the latter can differ from world to world. One could argue that even the Law of Non Contradiction -this milestone of Western thought- has been criticized and so its validity is not to be taken for granted.[17] This is true but to our mind it does not mean that there are two possible worlds, in one of which the Law obtains, while in the other it does not. We do not *a priori* exclude this but we do not think that it follows from the fact that there are two philosophers disagreeing upon the matter. We think rather that the laws of logic are what all possible worlds share in common, and as such they must be deducible out of any subset of AS. Now the very fact that possible worlds have something in common is analytic: they all are (can be reduced to) subsets of AS. Let us, at this point, define logical laws (or logically true sentences) as the laws (sentences) that all possible worlds share in common. It is clear that this definition stays neutral with respect to what these laws are - i.e. if the Law of Non Contradiction does not obtain in all worlds, it is no law of logic; if, in every world, Constantinople falls in 1453, this event is represented by a *logically* true sentence. In that sense one can also say that even sentences that for the working mathematician, or even the historian, are not considered as logically true, must be considered as such, in case they are true in all possible worlds. For example: if the speed of light is a constant in all possible worlds, then Special Relativity makes a part of logic, not of physics. *A fortiori*, the same obtains for 7+5=12 and the question concerning the discipline it belongs to (i.e. logic or mathematics?).

At this point, we will posit yet another essential parameter of our general argument. As we said in chapter 2, causal implications imply the truth of the corresponding simple implications,[18] but they are not implied by them. We here assume that causal implications, even if not primitive, can be analyzed into well-formed compound sentences containing only atomic sentences and primitive

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Preface of the *Begriffschrift* (G. FREGE, [1964]), Frege suggests that not only mathematics, but also every other science could be founded upon a calculus like his, and when Wittgenstein (cf. L. WITTGENSTEIN, [1922]; 4.25ff) asserted that every other proposition follows from elementary propositions, he meant his words *stricto sensu*. In order to grasp the extremity of such a claim, just think that the Law of Non Contradiction is no elementary proposition, and as such it has to be provable, if true, on the basis of the elementary propositions.

[17] Cf. G. PRIEST - R. ROUTLEY - J. NORMAN, [1989], and, surprisingly enough, W. O. QUINE, [1951], reprinted in W. O. QUINE, [1953a].

[18] We call these implications "simple", because although more general than the causal they are not material implications. See the last part of this chapter.

connectives.[19] We have to assume this for the moment because if the physical world is governed by some causal apparatus, some causal conditionals have to be true and as such they have to be deducible from the set of atomic sentences. We further stipulate that all causal conditionals have the same truth-value in all possible worlds. This last stipulation, combined with what we were saying in the previous paragraph, yields that the causal conditionals that are true are *logically* true. Intuitively this does justice to the claim that whereas there are possible worlds, where -e.g.- Hitler does not enter Poland, and worlds, where there is no World War II, in no possible world is it false that, *if Hitler enters Poland*, then, a World War follows.[20] And no matter how convincing this example might sound, the general assumption is that there is such a thing as the *cause* of World War II, and wherever this is present the War necessarily follows.[21]

Let us pause for a moment and claim our ground. We have posited that whatever is true in all possible worlds (i.e. whatever can be deduced out of all subsets of AS) stands for a logical truth and that there are no other logical truths than these. Here we have tentatively classified all causal conditionals. The rest of true sentences about the world (i.e. the contingently true ones) come out of the set of the logically true and what contingently obtains - e.g. if "Hitler enters Poland $\rightarrow_c$ the Second World War is a reality"- belongs to the set of logically true sentences and "Hitler enters Poland" is (contingently) true then "The Second World War is a reality" is equally true and equally contingent.

We recapitulate: There is a set of non-contingently true sentences: the set of logical truths. This set when combined with any subset of AS yields the entire truth about the world this subset of AS represents.[22]

[19] This assumption will be subsequently refuted with respect to all non deterministic universes. As we said earlier, the main objective of this part is to show that indeterminism is incompatible with logical atomism.

[20] We, personally, do not subscribe to the view that such a science is humanly or even logically possible. Causes and effects are, in practice, defined inductively (this involves experience and the passage of Time), and, so, there must be many problems in the assumption that they can all be gathered together within a set of logically true propositions. In a sense truths by induction are never *logically* true. Of course, the atomist would counter-argue that science finds rescue to induction because scientists do not know the set of true atomic sentences. This whole discussion, however, is out of context with respect to our present worries. The reason why we present things the way we do is that we want to introduce a possible way out of the fantasy in the context of an extremely rigid (objective) science. And, once more, we do this not because we commit ourselves to the existence of one such science but because if our proposed way out works well within this rigid context it will also work when one relaxes it into a more humanly possible one. Cf. part I.

[21] For the possibility of a subtler analysis see note 30 of chapter 2: the part about causes being intensional objects.

[22] Since the set of non-contingently true sentences (i.e. the set of logical truths) can be deduced out of any subset of AS, it also follows that the subset of AS that corresponds to the actual world yields *simpliciter* any truth there is. This is to say that it is not that the set of logical truths lies over and above the set of contingently true atomic sentences. It is rather that it is yielded by this latter, no matter which one this is. See also note 14.

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We have previously asked ourselves about what kind of sentences are those that are true in every possible world - i.e. that are generated from every subset of AS. We will now ask the mirror question: what kind of sentences are those that are contingently true - i.e. true in some possible worlds and false in others?[23] Since, as we have previously seen,[24] the power set of AS represents the set of all possible worlds, atomic sentences are such sentences. For there has to be (for every atomic sentence) at least one world where it is true, and at least one world where it is false. (Each atomic sentence belongs to some of the members of the power set of AS and does not belong to others.)

From what we have assumed up to now it also follows that the rest of contingently true sentences about the world are exactly the sentences that can be relevantly deduced by the application of the set of logically true sentences to this particular subset of AS, which happens to be true.[25] One aspect of these sentences is that they are tensed. Why is that? For a trivial reason: according to our approach towards sentences in chapter 2, there are no other sentences than tensed.[26] In fact, by a complex argument, one can arrive at the stronger conclusion that, in the contexts of the present analysis, different possible worlds and different possible moments are one and the same thing. First, assume two sets of sentences representing what obtains at two different moments. They have to differ with respect to at least one (tensed) sentence.[27] If so, they have to be the logical product of two different subsets of AS, and consequently, they have to represent two different possible worlds. Second, assume any two different subsets of AS (i.e. any two possible worlds according to our definition at the beginning); they do differ with respect to at least one tensed sentence since they differ with respect to one sentence by definition, and all sentences are tensed according to chapter 2. Finally, assume that all subsets of AS happen to be true, at some possible moment.[28] The three stages above yield that different possible worlds are different possible moments. More precisely, the needed premises are:

(i) any two maximal sets of sentences describing two different moments are reducible to two different subsets of AS,

(ii) any two different subsets of AS differ as for one tensed sentence at least,

(iii) possible worlds are subsets of AS,

(iv) different moments differ as for one tensed sentence at least, and

[23] The same question was put a couple of pages ago; the "contingently true sentences" were the sentences that differentiate worlds form one another.

[24] See pp. 40-41.

[25] "Relevantly" is crucial here because if not the logical product of this application will again contain the set of logically true sentences.

[26] Cf. ch. 2; part I.

[27] Even in cases of circular Time they do differ with respect to the (tensed) answer to the question "What time is it?" Once the circle completed and if constituted out of n moments, the answers would differ by n numerals: "It is m", "It is $m + n$", etc. Otherwise, when even this difference is not discernable, the moments are, by Leibniz's Law, identical.

[28] This is a generalized form of the *principle of plenitude* (see *infra*).

(v) all possibilities are realized somewhere in possible Time.[29]

Possible worlds, we have argued, are different possible moments of Time. Now the question arises as to whether these moments belong to the same or to different possible Histories. The question in itself is a central and difficult one, and becomes even more difficult because of a certain ambiguity involved in the expression “different possible Histories”. We are now going to turn our attention to this ambiguity.

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Our having identified possible worlds with moments of Time can give the impression that two among the atomistic dogmata have to be relaxed, if not abandoned. The first is (iv): No atomic sentence's truth-value depends upon the value of another atomic sentence. If we do not inhabit a completely chaotic universe, the shape of the world during a certain moment has to (i) determine some aspects of the world at subsequent moments, and (ii) provide some evidence of how the world looked at previous moments. The above says that some worlds (i.e. moments) causally determine some parameters of others (i.e. their successors) and give evidence of some causal impact other moments (i.e. their predecessors) have had upon them. Suppose, now, that a certain subset of AS is our present world. Some things about the future have to be excluded because of the *now* actualized strong causal chains; but this is tantamount to saying some subsets of AS have to be excluded from possible successors of the present. Thereby, one might think that not all combinations of atomic sentences are equally permitted and this is equivalent to saying that some atomic sentences are functions of other atomic sentences, on the occasion they are functions of the atomic sentences that *now* obtain. This relaxation of atomistic principles is only apparent. First, let us note that each combination of atomic sentences is, still, equally possible, if considered in isolation. This formally says that still no contradiction can ever emerge from merging together all logically true sentences with any subset of AS; it is a trivial result. The apparent interdependence between atomic sentences comes from the way our understanding of a “possible world” has made some worlds appear to be (to some extent at least) functions of other worlds. Hence, the provisory conclusion that some atomic sentences (i.e. those true in the present world) determine the truth-value of other atomic sentences (those true in some descendants/ancestors of the present world). But, if we are *now* -say- at n, any sentence about any state of a moment other than n will have to either contain temporal indexes, or tense operators, or both. One is inclined to say that the above kind of sentences are not atomic. “It rains” must be more primitive than “It rains at n” or “It will rain”. In reality, if we remain attached to our previous commitments, according to which the only possible perspectives are from within Time, and therefore from the occasional present moment, and (also according to which), the present is more primitive than the rest of the tenses, and the indexless more primitive than the indexed, we see that at each occasional present moment sentences about what (atomically) obtains

[29] e.g. proof by contraposition: Identical subsets of AS do not differ with respect to any (tensed) sentence, and the maximal sets representing what obtains in the same moment are reducible to the same subset of AS; finally, (v) guarantees that there is no subset of AS that is not true at some possible moment.

at moments other than the present can never be atomic.[30] They all are (will be) atomic, when -so to speak- “their moment comes”, but they are not now. And, since there is no other perspective than the occasional present, it follows that no sentence about moments other than *now* is atomic.

In deterministic universes the above conclusion suggests the following argument: since every event is causally accountable, any moment determines completely the entire History to which it belongs. Now, by the fact that moments are worlds and worlds are subsets of AS, it follows that any subset of AS can logically generate a maximal and consistent set of tensed sentences. And, finally, since we do not assume atemporal sentences, any subset of AS generates a maximal and complete set of sentences. Therefore as for deterministic universes the answer to the question we asked in the last paragraph but one goes as follows: possible worlds are different moments of the same History. Why is that? Because for such universes there are no other possible Histories. We, however, who are interested in the Sea-Battle, and aim for the possibility of an objective and not merely epistemic indeterminacy, are forced to relax the deterministic proviso, according to which any moment contains some singularizing effects of everything that has happened during its entire past together with the sufficient conditions of everything that will take place during its entire future. Therefore, we have to allow that at some moments some sentences cannot be logically generated, and -this is the important part- neither can their negation; e.g. if tomorrow’s Sea-Battle is, by today’s standards, an indeterministic event, and AS' is the subset of AS determining the present moment, we have to admit that AS' can neither generate, “There will be a Sea-Battle tomorrow” nor, “There will be no Sea-Battle tomorrow”. If, at this point, we go back to Russell’s optimistic prophecy according to which, “Given all true atomic propositions, together with the fact that they are all, every other proposition can theoretically be deduced by logical methods”, it looks as if we have to take some distance. For on the occasion one out of, “There will be a Sea-Battle tomorrow” and, “There will be no Sea-Battle tomorrow” is true, but AS' is unable to spot it. These criticisms, together with the apparent need to relax the atomistic apparatus, fall short in the following account. They both assume that worlds are indeed consistent *and* maximal sets of sentences. If they are really such, there is certainly a problem in the fact that AS' is not able to attribute the truth-value True to whichever among the above contradictory pair this value applies. But, if worlds are not maximal, there is no problem in the first place. The reason that AS' can generate neither the one nor the other is that neither of them belongs to the set of sentences that are true about our world at present. “Every <sc. true> proposition can be deduced”. Yes, but neither of them is! This remark has to be seen in combination with what we were saying in part I about there being no problem for a scientist not knowing something, in case this “something” does not exist. And, as for our case, we take it that neither the Battle nor its absence *now* makes part of the world, although they both would have made part of it, if there had been sufficient conditions for either of them, and no matter whether they concern events that actually take place or future events.

Contemplated from that angle, the apparent difficulty not only vanishes but also provides us with some hints about what the aforementioned ambiguity

[30] In chapter 4, the priority of the present tense with respect to the rest of the tenses and the priority of the index-free with respect to the indexed will be formally accounted for.

within the expression "different possible Histories" could consist of.

To start with, let us adopt the emblematic Wittgensteinian principle according to which the world is everything that is the case. Since, for this analysis, we take worlds to be moments in Time, the claim amounts to saying that every world can be defined as everything that is the case during the moment this world corresponds to. But -one should ask- what can be meaningfully said to be the case during a moment of an indeterministic History, when not everything has been settled for the future? There are two possible answers to this question and hence the ambiguity.

In the first place (this is the first type of answer), one can say that although there are *now* no sufficient reasons either for the Battle or its non occurrence we are actually following a possible History that (already) contains one of these alternatives. In other words, one can say that it is (now/already) the case that -e.g.- tomorrow's Sea-Battle will take place because it happens that we are moving along this particular possible path. We are certainly moving along a path, and the path we are moving along could indeed be the one containing this particular event. In that case, and for as long as there are no sufficient conditions for the Battle, we would be *a priori* excluded from any knowledge of it, but, all that time, the future advent of the Battle would have been something that is the case about our world and the sentence asserting it would have to be true. Concerning the question of the semantic import of our calling this event "indeterministic" the explanation would have to go as follows: the present world is a maximal and consistent set of sentences and, in that respect, the present world contains already one sentence out of the contradictory pair. Nonetheless, there are no sufficient conditions at present which could give any account for the state this sentence describes. Therefore, there are at least two distinct possible Histories, which share in common identical pasts and presents and which diverge at some point between *now* and tomorrow.[31] In one, the Battle takes place, in the other, it does not. In any event, these two *now*-s are not identical, for, although they share an identical past and present, they (already) diverge as for their respective futures, and this divergence is already depicted in what they are. They are maximal and consistent sets of sentences and (thus) differ in at least one sentence: the sentence about the indeterministic event. Strictly speaking, even to say that they share identical presents and pasts is, in this account, formally wrong, because all moments of these Histories, no matter how far in the past they lay, contain all the true sentences about their futures, and, so, even their presents and pasts diverge. It is as if these moments (worlds) have been eternally "charged with" whatever will be the case in their futures. Of course, since these worlds are indeterministic, no temporal being could have known which History is the one he is living in before the advent of the occasional self-caused event but the difference would indeed have been there. The question is in which sense one is entitled to assert that there are truths about the world that no temporal being can *now* grasp. It is only on the assumption of an atemporal nature of things that one is entitled to say that the states obtaining within a world can contain states for which there is not yet any sufficient explanation. This is because, according to the same assumption, Time

[31] i.e. at the point, where the cause of the one or the other possible outcome becomes part of the world.

and causality are not essential to things. Therefore, our reality is not in itself temporal, and so, Time is not to be counted among the parameters determining the truth-value of sentences. All of these (truth-values) are atemporally there, independently of whether there are, at the occasional present, sufficient reasons for what these sentences represent. From that angle, it is only for temporal observers that indeterminism could legitimately be said to be "objective" and not "epistemic" - i.e. although it is not the case that another temporal being would have "known better", an atemporal being (contemplating the very same reality that we do, but atemporally) would! So, the indeterminacy does not concern the reality *per se* but the way we *a priori* approache it.

Now what about the question of whether our different possible worlds are different moments of the same, or of different Histories. It is obvious that if one believes that worlds are maximal and consistent sets of sentences the answer must be: of different. If so, no two same moments of any two Histories could be identical. They (i.e. possible Histories) would all differ in at least one state, and this minimum difference would be depicted in all their moments. Graphically, one should, in order to represent such different possible Histories, draw parallel straight lines, each of which stands for each different History.[32]

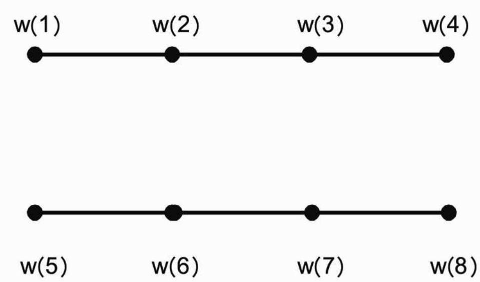

Figure 1.

w(2) = w(6), w(3) = w(7)

The points of these lines are the moments (worlds), and, in these conditions, it is inappropriate -strictly speaking- to say that any two individuals inhabiting (or any two events taking place in) any two of them are identical. It would be more appropriate to speak of counterparts[33] of individuals, events, etc., populating, occurring, etc., in Histories other than the occasionally real. Thus, if it is true about a future moment n that a Sea-Battle takes place then, but this is not necessary at present, then we should postulate another possible History where what does not happen there is not the same event that happens here,

[32] These kinds of representations are sometimes called $T \times W$ (cf. R. THOMASON, [1984]), because they are mappings of moments (T) onto Histories (W), or worlds as "W" suggests. I prefer to take the mappings to be onto Histories, for "world" is ambiguous here and can either mean a point of the model, which (on Thomason's context) is misleading, or a complete maximal History of such a moment, which is exact. These representations are to be contrasted with the branching kind of representations that seem to have the same semantic import, but they do not - either philosophically and formally. See figure 2 and *infra*.

[33] In the sense of Lewis. Cf. D. LEWIS, [1986].

but its counterpart in that History. The picture is more straightforward for persons. If I am going to decide to leave my wife tomorrow, and this decision is indeterministic, the person who in another possible History decides to do the contrary is not me, but my counterpart in that History.[34] This does justice to the semantic fact that in these representations the description, "he who will decide to leave his wife tomorrow" is true about me, no matter whether I know it or not. The picture suggests that I "travel" upon this or that History and that that History is alienated from all others. I and my other possible "selves" are not the same person.

From what preceded it is clear that there is a certain sense in asserting that the many possible worlds are different moments of *many* possible Histories. One may say that each such world represents totally the History it is a part of.[35]

On the other hand, if one examines the nature of the commitments we have gone through in order to give sense to that claim,[36] one will immediately guess that this is not the answer I personally prefer. To the question, "are these commitments really necessary to give sense to the same assertion?", the answer is positive; but this will not become clear until we fully stress the ambiguity of the expression "many possible Histories".

Let us, to that end, go back to my preferred way of seeing things. Let us again assume that Time and causality make up part of things themselves, that there is no other perspective than from within Time, and that there are no eternal sentences. Let us also go back to our atomistic assumption that worlds (i.e. moments) are reducible to subsets of AS, and that if our world is indeterministic some subsets of AS do not generate maximal sets of sentences. What results from these assumptions is that many of the worlds upon the different parallel lines of figure 1 are identical (i.e. indiscernible). Consider again the Sea-Battle case. If there are two possible Histories in one of which the Battle occurs, while in the other it doesn't, the present moment of "this" History and the present moment of "that" History are represented by identical sets of sentences. This is because the event differentiating these two is tomorrow's Sea-Battle and since there are *now* sufficient reasons for neither its advent, nor for its non advent, neither, "There will be a Sea-Battle tomorrow" nor, "There will be no Sea-Battle tomorrow" can belong to any of the sets representing these moments. As we said earlier, there is not (as yet) any state corresponding to the Battle or to its absence in any of these two alternative present moments. But, in such a case, and by an identity of indiscernibles argument, one could claim that these two "alternative presents" represent not two but one single world. And, by generalization, one could also make any two such points in figure 1 graphically identical. The identification being completed, we would be left with figure 2, or -more technically- with a "branching" temporal model. The fact that any

[34] The picture becomes straightforward even for events if one considers the possibility that they can be analyzed into the doings of the individuals (not necessarily animated), who participate in them.

[35] We would get a Leibnizian view, if we assume that possible worlds, instead of being the moments of these Histories, are the Histories themselves before God choosing the best among them.

[36] i.e. mainly to the atemporal nature of things.

two "identical" worlds in figure 1 occupy, in figure 2, same points depicts the fact that they are to be taken as the same world. The fact that any one of these "same" worlds can evolve in more ways than one is depicted by the future branches having as origins moments in which the matter has not been decided as yet.[37] (The backward branching in the figure suggests that the "identical" worlds might have had more that one possible past.)

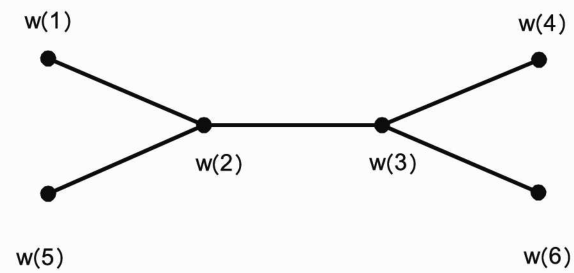

Figure 2.

Now, one could object here that these branching diagrams of indeterministic Histories could, in a sense, also be used by one who favors the previous *Weltanschauung*, and assumes the atemporal nature of things, and one could also object that $T \times W$ diagrams can, in a sense, be used by an advocate of the present *Weltanschauung*. These objections do not hit the target, and this is for both technical and philosophical reasons.

Let us see, in detail, which are the objections and which are the reasons why they fall short.

One could think that, as far as figure 2 is concerned, a specific branch emerging out of the origin represents the real History, while all the others, the merely possible Histories. On the other hand, one could also think that the many "different" points of figure 1 are in fact identical, when seen in isolation. The only (accidental) thing that separates them is that they belong to different Histories; hence, their different coordinates in the diagram. Both these suggestions fall (intuitively) short on the following account. Just as in mathematics (not to say in agriculture), the origin of many branches does not belong to one of them more than it belongs to the others -i.e. just as the origin has no (so to speak) "privileged" relation to any of the branches of which it is the origin- in our branching temporal models, it is somehow inappropriate to say that there is now one among the branches that *is* the actual future.[38] For consider it this way. Suppose that, at the origin of the branch, it is already true that the Sea-Battle will take place tomorrow. If so, what does the line connecting this origin with the other "possible" tomorrow stand for, if not for our present incapacity to know which is which? But we have posited that differences between Histories

[37] In fact the origins of the branches are the last moments before the matter becomes decided.

[38] The idea criticized here is sometimes called the "thin red line" approach, See N. BELNAP, [2001].

are not to be taken as mere epistemic.[39] On the other hand, to say, about figure 1, that there are many identical worlds followed by different futures is also inappropriate, because, by making them belong to many of these different parallel lines, we have already -so to speak- "charged" them with a fate that has not as yet become their own; e.g. if w(3), of figure 1, is no more inclined to evolve into w(4), than is w(7), why did we then connect the first two by a straight line in the first place? This is why I think that figure 1 is appropriate for depicting Histories containing (also) atemporal sentences and figure 2, appropriate for depicting Histories not containing such sentences at all. (Formally, it can further be proved that $T \times W$ infinite models are richer in theorems than branching ones. Which means that the transcription of 1 to 2 might, on some occasions, fail.)[40]

We now go back to the ambiguity. Once the transcription of figure 1 to figure 2 is successfully completed,[41] one can regain the question as to whether the many possible worlds are moments of the same, or of different Histories. One possible answer is to say -just as in the previous case- that they belong to many Histories. For one can, theoretically, follow all maximal branches and say that one has gone through all (possible) Histories. Nonetheless, it is to be debated whether the parallel lines of figure 1, after having merged into figure 2, have remained exactly what they were in figure 1 - i.e. different lines standing for different Histories. For consider it this way; choose a random origin of two branches. To commit ourselves to the idea that the different maximal branches represent different Histories would have condemned us to a peculiar (contradictory?) overvaluation of the origin. Suppose that the two branches coming out of the origin differ as for the truth-value of the sentence p. Then, the origin, *qua* part of the one branch, would have to be part of a History that validates p. And the same origin, *qua* part of the other branch, would have to be part of a History that validates $\neg p$. The situation becomes even more puzzling once we see things from the angle of the points that come after the origin of the branch. Suppose that it is p that finally becomes the case. Then, where is the History where it does not? Out of the perspective of the branch, where 'p' is true, and after having become such, the History where it is not true is no (possible) History. Diagrammatically, this says that branches become fewer and fewer as the world evolves into the future.

The above and similar examples invite one not to sustain the same theoretical attitude as to what Histories essentially are throughout figures 1 and 2. What I mean is that, in figure 1, Histories are sequences of points, each of which completely determines the rest of the sequence to which the point belongs. But,

[39] This claim is not unconditional here. It depends upon the earlier assumption that whenever an ideal knower living in Time does not know p and does not know $\neg p$ either, it is because there is no p and $\neg p$ to be known in the first place; not because one of the two obtains, but he -*qua* temporal- is cognitively closed with respect to the question (cf. p. 31). Of course, this assumption is also open to criticism, for it depends upon the assumption that there is no outside of Time standpoint. However, if one assumes that, one can understandably reach the conclusion that, when the future forks, it does not fork because we cannot cognitively discern between the real and the unreal future. No one of these two can be said to be such at the present moment, and so, their differences have to be accountable for by the nature of things *themselves* not by our cognitive shortness towards them.

[40] Cf. R. THOMASON, [1984], pp. 151ff.

[41] The transcription might sometimes fail because of the above-mentioned technical discrepancy.

then again, what guarantee do we have that Histories are essentially that kind of thing? One might say -in fact I think that one should say- that, according to the *Weltanschauung* underlying figure 2, Histories are sequences of points, some of which underdetermine the rest of the sequence - i.e. one should not assume that what is involved in figure 2 is many fully determined sequences, but that there is only one underdetermined sequence that tends to become fully determined as one moves from left to right, and no backward fork emerges. Such an assumption drastically changes our notion of what a History is. For, from that angle, the many branches of figure 2 are not put there because they stand for many different fully determined Histories, but because they represent, in total, one and the same History that is not (*essentially* is not) fully determined as for some moments of it. For example, while, in the *Weltanschauung* underlying figure 1, we were forced to conclude that individuals and events in any two different Histories are always distinct, we now have to concede that there is always one and the same individual that does, or does not, do this or that, and one and the same event that does, or does not, take place. It is only that whether or not the individual will, or will not, undertake that action, and whether or not the event will, or will not, occur is, on some occasions, an open matter. Figuratively speaking, the first of these two conceptions of History resembles a number of non-connected and forkless roads upon which similar cars are traveling; while the second resembles a single, unified road network, with many different roads but a single vehicle. The roads that the driver chooses not to take do not, in the end, make a part of his entire *real* journey.[42] Contrary to that, every single driver of the first model has a different tale to tell.[43] These two notions are so alike that one can easily consider them to be identical. To identify them, however, is, to our mind, misleading. For we think that there is a slight difference between saying that indeterministic Time consists of many linear Histories that share some moments in common, and saying that it consists of one History, which, nonetheless is a partial and not linear order. More precisely, the difference comes down to the fact that according to the first, Histories are sequences of moments, each of which has a single immediate successor appointed to it, while according to the latter, although the successor is/will be unique, the appointment can be postponed for a later stage. In fact, this is the very reason that we proclaim that the absence of a state from a world does not entail the presence of its negation - i.e. that worlds are not maximal. One of the very few ways that one can proclaim that the absence of p from within a universal state does not imply the presence of $\neg p$, and not be sent to the logician's asylum, is to postpone a Time horizon. Neither p, nor $\neg p$ make part of the world, because worlds are moments of Time, and the matter of the presence of $p/\neg p$ has not been decided as yet (i.e. up to the present moment).

In the rest of this book, I am going to further elaborate that notion by progressively substituting the notion of many possible linear-order Histories for the notion of a single History, which, however, is a partial, and not linear order.

[42] Complications will emerge, since we allow for backward branching, but we put them aside for the moment.

[43] Another intuitive example could be the following: contrast a set of parallel lines of different colors with exactly the same set of lines, once rearranged in the way figure 1 has been rearranged into figure 2. At each point, the lines that the past has -so to speak- "absorbed" provide a new shade of color. But, before being absorbed by the past, the "different" presents cannot tell which shade this will be.

The time has now come the address fully the question of whether moments of Time (i.e. worlds) belong to the same or to different Histories. According to the last notion of what Histories are, we would say that they belong to the same History, keeping, however, in mind that by "History" we mean a partial, not a linear order. Therefore, our answer differs from the "same" answer that the determinist provides to the same question. For him, History is one, because there is only one possible linear order. That is to say that he too claims that different possible moments belong to the same History, but, if his determinism were refuted, he could possibly change his mind and say that they belong to different Histories. In fact, he would *have* to claim that, if he were under the spell of the *Weltanschauung* underlying figure 1.

This is the ambiguity we talked about.

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Before passing to yet other intermediate considerations, I would like to clarify a technical point about these Histories being primarily partial and not linear orders, and tending to become the latter as Time goes by and no backward fork emerges. The picture of a single vehicle moving within a unified road network suggests the idea that it is possible, or, at least, has been possible, for the vehicle to take any of the roads of the network. This comes from the assumptions that the network is unified, and that there are no other roads outside it. For if there were roads left outside this network, there would also be destinations beyond the reach of the vehicle. So, if there are no such destinations, the driver can always describe every point of every road as a point where either he was, or might have been, or will be, or might be. In technical terms this says that in the temporal model, of which the road network is a metaphor, any point is connectible to any other point by a series of tense operators.[44] However, in General Relativity, this idea needs to be abandoned. For there are, in curved Space-Time, some pairs of world-lines that will never receive any information from one another, and, so, no potential observer inhabiting one will ever attend (could attend) the events that the observer inhabiting the other attends (could attend). For such cases, we need to reformulate our assumption that possible worlds are points of a single partial order constituting a unique History. We need to concede that there might be more than one History, even if these latter are partial and not linear orders.[45]

Another case, for which one could understandably claim that there is more than one (partial order) History concerns fiction. "The series of adventures of Don Quixote", McTaggart says, and Prior quotes, "does not form part of the A-series. I cannot at this moment judge it to be either past, present, or future".[46] This is equally disconcerting with respect to our assumption that

[44] According to the metaphor the driver can always denote the points of the network by sentences like: the point where I would *now* be, if n miles back from where I *now* am, I had turned left instead of right, then taken the first turn to the left etc. It needs to be said, though, that, since we will allow for backward branching, there will be occasions, where the driver does not remember exactly what itinerary he has taken. Cf. *infra* and chapter 4; 0.2.

[45] A. PRIOR, [1967], pp. 203-205, and Appendix I of our treatise.

[46] A. PRIOR, [1967], p. 199.

there is only one History. Fiction seems to have provided humankind with temporal orders that cannot be connected with any present of ours *via* any series of tense operators.[47] This case, however, is not as alarming as the last, because one can always call for a generalized principle of plenitude,[48] according to which everything that is not logically impossible takes place, upon a point of the partial order we are traveling in - e.g. that at some point of the order a monkey sits down in front of a computer and writes the Odyssey; that at another Don Quixote fights the windmills etc. Of course, the Don Quixote example has more difficulties in it since it raises the problem of proper names of fictitious characters but what I mean is relatively clear. I mean that, if some B-series are logically possible, they have to occupy a linear substructure of the order,[49] and, if they are not, they are no temporal series in the first place, and, so, they are no B-series either.

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Now, there is a last argument that points towards the need to relax some among the fundamental atomistic principles. We have left it to the end, because here the need is even more than pressing. The sense in which our model can be said to be atomistic has finally reached its own limits, and by the end of this chapter we will have abandoned our early assumption that all subsets of AS can generate all causal conditionals. In a sense we have arrived at the central point of this chapter, since we are going to advocate the idea that no matter whether or not logical atomism in its rigid form is an inconsistent project, it is inconsistent with indeterminism, and needs to be relaxed in order to be applicable in indeterministic contexts.

"There will be a Sea-Battle tomorrow" is, according to what AS'[50] can generate, neither true nor false. And yet it is supposed to be well-formed, and so, analyzable into a well-formed sequence of atomic sentences and connectives. Therefore, one should, in principle, be in a position to compute its truth-value from the truth-values of some members of AS' and the tables. The atomistic dogma, which is challenged here, is that there is a unique and final analysis for every composite sentence and that at the bottom of this analysis one encounters a sequence of atomic sentences and connectives. If so, the truth tables should give a truth-value to, "There will be a Sea-Battle tomorrow". But they do not![51]

There is an obvious sense in which gaps and logical atomism are incompatible: (i) No atomic sentence stands for a gap. (ii) The truth tables function perfectly. (iii) All sentences can be analyzed into atomic sentences and connectives. *Ergo*: no sentence stands for a gap. Challenging this result presupposes

[47] Although they can be connected with various fictitious present-s - i.e. the present-s of the story itself. Cf. G. SEEL, [2006].

[48] Postulating not that every possibility is actualized in real Time, but that every possibility is actualized in *possible* Time.

[49] This does not ontically commit one to the monkey which writes the Odyssey and the like. We remind the reader that in branching models there are not many, but only one "tale to be told". The other possibilities become extinguished as Time goes by, and as long as past forks do not emerge.

[50] The subset of AS that happens to be true now.

[51] The present puzzlement can also be seen as the mirror image of our previous worry about AS' not being able to spot which, among, "There will be a Sea-Battle tomorrow" and, "There will be no Sea-Battle tomorrow" is true.

challenging one of the three premises above.

We begin by trying to challenge the first.

Suppose that the atomistic analysis of "There will be a Sea-Battle tomorrow" is $p \otimes q \# \ldots \bullet r$, where $p, q, \ldots, r$ are atomic and $\#, \bullet, \ldots, \otimes$ are primitive connectives. Then, if there are no gaps among $p, q, \ldots, r$, the tables would have to provide a value for the whole sentence. This, however, must be excluded, because, in such case, one would theoretically be in a position to calculate today the truth-value of, "There will be a Sea-Battle tomorrow". This contradicts our tacit assumption that the event is genuinely indeterministic. On the other hand, one should also retain a skeptical attitude towards the assumption that one of these atomic sentences stands for a gap. As you remember, we have posited that atomic sentences contain no future/past references. So, if atomic sentences do indeed refer only to the present, then, if there are gaps among them, the present must be partially underdetermined. And, as we are going to see later, the determinateness of the present is a *sine qua non* for the intuitionistic dissolution of the fantasy. So, we need to assume that there is a non further-reducible reference to the future within these atomic sentences. At the end, we have either to abandon the idea that future tensed sentences are never atomic, or to abandon the idea that the present is fully determined. Doing the latter generates insurmountable problems for the intuitionistic solution to the fantasy, and, so, we assume the former.[52] The same way out can be advanced in a more "causal-centered" manner. Suppose that $p\&q\&\ldots\&r$ is the conjunction of all necessary conditions for the Battle, and that $p, q, \ldots, r$ are atomic and & primitive. This sounds decently atomistic, because it suggests the idea that the sufficient condition for the Battle has n necessary conditions, which are, in turn, constituted out of a number of necessary conditions of their own, and, so on, until one reaches a sentence constituted by atomic sentences and &-s. For exactly the same reasons that $p \otimes q \# \ldots \bullet r$ has to be truth-valueless, $p\&q\&\ldots\&r$ must be too: If it is not, it contradicts (manifestly, this time), the underlying assumption that the event is *now* underdetermined. One has to assume that there is at least one atomic necessary condition, which is *now* neither fulfilled, nor made impossible, and, thereby, that the (atomic) sentence representing it is neither true, nor false. By the present's determinateness, we, again, conclude that the sentence concerns the future and, finally, that some atomic sentences have a non further-reducible reference to the future. (Notice here that we do not postulate that $p\&q\&\ldots\&r$ is -should be- the atomistic analysis for the event,[53] although it is clear enough that $p\&q\&\ldots\&r$ and whatever is the atomistic analysis of the event should have identical truth-conditions.)

[52]A more cunning argument, along the same lines, would be to assume that all ultimately analyzed sentences will be revealed to be sequences of simple names, and abandon the idea that all such names necessarily denote an item (i.e. that they denote an item at all possible moments). This is, again, a relaxation of atomistic principles and is not without some problems of its own - e.g. how is one, then, supposed to state the fact that a simple name denotes, or does not denote something, by other sequences of simple names - i.e. by other atomic sentences?

[53]i.e. we do not postulate, in general, that the atomistic analysis of sentences representing states consists in the statement of their causes. But see the analysis concerning relativity in Appendix I.

The second way to escape the problem is to assume that the truth tables can fail, sometimes, to give a truth-value - i.e. that they do not behave monotonically. This, however, comes very close to the first in as much as it also acknowledges that the future is partially independent of the present. This comes as follows. Under this notion, one needs to assume a metalanguage strong enough to describe the exact conditions under which this failure occurs. And, again, the same need condemns one to state what kind of thing a future indeterministic state is. We cannot escape this, because it is for sentences about these states that the tables would have stopped behaving as usual. One thing that has to be stressed about this approach is that sentences about future indeterministic events could no longer be atomic, since they would have to be well-formed combinations of other sentences that have a truth-value but fail to provide an overall value by the time they are put together in the sentence representing the indeterministic event. In such a case, one would need to specify what kind of state is the one they describe, when they come together within that crucial sentence, and, as a consequence, make the tables behave in a non standard way. This says that one would have been forced to declare what it means for something to be a state that can neither be affirmed, nor denied, by the present universal condition. This, in turn, would have forced one to acknowledge that there is something in the future, which is non further-reducible to anything else. The future would, in some aspects, be independent of the present.

The third option also concludes (just like the other two) that there is a non further-reducible element within sentences that represent future indeterministic states, but no longer calls these sentences, “atomic” or, “built up upon atomic sentences and primitive connectives”. More precisely, the last option overrules the very idea that the truth-value of these sentences can be calculated by the tables, and, thereby, overrules the idea that they can be analyzed into sequences of atomic sentences and connectives. This means that they are neither among the things that are primarily true about the occasionally present world -these things are the things described by the atomic sentences- nor among the things that depend upon these latter, and the tables. As we will see very shortly, the intuitionistic device consists in exactly that: it takes the partial order that a world/moment belongs to as a non further-reducible parameter for the calculation of the truth-value of some compound sentences. (“Non further-reducible”, meaning here: the truth-value of these sentences is, at the present moment, no function of the atomic sentences and the tables). The above change of perspective consists -intuitively- in our gradually moving away from the notion of a completely truth-functional conception of truth, to an only partial such.

But let us, for the time being, keep only in mind that the non-reducibility of the future seems to be (as for indeterministic universes), a one-way road, it is a common denominator of all three options above. If we allow for past indeterminacies, the same applies as for the past.

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It is now time to introduce the formidable scientist who will -during the next section- perform the thought experiment that leads to our supervaluation intuitionistic results. Some preparatory moves are, however, still needed.

At this point we will assume that there is a "final scientific theory S", such that it allows one to deduce at every moment whatever is true at that moment, and nothing else. We call it "final" because it is assumed not to depend upon the present stage of scientific development - i.e. we assume that it will never ameliorate itself for the simple reason that the only things that cannot be deduced with its aid are the things that do not obtain at the (occasionally) present moment during which we apply it. This does not exclude that in subsequent moments humankind will know more than the things that S allows it to deduce now.[54] It only says that no science could be established in the future such that if one were to come back to this present moment, he could deduce more than those things he now deduces by means of mastering S.

Now, beside the tautological -on the previous definition- claim that S allows one to deduce whatever is true about a world, one can, again, wonder what exactly is (could be) the specific content of these truths. We assume here that S allows one to deduce, at any moment, everything that can be deduced by the atomic sentences true at that moment and the tables *plus* the set of true causal conditionals, which is taken to be a trans-world constant. These three things are the starting point of S. It is important to notice here that when we exclude from *AS* references to the future, "everything that can be deduced by the true atomic sentences and the tables" cannot contain the complete corpus of causal conditionals. This is because causal conditionals in indeterministic worlds contain sentences of the form $a \rightarrow_c a$ where a describes a self-caused future event and, as such, contains elements that are non further-reducible to the occasional present.[55]

To other general questions concerning the kinds of truths contained in the occasional logical product of the atomic sentences, the answers are, in general, quite indifferent. If one is an intuitionist in mathematics one will perhaps assume that this set never contains the "totality of *all* mathematical truths". Nonetheless, and as we are going to see during the following chapter, the present system can be said to be "intuitionistic", even on the basis of its physics and history. What we have in hand will do. And what we have in hand is that (i) S can generate, at any time, the logical product of all atomic sentences true at that moment, that (ii) this moment in itself[56] is not underdetermined and that (iii) S validates, at any moment, the set of true causal conditionals,[57] and this set is constant for all moments.

We now introduce the idealized scientist, whom we will call the "ideal agent of S". We assume him to be able to compute every truth that can be deduced by

[54] It will possibly know -e.g.- that the Sea-Battle took place.

[55] Notice that if, alternatively, we do allow for atomic sentences referring to the future, they have to be truth-valueless, when about indeterministic states.

[56] What exactly "in itself" here means will be clarified in the next part. The rough picture is that the "in itself" excludes any references to moments other than the present.

[57] Not on a truth table basis though. See *supra*.

the aid of S.[58] Another thing that has to be stressed here is that we situate him in a non-relativistic universe, timed by a universal clock, and where information about events can (and in fact does) reach the agent in no time, wherever this latter might occasionally be located with respect to the events (i.e. we assume action at a distance).[59] This makes some difference but only as for the self-caused events and the events depending upon them,[60] because, for all the rest, he can always anticipate their occurrence, thanks to his perfect knowledge of the causal conditionals.

We also assume that the ideal agent of S, though immortal, has not necessarily a perfect memory. In that way, we allow for converging Time - i.e. we allow for the possibility that with respect to some states there are future moments at which no evidence (i.e. singularizing effect) remains in the universe. If so, after these moments the states will have left no trace to in the mind of the ideal agent.[61] Moreover, since by the time that the future converges no evidence of them will have remained we can easily conclude that if Time is not circular our ideal agent will never regain any of his lost "knowledge".[62] Finally, we will

[58]In many respects, this super-knower resembles the *Creating Subject* of Brouwer's, or even Husserl's transcendental ego. The main difference separating our "ideal agent of S" and the *Creating Subject* is that the latter is an ideal agent for mathematical activity living in Time, while our "ideal agent" is also a temporal agent, but an agent of any "scientific activity", with a certain inclination to the study of events. While the *Creating Subject* establishes mathematical theorems and creates new elements for his mathematical universe, our "ideal agent" gathers information about events and causally interacts with reality. The ideal agent resembles also Husserl's transcendental ego, although this latter seems to be ontically less generous towards free choice sequences or towards sequences of indeterministic events. Cf. D. VAN ATTEN et al., [2002].

[59]The above is a logical consequence of the proviso that he can know all the atomic sentences true at the occasional present, combined with the proviso that these latter determine the occasional present completely. The "scientific" problem with that picture is our needing to assume action at a distance, in each case an atomic sentence represents an atomic indeterministic event occurring somewhere else in the universe. The same parameter provides evidence in favor of the view that logical atomism favors, by and large, the *sub specie aeternitatis* approach to sentences. It is only *sub* that *specie* that the problem does not emerge.

[60]For example: when the universal clock shows n, and there is a true causal conditional like "if p, at n, then, q, at m", and p is a self-caused event that takes place at n, the ideal agent of S knows that q-at-m, from n onwards, no matter how spatially separated he is (at n) from p.

[61]We can straighten this clause, by imposing that future convergence is not only possible but also necessary for some events. If so, the ideal agent will not only become forgetful with respect to these, he will *necessarily* do so. This is tantamount to saying that he will, on some occasions, be in a position to do nothing about his future forgetfulness, he will not even be in a position to keep a logbook. Note here that this future convergence with respect to events is more understandable than a similar future convergence with respect to discovered theorems of mathematics. An event is an item belonging to a causal chain, and memory is always caused by something: the thing it is the memory of. Therefore it is understandable that if there are events that after some time leave no trace they will be eventually erased from the ideal agent's mind as well. See also, chapter 4; 0.2.

[62]Notice that the presence of forks (independently of whether they are in the future or the past), goes hand in hand with the absence of a causal chain: strong as for the future, weak as for the past. The substantial difference, however, separating forward from backward forks is that the former will necessarily be left behind, as ones evolves into the future while the latter, once they emerge, they are here to stay. I want to insist in this fundamental difference because logicians who disallow backward branching (see *infra*) often claim that, if there were such (past) forks, future and past would be indiscernibles. They think -i.e.- that forks are the specific difference separating the future from the past, and as such determine Time's arrow. This, however, is not exact, because these two forks belong to different species of forks, and this difference is manifested in the above-mentioned essential characteristic: future forks will be eliminated, past forks will stay forever. To claim that future and past are indiscernibles,

assume that any two branches coming out of a past fork have a common origin further back in their pasts. In our ideal-agent model this says that for every loss of memory he can remember a universal state further back in the past, where the past had not divided itself - i.e. this last universal point is a predecessor of both the exclusive alternatives he occasionally does not remember.

These backward branches will complicate our models to a considerable degree, but, from the way we see things, these complications are unavoidable. Logicians (even if indeterminists in their physics), do not usually treat the future and the past alike, as far as the presence of forks is concerned.[63] What indeterminists usually do is to allow for the possibility of non yet settled future states -it is these states that generate the forks- while, at the same time, they exclude the possibility of any fork in the past. This asymmetry reveals a contradictory attitude. For consider it this way: why is it, in the first place, that the future is allowed to fork? It is because -indeterminists hold- there is no logical law imposing that there are always sufficient reasons for everything that will follow. This is understandable, and, more than that, I would like to think that it is true. But, then again, to assume that no past forks can exist is equivalent with assuming that there is a logical law that guarantees that there are, at every moment, singularizing effects of everything that has been the case. In other words, it is equivalent to assuming that it can never be the case that, at some moment t, it *stricto sensu* makes no difference whether the state p has been the case, at some previous moment, or the state $\neg p$ has been the case, at the same previous moment. Excluding this possibility is not only a view equally dogmatic to the deterministic exclusion of future forks, but moreover, it is in obvious disagreement with the way indeterminists themselves treat the future. For, when the indeterminist excludes such a possibility he is, at the same time, allowing that the future is independent of the present, and disallowing that the past can ever be independent of the present. As a matter of fact, it appears that one is instinctively more ready to let go of the idea that there are at present causes for everything that will follow than of the idea that there are singularizing effects of everything that has preceded, while clearly these two ideas are two sides of the same coin.

Let us scrutinize the same point for a little longer by using the Sea-Battle example. On the one hand, these indeterminists assert the possibility that in the present universal state there can be no evidence of whether or not there will be a Sea-Battle tomorrow. If this universal absence of evidence concerns the world *per se* and not our epistemic limitations concerning the world, then there should be a fork in the future. Now, symmetrically, it is possible that the present universal state gives no evidence of whether a past Sea-Battle took place; this absence of evidence says that there is now no singularizing effect of what has been the case in the past as for the matter of that (possible) Battle. As there can be strong causal chains that have not yet begun, there can also be weak causal chains that have stopped dead, leaving no causal imprint upon the present world. For it can logically be the case that there are situations

on the basis of their both having forks, strikes me as absurd as claiming that, from the fact that our parents are human beings, no one among them is "our mother", and no one is "our father".

[63] But see Lukasiewicz reaction to that in J. LUKASIEWICZ, [1922] and A. PRIOR, [1967], pp. 27ff.

where there is no contradiction in assuming that the present is the decedent of a world with a Battle, and neither is there any contradiction in assuming that the present is the decedent of a world without a Battle. The latter is the mirror image of what an indeterminist asserts when asserting that today is possibly an ancestor of a tomorrow with, and of a tomorrow without a Sea-Battle. Now, following a *rationale* that looks superstitious to me, the targeted indeterminist, although he has, sooner or later, to accept that causal evidence might be absent both for causes and for effects, still upholds that a past state which has now left no causal imprint, is somehow more substantial/real than a future state for which there are now no sufficient conditions. For -as he might argue- *historical necessity* imposes that the past cannot change and whatever has been the case will be the case forever. This, in turn, is for him an ambiguous claim meaning either that the event cannot stop having some singularizing effect or that the sentence describing it cannot loose the truth-value it has. He finally tends to dissociate these two by claiming that although the state might indeed stop having any (singularizing) causal impact upon the world, the sentence representing it will have the same value until the end of Time.[64]But, the rhetorical force of "what has been, has been" can effectively be called into play, if, and if only, the symmetrical (rhetorical) force of "what will be will be" is also activated. In other words, the *motto* "what has been, has been" cannot legitimately be used by he who assigns a gap to sentences referring to some future events. It cannot be used by that person because the same person claims that the present is undetermined with respect to the future. If truth-values are independent of the causal apparatus of the world they have to be equally independent for both directions of Time. Otherwise one direction would rejoice in an unexplainable (i.e. superstitious), priority over the other. There can be no convincing argument in favor of the exclusion of backward branching, except from him who excludes forward branching as well.[65] Finally, we have to note that even if one allows for past forks the past will not change -so to speak- "drastically". By this, we mean that first level sentences about the past cannot become true from false and *vice versa*.[66] For example, if there has been a Battle in the past, the most that can happen to the sentence representing the Battle is for it to become truth-valueless; it will never become false. Moreover, after the emergence of the fork, it will be senseless to say that the *true* sentence describing the past Battle has become valueless. From then on, one could only say that

[64]The reasons indeterminists usually approach the matter in that way is -I think- very complicated and has much to do with philosophical anthropology and our Judeo-Christian tradition. A good Christian is supposed to believe both that (i) he is free to choose between good and evil (hence future openness), and that (ii) justice will be given at the end (Second Coming) not on the basis of the things he has chosen to do and of which (things) some evidence still remains, but upon the things he *has* chosen to do. I make a caricature: imagine Cain wanting to be acquitted upon the basis of "I have murdered my brother" being truth-valueless. An opposite approach towards moral responsibility is depicted in J. LUKASIEWICZ, [1922].

[65]I take this general argument to be valid, irrespectively to what one chooses to do in order to cure the asymmetry. Dummett -for example- (see M. DUMMETT, [2004], [2007]) uses some same arguments in order to criticize his own previous views (M. DUMMETT, [1969] and [2007]-partially), according to which one may not claim that a sentence about the future can be true, if one has no evidence of its truth, while one has no such restriction as for the past. Dummett (op. cit.) draws the opposite conclusion to the one I draw, since instead of disallowing the claim for the past too, he allows it for the future (too). His argument however still speaks in favor of the general claim for symmetry between future and past.

[66]For the sentences of the metalanguage that do indeed suffer such an alteration, see chapters 4 and 5.

the either-true-or-false sentence about the Battle is now valueless.

III

We now proceed with the informal demonstration.[67]

Assume that the ideal agent of S lives at the present moment. Since he is capable of knowing everything that S can prove at this moment, he is capable of knowing all the atomic sentences that are now true as well as their logical product. He now can, as he can at any moment, use the constant set of true causal conditionals and he is also aware of the fact that any causal conditional implies the simple conditional.[68] We now assume that -based on this epistemic content- he can prove the following five things, and that he can prove them no matter where he is situated within the model - i.e. he would have known them, no matter which moment is "*now*".

1) He is aware of monotonically advancing towards the future. One might wonder how this awareness grew in him. This is no central point of the experiment; just assume it in the way one assumes that to travel back in Time is impossible.

2) If the moment n is ahead he will necessarily arrive at n, and he will necessarily arrive at n, after having passed from m moments - m being a positive integer. This makes his Time discrete. With Time continuous, he could (?) endlessly postpone the arrival of a certain future moment, when an indeterministic event "will" happen, by occupying an infinite number of intermediate points. This -unless one is Zeno- sounds rather absurd, and we can be just as happy with the clause: if the moment n is ahead, he will necessarily arrive at n, no matter whether Time is discrete, or continuous.

3)For every past or future moment he thinks of he always finds one past/future moment lying in its respective future or past. Had we been less intuitionistic we could have just said that he is aware of Time being infinite in both directions. How has this knowledge crept into him is, again, of little importance, but one could try the explanation that he is aware of a causal chain of which he cannot find any beginning or end.[69] In any event, even if Time had a beginning or an end, the solution of the fantasy that he is about to elaborate still works. Notice, though, that in this last case the underlying logic would have been classical, and not intuitionistic (see chapter 4).[70]

4) If his History were constituted out of a single moment, and no matter which moment this moment was, there would have been no pair of contradictory states such that neither of them would have belonged to this "History". This last knowledge of his guarantees that despite the fact that the future or the past might be underdetermined the present never is, unless considered as *a present having an underdetermined future or past.* It is never underdetermined

[67]In the statement of the conditions of the thought experiment, I describe a framework more rigid than necessary. I will indicate, *in passim*, any possible relaxation of it.

[68]Cf. note 18.

[69]For the future this chain has to be strong while, for the past a weak one would suffice.

[70]Notice also that (2) and (3) yield that he cannot die and that he was never born.

when these two parameters are not taken into consideration, or, as we will say, it is never underdetermined "in itself".[71] The ideal agent might have come to this conclusion by knowing that all atomic sentences are present tensed and all the rest of present tensed sentences containing no reference to either the past or the future are well-formed sequences of atomic sentences and primitive connectives. Intuitively this knowledge depicts the obvious exegetical fact that Aristotle would never have written *De Interpretatione* 9, if it had been about "today's Sea-Battle".[72] Besides, as we will see later on, the particular solution to the Sea-Battle that I will present does not work when the present is not fully determined with respect to sentences that do not assume references to other moments.

5) He always knows what time it is. This is not trivial since it implies that each set of atomic sentences suffices by itself so as to account for the exact moment the occasional present belongs to.

Besides these general truths, he also knows that:

A) Among the necessary conditions for the Sea-Battle is a certain future self-caused event.

B) He has enough time before the event in order to make the following speculations:[73]

"*There are now no sufficient conditions either for the Battle, or its non occurrence. This I know from the fact that the Sea-Battle depends upon an indeterministic future event, and so, is located in a non yet actualized strong causal chain. To know that there are sufficient conditions for the Battle is impossible without the knowledge of this future event. But, since this is a self-caused event the antecedent is the same as the consequent - i.e. they are both the same indeterministic future event.*[74] *By that, I conclude that no sufficient conditions for this future event can exist in the present. On the other hand, neither can any sufficient conditions for its absence exist in the present, for the actual presence of sufficient conditions for the absence of the event would have meant that the cause of the self-caused event can never be actualized, and so, it*

[71] There is a systematic ambiguity when one tries to isolate what concerns a certain moment and that moment uniquely. For, in a sense, what will happen tomorrow also concerns today because it states a fact about today - i.e. it says something about the order today belongs to. Thus, in a sense, the possible truth of "There will be a Sea-Battle tomorrow" says something about today as well. Prior is aware of this and uses a double category of sentential variables. In the first, he disallows any occurrence of "F", and so restricts sentences down to a set almost identical to the one which, according to our idiolect, represents a "moment in itself". Cf. A. PRIOR, [1967], p. 124.

[72] We do not mean that the possible indeterminacy of "today's Sea-Battle" has no philosophical interest of its own. Vagueness or presence's indeterminacy, however, lies outside our worries here, as it also lay outside Aristotle's worries in *De Interpretatione* 9.

[73] (B) is intuitionistic and is necessary only if one does not think that the agent is capable of any infinite act of intuition. For if he is, he would always have time to reach the conclusion of any valid argument. For simplicity, one can just make him capable of infinite acts of intuition and ignore (B).

[74] Since among the conditions for the Battle is a non yet settled indeterministic event, and any actualized strong causal chain leading to the Battle contains this self-caused event, no actualized strong causal chain leading to the Battle can be there at present.

does not (cannot) depend on the future. But it does depend on the future, since this self-caused event is also a "future" event.

Therefore, the sentences "There will be a Sea-Battle tomorrow" and "There will be no Sea-Battle tomorrow" are neither true nor false: Since neither of them belongs to the present, neither of them is true, and, since the one is the negation of the other, neither of them is false.

Now, as for their disjunction: despite the fact that I cannot, at present, find sufficient reasons for any of the disjuncts, it is certain that in a finite period of Time I will necessarily find some for one of them. To begin with, I cannot die. Since, I can neither stay forever at the same moment, and, moreover, since I cannot go back to the past, and, finally, since the moments separating me from any other moment are finite, I will necessarily arrive at a stage where my universal calendar will say that it is the day that I now call "tomorrow". On this day, and since the present in itself is never underdetermined, I will certainly know one of the two following things: that there is a Sea-Battle, or that there is no Sea-Battle. On both these occasions, I will also know (by the truth tables), that the disjunction is true. Now, since I cannot by-pass tomorrow, I cannot go on living without eventually proving the disjunction in a truth table manner. Tomorrow is a bar for my evolution in Time, and upon that bar, I necessarily prove the disjunction by the truth tables. Therefore, I find no reason why I should not consider the disjunction as already proved.

With this general kind of reasoning, I can assert something similar about the events I have forgotten and have taken place at some definite moment. The disjunctive sentence constituted out the sentence asserting that they took place at that moment, and the sentence denying that same thing, is certainly true. During my endless life I have certainly passed a point where this was true that moment. Unfortunately, I cannot say the same about any of the disjuncts. I might equally have been at a point where they did take place and at a point where they did not. More unfortunately, and unlike future indeterminacies, I will never, in the future, settle the matter about which was which. No singularizing effect of the event or of its absence remains in the world. This is why I can neither remember its presence nor its absence. If I could, there would have been such an effect in the world: a trace of it within my memory.

A trickier case is presented by boundless future predictions like "There will be a Sea-Battle in the future". If I have no evidence for the presence of the event at any future moment, and nor have I any conclusive evidence for its absence, I am in no position to assert the disjunction. For I am not able to say that I will certainly pass from a moment where I will have evidence that the Battle happens nor am I able to say that after a certain moment its advent will have become impossible. From the moment a Battle becomes the case, or becomes impossible, the matter is settled and the disjunction proved. But what if, day in, day out, none of these happens? It is not contradictory to suppose this, and so, unlike with tomorrow's Sea-Battle, I will not be able to assert the disjunction."

Do the above meditations consist of a sound piece of reasoning? I think that they do, and I will try to give a formal justification of that claim in the following chapter, where I will present the same argument in a non-anthropomorphic way.

As an introduction to this presentation, let me stress the fact that the reason that the logic behind the argumentation of our ideal agent allows for, "There will be a Sea-Battle tomorrow" and, "There will be no Sea-Battle tomorrow" to be untrue when their disjunction is true -i.e. the reason it allows for the fantasy- is that the agent evaluates the disjunction not by the standards of the world he is occasionally living in, but on the horizon of the spectrum of worlds that he will encounter during his future. Or, to put it better, the agent treats the disjunction in a way that can be considered as partially truth-functional and partially not. It can be seen as truth-functional on the basis that the argument proving that there will either be a Sea-Battle tomorrow, or not, partially depends upon the fact that tomorrow (when tomorrow comes) will validate the disjunction on a truth table basis. On the other hand, the overall argument is not truth-functional because it asserts the disjunction already by now, and now none among the disjuncts has a value. It is as if the flow of Time guarantees that the future will make truth-functional every truth that is not truth-functional at present. Based on that guarantee one can reasonably assert the truth of compound sentences containing gaps. The above argument shows how important was point (4) above. With (4), we have relaxed the atomistic apparatus by allowing some sentences not to be analyzable in the way the classical apparatus would have wanted them to be. Had we, nonetheless, relaxed it too much, and allowed sentences about the present in itself not to depend on the tables, the solution would not have worked. If "There is now a Sea-Battle" can stand for a gap, then, one has no warrant whatsoever that the advent of the Battle will ever be settled, for it would no longer have been necessary that tomorrow "There is now a Sea-Battle" will have a value. This means that an indispensable parameter for our solution to work is that the present (each occasional present, when "its time comes") is/was/will be decidable.

This peculiar and fruitful combination of truth-functional and non truth-functional reasoning illustrates also what we were saying earlier on about the partial order that our occasional present belongs to. We claim that this is a primary feature of our world, which (feature) is non further-reducible to the atomic sentences and the tables.[75] It is non further-reducible to them, because although it will eventually be thus reduced, this reduction cannot happen until some time has elapsed. Another way to say the same thing is by wondering about where the newly found truth of $T(p \vee q)\&\neg T(p)\&\neg T(q)$ belongs. It does not belong to the logical product of the set of the *now* true atomic sentences, since the argument leading to it assumes causal conditionals about future indeterministic events. It does not belong to the causal truths either, since it assumes the Excluded Middle, which is taken to be valid for present tensed atomic sentences and any sentence that can be generated out of these. On the other hand, the same truth does make part of S, since it reveals a truth about the world, and S has been defined to be a science dealing with everything there is to know. It is -figuratively speaking- as if S is the logical product of the distribution (spread) of subsets of AS along a partial order, and of the awareness that somewhere upon this order the present world is to be found, and that this world is forced to constantly move from one point to another, in the same direction. The fact

[75] The silent investment we have made, in page 45, where we have said that worlds of the model are not completely independent from one another begins to pay off. Worlds of the model are not isolated. Some of their sentences are functions of elements of other worlds of the model.

that the order is partial and not linear excludes a universal atomistic theory for, as we have seen, forks give evidence of well-formed sentences not analyzable by the tables. The fact that it depicts the movement of Time is also responsible for the previous results because it guarantees that as time moves on the tables will eventually justify either one of the disjuncts, even if for the time being they justify neither.

Finally, and in order to emphasize more the kinship between this solution and Intuitionism it will be instructive to notice that our agent is of the formal opinion that a true implication does not necessitate that either the antecedent is false, or the consequent true. See -e.g.- how he treats causal conditionals. For every indeterministic future event a, he asserts $a \rightarrow_c a$, and, because of that, he also asserts $a \rightarrow a$. But neither the antecedent of the above implication is false, nor the consequent true - they are both truth-valueless. His *sui generis* implication means rather that once the antecedent is at hand the consequent cannot fail to be at hand as well.[76] This explains why I have in chapter 2 avoided saying that the causal implication implies the material implication, but preferred to use the neutral term "simple". Notice also that the remote reason that the conditional is allowed to be true, while formed out of gaps, is, again, the Time horizon of the model.

For similar reasons, the negation used by the ideal agent behaves non classically as well. For it is not the case that a sentence is false, if not true. Think again of the Sea-Battle. In order to assert the falsity of "There will be a Sea-Battle tomorrow", he needs to consider himself able to show that tomorrow's Sea-Battle is S-contradictory. Its simple non-truth (i.e. absence from the present world) does not suffice.

It is only for conjunction that his apparatus is classical. Remember of $p\&q\&\ldots\&r$, where $p, q, \ldots, r$ were supposed to stand for all the necessary conditions for the Battle. They all would have to be true *now* -i.e. to belong to the present world- in order to allow one to assert the conjunction.

Add to these remarks the straightforward result that the four fundamental connectives used by the agent are independent of one another, and there you have all the makings of an (almost) intuitionist.[77]

[76]A more clear-cut case is when one considers how an event which is upon a non yet actualized strong causal chain, depends on the (future) origin of the chain. When the agent asserts that, if the admiral decides in favor of the Battle, the Battle will take place, he does so, on the basis of excluding that the admiral can decide in favor of the Battle and the Battle does not follow, after such a decision; not on the basis of 'There will be a Sea-Battle tomorrow' being true, or 'The admiral will decide in favor of the Battle' being false.

[77]*Pace* the falsification of the Disjunction Property and some further non intuitionistic aspects that we will elaborate in the following chapter. These explain the "almost".

Chapter 4

The Intuitionistic Solution - Formal Account (an outline)

In part I, I present the model corresponding to the intuitionistic solution. This model is a variation of Beth models (cf. E. BETH, [1956], and D. VAN DALEN [1978], [1986a]=[2002], [2001]). In general, while Beth-Van Dalen models are mathematically oriented, the model presented here focuses on sentences concerning events, and as such, transposes the weight of the theory from the body of mathematical theorems to sets of sentences constituting Histories. In part II, I present my solution to the fantasy. A comparison between the informal account of chapter 3 with the formal account of this chapter is attempted in part III.

I

<u>S-models and Beth models</u>

0. Those that are acquainted with models of intuitionistic logic, and especially Beth models might be helped by the following preliminary remarks so as not to misunderstand the general purpose of S-models. Others may skip the present section and go directly to 1.

0.1. The *Creating Subject* of Beth-Van Dalen models is here replaced by the ideal agent of S of chapter 3 (here the *Ideal Scientist*). This in itself is suggestive. The *Creating Subject* is an ideal mathematician who can at any moment enlarge the universe of constructions by his own activity. What this universe is depends upon the activity itself. The *Ideal Scientist* looks much more like an ideal recorder. As time goes by he accumulates more and more information on the physical universe he is living in. Moreover, ideal as he is, he cannot fail to record any new information, if the information crosses his location.[1] This implies that in indeterministic universes there are moments when the *Ideal Scientist* unavoidably enlarges his amount of knowledge; these are the moments when the information about the occurrence or non-occurrence of the indeterministic event reaches his spatio-temporal coordinates. This also explains why, while

[1]The picture is more vivid in relativistic universes. Cf. Appendix I.

the standard Beth models are equivalent to Kripke models,[2] our S-models are, in general, not. There are Kripke models where the *Creating Subject* keeps the same amount of knowledge *ad infinitum*. This is because he can endlessly remain at the same node of the model. Contrary to that, in a Beth model, the *Creating Subject* is forced to constantly "jump" from one node to another as time goes by. Now, there are Beth models, where the *Subject* can endlessly jump from world to world in such a way that he endlessly avoids the acquisition of any new knowledge, imitating thus, the "stable" inhabitants of Kripke models.[3] In an S-model the *Ideal Scientist* cannot always do the same. Moreover, if tensed sentences are allowed, or if indeterministic events are posited, he can *never* do the same! The reason is the aforementioned. After a certain point, he will unmistakably learn about the advent or non-advent of the indeterministic event, or he will unmistakably experience the truth of a "presently" true atomic sentence that was not true "a moment ago". The march of Time necessitates some changes in the truth-value of tensed sentences.

The impossibility of retaining the same amount of knowledge *ad infinitum* affiliates the S-models to their Beth, rather than to their Kripke kindred.

0.2. Intuitionistic models (both Kripke and Beth) usually adopt linear pasts. The intuitive motivation for this is obvious. Ideal mathematicians have to have perfect memory. Their universe of constructions is, as time goes by, monotonically augmenting. Our S-models allow for backward branching. Does this mean that *Ideal Scientists* suffer from imperfect memory? In a sense it does, and in a sense it doesn't. The backward branching of the S-model is supposed to capture a basic intuition of chapter 3 - i.e. that there is no logical law imposing that all events have to have at least one singularizing effect during all moments following their advent. If some of them stop having any singularizing effect after some point in Time, they will stop having any such effect in the mind of the *Ideal Scientist* too. This amounts to the claim that the *Ideal Scientist* will, after a certain point, "forget all about them". For how can he remember something about them, if they have left no singularizing trace in the entire world, and, consequently, in his own mind either. Backward branching depicts just that. On the other hand, the *Ideal Scientist* has perfect memory, in the sense that he cannot forget any past event that has left even the slightest singularizing trace in his actual present. This illuminates yet another aspect of the S-model. Any event, once forgotten, can never be regained by the *Scientist*. It is not as if the event has left a singularizing trace in the world that the *Scientist* is not aware of at present, and when he becomes aware of it memory will creep back into him and he will remember. The event has left no trace whatsoever, and therefore unless circular time is assumed, the *Scientist* will never remember it anew. So, the assumption of imperfect memory for traditional intuitionistic models is a quite different assumption from our own. In a classical intuitionistic model, imperfect memory means that the *Subject* can forget a certain construction. This *eo ipso* does not exclude that the *Subject* will one day arrive at the same construction, and by doing so, "remember" the knowledge that he has temporarily lost. The *Creating Subject* experiences (knows) mathematical theorems, not historical facts, and he creates numbers, not events. The difference

[2] Cf. See S. KRIPKE, [1965]
[3] Cf. D. VAN DALEN, [1986b].

between the *Subject* and the *Scientist* is best exemplified, if one considers -for argument's sake- that the ideal knower of the model is both a *Creating Subject* and an *Ideal Scientist*. Suppose that at a certain moment he establishes a theorem which, at some subsequent moment, he forgets. What he is always capable of is to "reconstruct" its proof; the one he has forgotten or another, it is of no importance here. What he can never do is to remember the exact circumstances that lead him to the first construction,[4] for this latter is an event which has left no singularizing effect within the world. Since he is here assumed of having forgotten the constriction, he must be also assumed of having forgotten the exact circumstances that have led him to the proof. If he had them in his grasp, he could also have the construction. The above was an example that aims to illustrate the difference separating these two "super-minds". Otherwise what we will systematically do in our presentation is to abstract any kind of non-historical knowledge by assuming that the *Ideal Scientist* omnitemporally knows any logico-mathematical truth there is. In other words the model will from the logico-mathematical point of view be supposed to be classical. The surprising fact -as we shall see- is that the underlying proof theory remains intuitionistic because of reasons having to do with the events themselves.

0.3. The reader must also bear in mind that our S-models can be seen as all round classical models of temporal logic with forward and backward branches (i.e. not as models of intuitionistic logic), with the sole exception that quantification over worlds of the model is sometimes intuitionistic; not classical. (We will say more on that point in 5.1) This, combined with the fact that the metalanguage belongs to the model itself (see 0.4), accounts for its being intuitionistic.

0.4. A last, but very important, difference that separates intuitionistic models and S-models is the following. Unless, in a somehow implicit manner,[5] the metalanguage about the intuitionistic models (either Kripke or Beth) makes no part of the models themselves. On the other hand, the metalanguage about S-models belongs to the models. Let us use a picture: the *Ideal Scientist*, apart from the new things he learns about the world, also learns that he learns these things, and he can also speculate about (and assert) the way he learns them, the truth-conditions on the basis of which he asserts them, and so on and so

[4]Although he will, possibly, be in position to infer them from the evidence provided by the reconstructed proof, when he will have reconstruct it. In everyday mathematical experience these occasions arise quite often - e.g. one forgets in the morning a solution he has found in the evening. When he rememberers it anew, he also remembers the exact line of thought that has led him to it the previous day, the accidental incidents that provided the "illumination", other things that were passing through his mind back then, etc. Notice that the new construction need not be identical with the old one. Even if different, it can trigger the same recollection. Such cases however will never arise in our S-models because we consider that the *Ideal Scientist* is constantly in possession of the entire corpus of logico-mathematical truths (see *infra* and 4.4.3.2). Therefore, the events he forgets will never be inferable by future evidence: the truths he forgets concern the events themselves, not some atemporal theorems that are "discovered" by these events and can only be re-"discovered" by routes of the same or similar kind, allowing thus the recollection of the forgotten events in the way described above

[5]See esp. the *Creating Subject* variants of the models in D. VAN DALEN, [1978]; §. 5, [1986a], A. TROELSTRA - D. VAN DALEN, [1988]; ch. 16.3. It is asserted in many passages there that the *Creating Subject* can also think about his own activity and that this is in concordance with Brouwer's initial idea.

forth. A node in an S-model represents -just like a node in a Beth model- a possible moment in Time, together with an assigned set of sentences: sentences representing the things known to the *Scientist-Subject* at that moment. Among these latter are to be counted the knowledge about the knowledge of them, the ways they came to be known, the conditions under which they came to be considered as known, etc. Figuratively speaking, the present chapter -this presentation of S-models- is constituted out of sentences that belong to all the worlds of all S-models. This -abruptly put- is what is meant by the claim that the metalanguage about the model is not external to it.

Vocabulary and Syntax

1. We have at our disposal (vocabulary):

1.1. Individual variables: $x, y, z, \ldots, x_1, x_2, \ldots, x_n, \ldots, y_1, y_2, \ldots, y_n, \ldots$

1.2. Individual constants: $x_{(1)}, x_{(2)}, x_{(3)}, \ldots, y_{(1)}, y_{(2)}, y_{(3)}, \ldots$ (Schematically noted as $x_{(n)}, y_{(m)}, \ldots$)

1.3. World variables: $w, w^{'}, w^{''}, \ldots$

1.4. World constants:

$$w(n), w(n,m), w(n,m,s), \ldots, w-(n), w-(n,m), w-(n,m,s), \ldots$$

1.5. World constants depending on free choice sequences:[6]

$$w(\ldots\vec{n}\ldots), w(\ldots\vec{m}\ldots), \ldots$$

"$(\ldots\vec{n}\ldots)$", "$(\ldots\vec{m}\ldots)$" are open expressions posing an extensional condition on $\vec{n}$, $\vec{m}$, and "$\vec{n}$", "$\vec{m}$", are strings of numerals:$n_1, n_2, \ldots, n_{n\in\mathbb{N}^*}, m_1, m_2, \ldots, m_{n\in\mathbb{N}^*}$.

1.6. Sentential letters: $\alpha, \beta, \gamma, \ldots$, together with letters for their names (auxiliary names for the metalanguage). For simplicity, we will use the same symbol for both, when matters of ambiguity can be resolved on the spot.

1.7. Predicate n-adic letters: $\phi^n, \psi^n, \ldots$, for $n \geq 1$

1.8. Existential $\exists-$ and universal $\forall$ quantifier letters having as domains the items denoted by the constants of (1.2), (1.4)-(1.5), (1.6) and (1.10), according to the context.

1.9. Numerical variables: $n, m, s, \ldots, n_1, n_2, \ldots, n_n, \ldots, m_1, m_2, \ldots, m_n, \ldots$

1.10. Numerical constants. (Their schematic notation imitates the schematic notation of the individual constants (1.2) - i.e. $n_{(m)}, m_{(r)}, \ldots$)

1.11. The indicator $-$

1.12. The standard arithmetic and set theoretical connectives and operators:

$$=, >, \ldots, \emptyset, \in, \ldots, +, \times, \ldots$$

[6]For the notion of "free choice sequence", see esp. A. TROELSTRA, [1977], A. TROELSTRA - D. VAN DALEN, [1988]; ch.12.

1.13. Sentential connectives, other than 1.8:

1.13.1. primitive: $\&, \vee, \rightarrow, \rightarrow_c$

1.13.2. non-primitive: $\neg, \leftrightarrow$

1.14. Connectives between sets of sentences and sentences: $\vdash$

1.15. Connectives between worlds and sentences: $\Vdash$

1.16. Modal operators:

1.16.1. primitive: $\Box, \Diamond$

1.16.2. non-primitive: $\triangle$

1.17. Tense operators:

1.17.1. non-metric: P, F, H, G

1.17.2. metric: Pn, Fn

1.18. The constants *now* and $\perp$, and the function now.

1.19. Function letters like:

1.19.1. The star function:$(w(\ldots(\vec{n})\ldots))^* = (\ldots(\vec{n})\ldots)$

1.19.2. The f (rank) function:$f(w) =$ the rank of w

1.19.3. The ι (identity) function: $\iota(w) =$ the index identifying w

1.19.4. The ^{o} function, turning an economic into an objective index.

1.20. Expressions of the form $(\ldots)((w(\ldots(n)\ldots))^*)$, where the first pair of parentheses may contain any sentence or sentential letter.

1.21. Right and left parentheses, square brackets, and commas.

1.22. The ordinary English counterparts of the logical connectives and constants together with some of the basic facilities that usually accompany them in ordinary language contexts, such as copulative predicates, the "such that ..." connective, etc; "if and only if" will be abbreviated by "iff".

2. Syntax. We assume that the reader is familiar with the syntactic rules for many of the above. For the rest, the rules will be presented along the way.

Some convenient clarifications:

2.1. Parentheses can be dropped according to the following hierarchy: first one can drop parentheses containing entire formulas. Then the priority among the fundamental connectives goes as follows: $\neg$ binds more tightly than & and $\vee$, which bind more tightly than $\rightarrow$, $\leftrightarrow$ and $\rightarrow_c$. The bond between the two main pairs of parenthesis in 1.20 has priority over any other. All operators bind as tightly as $\neg$. Finally, we note that the difference between parentheses and square brackets is semantical (cf. 4.5.2); their syntactic rules are the same.

2.2. The syntax of $\rightarrow_c$ is the same as the syntax of $\rightarrow$.

2.3. The syntax of $\triangle$ is the same as the syntax of the rest of the modal operators.

2.4. "$(\ldots now \ldots)$" is well-formed, if, and only if, "$(\ldots w(\ldots \vec{n} \ldots) \ldots)$" is well-formed.

2.5. The Predicate Calculus letters of (1.7) will often be noted without their superscripts.

2.6. The indicator in (1.11) can only appear as in (1.4), and it is not the same symbol as the prefix of negative numbers.

3. Some more philosophical remarks concerning our vocabulary and syntax:

3.1. The schematic numerical constants of (1.10) appear awkward because they seem to mimic the schematic constants of (1.2). The problem arises from our having at our disposal the complete set of numerals and thereby not needing any $n_{(1)}, n_{(2)}, \ldots$ notation for the numerical constants, such as we need for the individual constants.

3.2. As already stated in (1.6), we will often abolish the distinction between an object and its name. We are confident that the matter will always be decidable in/by context.

3.3. The metalanguage about intuitionistic models is, most of the time, classical. This poses no problem because sentences belonging to the worlds of the model, and sentences about the model itself are usually strictly separated; the metalanguage about the model[7] does not belong to the model itself. Thus, one can safely live a double-life in and out of the model by evaluating the sentences of the model by means of intuitionistic criteria, and stating these criteria outside the model, and by some classical metalanguage. In an S-model, living the

[7] In most cases it is a semi-formal version of some natural language - e.g. ordinary English enriched with some formal operators which receive a (tacit) classical interpretation.

double-life is living at the edge. This is because as we said in 0.4, the metalanguage belongs to the model. It is of some importance to clarify here that the natural language counterparts of the connectives (i.e. "or", "iff","and", "not", etc.), will receive the standard classical interpretation they tacitly receive in any standard presentation of intuitionistic models.

3.4. It is convenient, as well as consistent with the atomistic *Weltanschauung* of chapter 3, to try to "construct" a deposit of atomic sentences according to the following procedure. We first assume a set that constitutes the universe of individuals which we call U, and a set of n-adic ($n \geq 0$) primitive predicate letters. (In this way, the members of (1.6) become a special case of (1.7): i.e. take $n = 0$).[8] Atomic formulas will then be all closed, non-quantified, well-formed formulas constituted by an n-adic primitive predicate letter and some corresponding n arguments (an n-tuple) taken from the members of U (the n-tuple would belong to the n^{th} Cartesian product of U). What we meant by, "an exhaustive description of the world" in chapter 3 is now identified with the set of atomic sentences that are true. The idea is that this is an "exhaustive description of the world" in so far as the truth-value of the rest of sentences can be inductively reached by the truth-value of these and the definitions of the connectives.[9] We hope that this parallel illuminates our previous insistence on the atomistic foundations of the model and the necessity for some historical introduction to it. How this atomistic ideal is supposed to relax, in order to cope with future contingents, will be stated fully in the semantics.

3.5. In (1.5), we used the controversial notion of a "free choice sequence". This terminology underlies the fact that "$\vec{n}$", within the open expressions presented in (1.5), is no variable but a specific item, which, nonetheless, is not determined -so to speak- "right from the start", or, "on the spot", or, "*in situ*". I very much doubt whether, at the present stage, the above explanation really helps, or throws more clouds around, but, in any event, I ask the reader to come back to it after having gone through the semantics.

[8] It is debated whether the early atomistic conception that we try to reproduce here would have assumed atomic sentences with 0 arguments. The alternative is that the Sentential Calculus is, in fact, constructed upon the atomistic Predicate Calculus - i.e. that atomic sentences of the former are n-adic ($n > 0$) open primitive expressions filled with n primitive names of individuals

[9] Here, we will take as primitive connectives $\rightarrow, \vee, \&$ together with the quantifiers.We can avoid the primitiveness of $\neg$ because we use the constant $\perp$. If we wanted to stay closer to the atomistic tradition we could reduce the connectives even further. Cf. the Sheffer stroke, L. WITTGENSTEIN, [1922]; 6, or the stroke function of Russell-Whitehead, A.N. WHITEHEAD- B. RUSSELL, [1910-13].

Semantics

4. S-models aim at corresponding with the final scientific theory S. By "validity", "soundness", "completeness", "correspondence", etc. the usual is understood. A (closed) formula α will be said to be valid in the S-model M (symbolically: $M \models \alpha$), if, and only if, α belongs to all the worlds of the model. A (closed) formula will be said to be S-valid, if, and only if, it is valid in all S-models. The desired correspondence between S and S-models is equivalent to the condition:

$(A \vdash_s \alpha)$, iff, (for every world w of any S-model, if, for every β, such that $\beta \in A, w \Vdash \beta$, then, w $\Vdash \alpha$).

The correspondence, if assumed, will make S less rich in theorems/axioms than classical logic, since the Excluded Middle will be shown to be non S-valid, but it will also make the metatheory of S less rich than the metatheory of intuitionistic logic, since we will provide S-models where the Disjunction and Existence Properties do not obtain. Both these "poverties" (esp. the second one), will work in favor of the present solution to the Sea-Battle problem.

***

An S-model is a septuple $\langle W, f, R, D, V, \Vdash, \text{now}\rangle$. W is a set of elements called "worlds". f is a function, which -from the intuitive point of view- assigns to the worlds "a moment". It functions like a clock; we will also call it "clock function".[10] R is the accessibility relation of modal semantics. R -intuitively- captures the "…is a possible present or future of—" relation. D assigns a subset of the universe of individuals (U) to each element of W. V assigns to these elements a set of atomic sentences (intuitively the set of atomic sentences true about them), together with the constant set of true causal conditionals (which are true for all worlds). $\Vdash$ determines inductively the value of the rest of sentences, if they have any value, and "now" is a higher order function, assigning the object "*now*"[11] to a single element of W. Intuitively it determines the occasional, "present moment".

4.1. W is a non-empty set of elements, called "worlds".

4.2. The function $f : \mathbb{N}(+/-) \to W$, assigns a number from the positive and negative integers *plus* 0 to each element of W, and obeys the condition that any set of elements that has been assigned the same number is denumerable;[12] the number assigned is called "rank" of the world it is assigned to.

[10] Since "f" is usually a predicate variable or schematic letter, another more transparent notation could perhaps be preferable - e.g. "r" from *rank* (see *infra*), or "c" from *clock*. We use "f", because it was the notation used by Beth in the original paper of 1956.

[11] We'll use italics for the object and non-italics for the function.

[12] i.e. $\{w : f(w) = n\}$ is denumerable for every $n \in \mathbb{N}(+/-)$.

4.3. The formal properties of the accessibility relation R are laid down in the following conditions from 4.3.1 to 4.3.8:

4.3.1. R is reflexive and transitive.

4.3.2. $(\forall w)((\exists w')(\exists w'')(w \neq w')\&(w \neq w'')\&(wRw')\&(w''Rw))$

4.3.3. $(\forall w)(\forall w')((\exists w'')(w''Rw)\&(w''Rw'))$

For the next two characteristics we need the following definitions:

4.3.4.

Definition 1. *w' is an "immediate successor" of w, if*

$$(wRw')\&(w \neq w')\&((\forall w'')(wRw'')\&(w''Rw') \rightarrow (w'' = w') \vee (w'' = w)).^{13}$$

4.3.5.

Definition 2. *A "maximal path through w" is a sequence of different worlds $\langle \ldots w', w'', w''', \ldots \rangle$, such that*

(i) $\ldots Rw'Rw''Rw''' \ldots$,

(ii) one of these is w,

(iii) if $\langle w, w' \rangle$ is a subsequence of consecutive worlds in the path, then, there is no world $w'' \neq w' \neq w$ in the model, such that wRw'' and $w''Rw'$ and

(iv) for every world w of the path there is always a world $w' \neq w$ such that wRw' and a world $w'' \neq w$, such that $w''Rw$.[14]

4.3.6.

Definition 3. *A "bar for w" is any set of worlds having an element in common with any maximal path through w. An "economic bar"*[15] *for w is any bar such that it is not the case that some of its worlds can be omitted in such a way that the remaining set is still a bar for w.*

4.3.7. Take "n" in "wR^nw'" to count the number of immediate successors in between w and w'. For every two worlds w and w', if wRw', then, wR^nw' for some $n \in \mathbb{N}^*$.

4.3.8. If $f(w) = n$, all immediate successors of w are of rank $n + 1$.

[13] in this chapter numbered definitions are the most important though not the only ones.

[14] We will call a path a "maximal semi-path from/to w" if only one out of the two subconditions of (iv) obtains and w is the first/last world of the sequence. A "path through w" is defined by conditions (i) and (ii).

[15] Also called "thin bar" in the literature.

From *axioms* 4.3.1-8 one can deduce the following *facts* about the model (4.3.9-18):[16]

4.3.9. The model is infinite in both directions (from 4.3.2, 4.3.7 and 4.3.8).[17]

4.3.10. The model is convergent in the backward direction (from 4.3.3).[18]

4.3.11. There are no separated frames in the model (from 4.3.3).[19]

4.3.12. The model is discrete (from 4.3.7 and 4.3.8).[20]

4.3.13. R is antisymmetric (from 4.3.1 and 4.3.8).

4.3.14. The only world that w sees and which has the same rank as w is w itself (from 4.3.1, and 4.3.7 and 4.3.8).[21] (This is the key to most of our subsequent results, and as such it will be called the "fundamental characteristic" of S-models.)

4.3.15. w is a bar for itself. (Think of w as a part of any path through w).

4.3.16. The set of immediate successors of w is an economic bar for w (from 4.3.7 and 4.3.8).

4.3.17. In each maximal path there is one and one world only out of every rank (from 4.3.7 and 4.3.8).

4.3.18. Every bar has an economic bar as its subset (set theory).[22]

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[16] The proofs for most of the above are trivial.

[17] However a kind of "circular Time" is not excluded, under these conditions. See 4.4.2 and Appendix III.

[18] i.e. for every three worlds w, w', w'', such that wRw'' and $w'Rw''$, there is a world w''', such that $w'''Rw$, and $w'''Rw'$. This characteristic can be codified by the motto, "no converging without diverging". In more words: there are no two paths in the model that converge without having previously shared some past in common. This condition covers also the more general characteristic of there being no two worlds in the model that do not share some past in common.

[19] i.e. sets of worlds belonging to the model of which (sets) no world belonging to the one sees any world that either sees or is seen from a world of the other. For this use of "separated" see G. E. HUGHES - M. J. CRESSWELL, [1996], p. 61. Cf. the "unified road network" metaphor of chapter 3.

[20] Notice that both 4.3.7 and 4.3.8 are necessary. 4.3.8 does not suffice for discreetness, for it does not see to the actual presence of immediate successors. On the other hand, 4.3.7 (without 4.3.8) would have allowed for constant ranks from the same numeral to convergent sequences - e.g. n (constant) $\rightarrow \{0, 1/2, 3/4, 7/8, \ldots\}$. I owe this point to professor D. Van Dalen.

[21] Note that (4.3.8) obliges all successors of w to have a rank superior to $f(w)$, and observe also that any world sees all its successors upon the maximal paths through it, and -apart from itself- these worlds only.

[22] Take b to be any bar for w. At least one element of any path through w belongs to b. If there are more, choose a representative. The resulting set is an economic bar for w. Observe that the Axiom of Choice is needed when the number of paths through w is infinite.

4.3.19. We now proceed to associate to each world, w, of the model, the structure $\langle W(w), f, R\rangle$, where $W(w)$ is the set $\{w'\colon wRw'$ or $w'Rw\}$ and f and R are the f and R of the initial S-model, when narrowed down so as to fit $W(w)$. We will say that the above structure is, "the substructure of W corresponding to w", and w will be called, "the origin of $W(w)$". We will also say that $\{w' : w' \in W(w) \& f(w') \leq f(w)\}$ and $\{w' : w' \in W(w) \ \& \ f(w') \geq f(w)\}$ are the two semi-substructures of $W(w)$. Our intention is to isolate within $W(w)$ the worlds that are possible presents, pasts and futures of w - i.e. to make $\langle W(w), f, R\rangle$ w's History, in the intuitive sense of "History", elaborated in chapter 3 and connecting it with partial, rather than linear orders. We also wish to isolate, within the semi-substructures, w's present and past and w's present and future, respectively. Observe that this procedure of narrowing down the frame can be iterated at will. Provided that $w \in W(w)$, one can construct the non-empty sub-substructure $\langle W(w)(w'), f, R\rangle$. These iterations will be helpful, in the following sense: $\langle W(w), f, R\rangle$ represents w's History. Now, in non-linear Histories things that -when the clock shows "$f(w)$" - are possible futures for w, might be impossible in subsequent moments. Say that they are impossible at $m > f(w)$. If $w' \in W(w)$, and $f(w') = m$, it is obvious that $\langle W(w')(w), f, R\rangle$ will be a proper subset of $\langle W(w), f, R\rangle$, and will represent w's History as narrowed down by the perspective of a moment ulterior to $f(w)$. This technique will be exploited in 4.5.2.

4.3.20. A denumerable set of indexes suffices for indexing all worlds of W.

This is an easy result and it comes from the facts that (i) the model is discrete (4.3.12), and (ii) any set $\{w : f(w) = n\}$ of the master-structure W is denumerable (4.2). Thus, any world can be identified by a pair of numerals. The first denotes its rank. The second is determined as follows: assume an indexing function from $\mathbb{N}^*$ to each element of $\{w : f(w) = n\}$,[23] the second numeral will denote the index assigned to the world by this indexing function. Thus there will be a different index in $\mathbb{N}(+/-) \times \mathbb{N}^*$ for all worlds of the S-model; the indexing is effected by ordered pairs, the first of which denotes the rank of the world and the second, its subordinate index, according to the indexing function. This way of identifying worlds concerns W itself (the master-structure) and will be called "objective".

4.3.21. At this point we will present three ways to identify worlds within the substructures of W. Consider the substructure: $\langle W(w), f, R\rangle$.

4.3.21.1. The first way to identify worlds within $\langle W(w), f, R\rangle$ is to use the already given "objective" and constant indexing of the master structure.

4.3.21.2. The second way is to apply the same general technique we have used for W, but to cut it down to $W(w)$ - i.e. to mimic the method of the "objective" indexing for W in $W(w)$. The identification will be achieved -again- by two numerals: the first will be the rank of w, and the second, the index appointed to w by an indexing function from $\mathbb{N}^*$ to $\{w' : f(w') = n \& w \in W(w)\}$.

[23] By (ii), it follows that no element of $\{w : f(w) = n\}$ is left without an index.

4.3.21.3. The third way identifies worlds by taking them to be elements of free choice sequences passing through the origin. The method follows each maximal semi-path emerging from/coming to the origin. By the *fundamental characteristic*, it follows that this suffices for identifying all the worlds of the substructure. The indexing is effected by an n-tuple, where n is identical with the distance separating the origin from the world to be identified, and each element of the n-tuple identifies a world upon a linear path. First, it identifies the origin by the numeral denoting its rank. This is effective because the origin is the sole element of the substructure with that particular rank. If the semi-path is directed towards the origin, the prefix "-" comes to be added in front of the sequence. Each other numeral in the sequence identifies the immediate successor/predecessor[24] of the previous element, according to yet another indexing function, which well-orders any set of immediate successors/predecessors. (Immediate successors/predecessors have the same ranks, and so they are denumerable by 4.2). The identification can always be achieved by indexes of the form (-)$(n, m, s, \ldots)$.

4.3.21.4. A feature that the two last methods share is that they readjust in economic ways the indexing function referring to the entire W. It is in this sense that they will be called "economic". In our exposition we will use the second of these two - i.e. 4.3.21.3. We will favor this method because it identifies the origin by means of a single numeral. This quality accentuates the intuitive foundation of our solution: there is only one possible present! It is this uniqueness that will subsequently generate the solution to the fantasy.[25] Another reason for preferring 4.3.21.3 is the following: from the philosophical point of view, it depicts more accurately the underlying intuition, according to which History can be represented in the form of a free choice sequence that tends to generate one final linear order.[26] This will become clearer during the elaboration of the so-called "forcing relation". We have to note, however, that the particular solution of the Sea-Battle that we will present here works just as well with the rest of the indexing techniques. Remember that the "objective" index is common to all worlds and so the schema, "the world having ... as its 'objective' index, has in the substructure ... the 'economic' index ..." is decidable. This suggests that there are decidable functions that transcribe any form of indexing into any of the other two, according to the occasional origin.[27]

[24] Depending upon whether it is a sequence towards or away from the origin.

[25] Against the generally held opinion that what solves the puzzle is "historical necessity", I believe that the mere "present's necessity" suffices. See also note 131.

[26] Cf. D. VAN DALEN, [1978], p. 4; Neither (i) nor (ii) of Van Dalen hold in our structures, but this is because of backward branching. Otherwise, one can conceive of our "free choice sequences" as sets of possible linear continuations of nodes. (Since, as we have seen, any set of immediate successors can be well-ordered, use this indexing so as to turn the free choice sequences of worlds into free choice sequences of natural numbers.) This is exactly the tale told by 4.3.21.3. The strings it uses present the portion of the total function (cf. ibid, p. 4, (ii)) that has been constructed, up to the world that needs to be identified. It is a path that unites the origin of the substructure to this world in the form of a sequence of (chosen) natural numbers.

[27] i.e. if $(n, m, \ldots)$ is an economic index of w in the substructure of w', there is a function that, when applied to $(n, m, \ldots)$, gives the objective index of w, and *vice versa*.

4.3.21.5. Some examples of indexing: An example of the second method is presented in figure 1.[28]

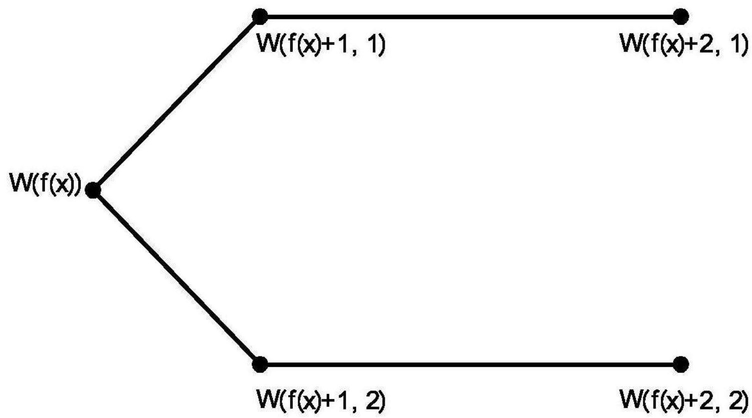

Figure 1.

$w(f(w), 1)$ is the origin, and $w(f(w)+1, 1)$, $w(f(w)+1, 2)$, $w(f(w)+2, 1)$, $w(f(w)+2, 2)$ are the rest of the worlds. This kind of numbering suggests that the worlds these indexes are attached to are: *the first world with rank* $f(w)$*; the first world with rank* $f(w)+1$ *and the second world with rank* $f(w)+1$*; the first world with rank* $f(w)+2$ *and the second world with rank* $f(w)+2$. This follows 4.3.21.2. But, on the other hand, the same worlds are also *the only possible* $w(f(w))$*; the first possible evolution of the only possible* $w(f(w))$ and *the second possible evolution of the only possible* $w(f(w))$*; the first possible evolution of the first possible evolution of the only possible* $w(f(w))$ and *the first possible evolution of the second possible evolution of the only possible* $w(f(w))$. This follows 4.3.21.3, and is depicted in figure 2.

[28] In the diagrams rank-numerals grow from right to left, and numerals ordering worlds with same rank grow from top down.

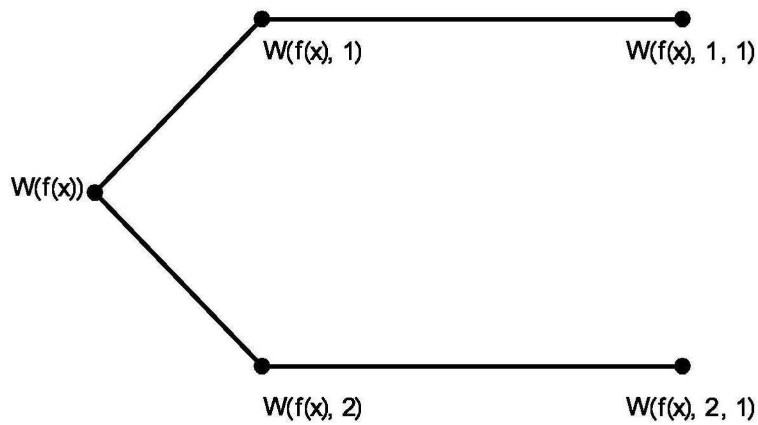

Figure 2.

Notice that by the second method a single world can have more than one legitimate *denotandum* (see figure 3).

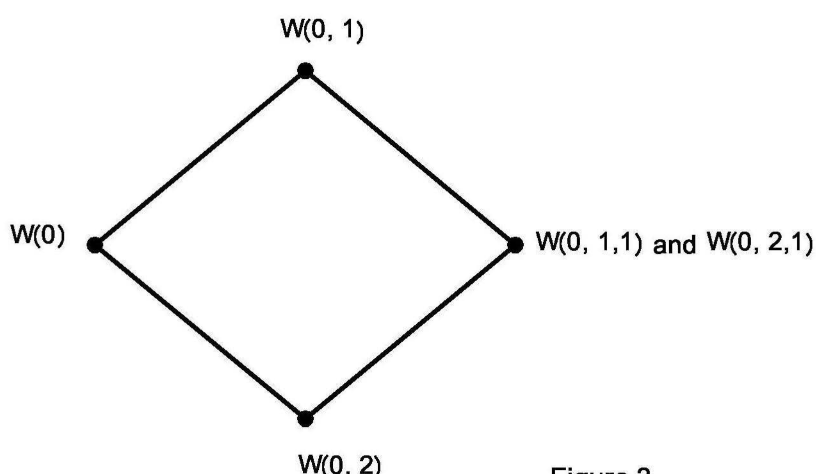

Figure 3.

It will never be, however, that two worlds can have the same *denotandum*, and it is this last possibility that would have been harmful.

4.3.22. Some examples of the functions of 1.19.1-4. We will apply them to the frame of figure 3. We adopt the method of 4.3.21.3.

$$\begin{aligned}
f(w(0)) &= 0,\\
f(w(0,1)) &= f(w(0,2)) = 1,\\
f(w(0,1,1)) &= f(w(0,2,1)) = 2.
\end{aligned}$$

$$\begin{aligned}
\iota(w(0)) &= 0,\\
\iota(w(0,1)) &= (0,1),\\
\iota(w(0,2)) &= (0,2),\\
&\vdots
\end{aligned}$$

$$\begin{aligned}
(\iota(w(0)))^o &= (0,1),\\
(\iota(w(0,1)))^o &= (1,1),\\
(\iota(w(0,2)))^o &= (1,2),\\
(\iota(w(0,1,1)))^o &= (2,1),\\
&\vdots
\end{aligned}$$

Observe that ι resembles the star function (1.19.2), although this latter has no semantical import. More precisely, if $(\ldots\vec{n}\ldots)$ identifies a world in the structure, $(w(\ldots\vec{n}\ldots))^* = \iota(w(\ldots\vec{n}\ldots))$. Notice also that $w(0,1,1) = w(0,2,1)$, which confirms (cf. 4.3.21.5) that if the second economic method is used ι is not functional. This as already said is not harmful because the converse of ι is functional; the domain of values of ι^{-1} consists of things (worlds), not expressions.[29] Observe, finally, an easy algorithm with which to calculate $f(w)$, when the second method of economic indexing is used. If the sequence identifying w has n numerals and begins with the numeral m, the rank of the w is $m+(n-1)$. This means that there is a decidable function g, such that $g(\iota(w)) = f(w)$.

4.3.23. From now on we will adopt the following helpful conventions. Since there are as many substructures as worlds, we will, instead of, "in the substructure having $w(n,m,r,\ldots)$ as origin, one identifies $w(n,m,r,\ldots)$ by its rank", simply say, "in the substructure having w$(n,m,r,\ldots)$ as origin, $w(n,m,r,\ldots)$ identifies itself by its rank".[30] This is harmless enough because in the rest of substructures it is not the world in question, which "makes the identification". In the above formulation "$(n,m,r,\ldots)$" is supposed to be an index for

[29] A delicate point concerns the indexing function according to the method of 4.3.21.3. The problem is that there might be more than one indexes identifying the same world according to this method. In order to overcome it, use the *fact* that all sets of different indexes identifying a single world can be well ordered and assume ι to identify the world by the index that comes first in the order. The *fact* that all sets of indexes identifying a single world can be well ordered is a corollary of the *fact* that there is no transfinite set of *paths* connecting two worlds, which, in turn, comes from there being no transfinite set of worlds having the same rank - cf. 4.2.

[30] Following the same stylistic idiolect we will often use the term "perspective" to denote the set of worlds that another world sees or is seen from - i.e. "w has w' under perspective" will mean that w sees w' or w' sees w. If not otherwise specified, the same will mean that w sees w' or is seen from w' in the substructure it is the origin of - i.e. the "perspective of w", also called "the visual field of w", will be the set of worlds it sees or is seen from in the substructure of which it is the origin.

the world in question, in some other substructure, and it is the origin of this "other substructure" that uses "$(n, m, r, \ldots)$", in order to identify this world. An apparent cost of our not using the "objective" indexing concerns ambiguities due to the plurality of origins. These ambiguities emerge at the time two or more substructures are juxtaposed. Consider, for example, what happens in figure 4.

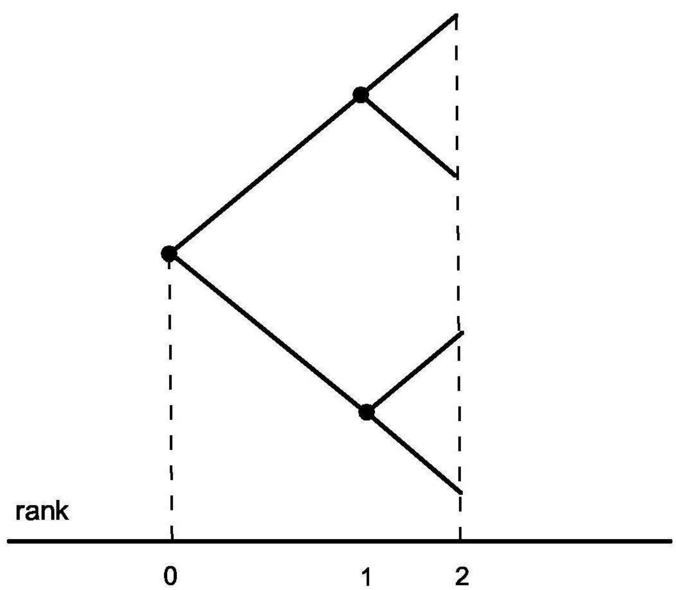

Figure 4.

To the substructure corresponding to the upper world with rank 1, "$w(1,1)$" denotes the first world from the top and on the right hand side. To the substructure corresponding to the lower world with rank 1, the same *denotandum* refers to the third world from the top and on the same side.[31] Now, in order to disambiguate things and instead of assuming an "objective" perspective one can just seek for the world, which both substructures share in common.[32] If we focus on the indexing corresponding to the substructure of this world any ambiguity disappears.[33] In that way ambiguities can always be decided on the spot without any need either to assume a common origin of all worlds or any need to use the "objective" indexing. Since there is a certain common past origin for any two worlds, any two worlds can rely on the "economic" indexing of this

[31]The example is more lengthy than necessary; even "$w(1)$" is ambiguous in the figure. Both worlds of rank 1 identify themselves by this expression.

[32]Because of 4.3.3.

[33]From the perspective of the first world on the left, the two $w(1,1)$ worlds are distinct. The upper is $w(0,1,1)$, while the lower is $w(0,2,1)$. Intuitively this says that despite the fact that -e.g.- from today's perspective the actual is the only possible "today", it has not stopped being the case that one day before, this very same day (i.e. the actual today), was one of many alternatives. The one fact does not contradict the other.

common, past origin. Figuratively speaking -and if we are to keep pace with the stylistic idiolect we have adopted for worlds- any two different worlds of an S-model can always find the means to "express themselves in an unambiguous manner", by "calling into the discussion" a world that both their substructures share in common. Now, since the model is infinite in both directions there will be in the pasts of any two distinct worlds an infinite number of worlds that belong to the substructures of them both: all the worlds on the semi-substructure coming to the origin of the substructure that has as origin the closest common past world. Out of economy, again, we will disambiguate things by the indexing of the substructure that corresponds to the closest common such world.[34]

4.3.24. The above observations can also be seen in connection with yet another characteristic of the model and this characteristic corresponds to the informal example of the driver in the unified road network that we used in chapter 3. If you remember, the driver could always identify all destinations as points that he either has, could, would, or will reach. 4.3.3 guarantees that the model is convergent in the backwards direction and that there are no separated frames in it. As a consequence, any two worlds of the model share a common past, and therefore any world "is aware" of which are the possible moments that are represented by the rest of the elements of the entire S-model, and so which are the elements that, in other substructures, have same indexes with worlds of its own substructure.

4.4. We now pass to the fourth and fifth element of the septuple: D and V.

4.4.1. First, we construct the set of atomic sentences (AS, henceforth). An introduction to this procedure took place in 3.4.

4.4.2. For the Sentential Calculus, one can skip D, and start directly with V and AS. In such case, V will be a function assigning a subset of AS to each element of W, and AS will be given beforehand. The relation from subsets of AS to worlds of the model is one to one.[35]

4.4.3. For the Predicate Calculus, one needs to be more analytic, but when this done one can consider the Sentential Calculus as built upon the Predicate Calculus.

4.4.3.1. First, assume

(i) U, the universe of individuals,

(ii) the set of primitive predicates letters, and

[34] If one postulates, in figure 4, a world of rank -1, the substructure corresponding to this world could also disambiguate the two "$w(1,1)$" of the previous example, but the indexes it uses are less economic than those of its successor.

[35] The last clause is for avoiding some kinds of circular Time. See Appendix III.

(iii) $\{\rightarrow, \vee, \&, \exists, \forall\}$ the set of primitive connectives. Assume also the constant $\perp$, called *falsum*. A well-formed atomic sentence is any formula

$$\phi^n(x_{(m)_1}, y_{(n)_2}, \ldots, z_{(s)_n}),$$

where

$$x_{(m)}, y_{(n)}, \ldots, z_{(s)}$$

belong to U and ϕ^n belongs to the primitive predicates. $\perp$ is also well-formed and atomic.

4.4.3.2. V assigns to each world w of the model and any $\phi^{n>0}$ belonging to the primitive connectives a subset of the n^{th} Cartesian product of the assignment of D with respect to w. More precisely, D assigns a (not necessarily the same) non-empty element $D(w) \in 2^U$, to all the worlds of W. Assume also that $\mathbb{N}$ is a subset of every $D(w)$.[36] V assigns to each $\phi^{n>0}$ and with respect to any w a subset of $(D(w) \times^{n-1} D(w))$, where $(D(w) \times^{n-1} D(w))$ is the n^{th} Cartesian Product of $D(w)$. V assigns also a (not necessarily the same) subset of $\{\phi^n : n = 0\}$ to all the worlds of the model.[37] Finally, V assigns $\perp$ to no world of the model, and V assigns the constant set of true causal conditionals to all the worlds of the model. In order to make life easier, we will also assume that V allows one to deduce the entire set of logico-mathematical truths. So, the *Ideal Scientist* is assumed to posses all of these at any time.

4.4.3.3. Truth-value assignments for atomic sentences are determined as follows:

a. for $\phi^{n=0}$, $\phi^{n=0}$ is true in w, [38] iff, $\phi^{n=0}$ belongs to the subset of $\{\phi^n : n = 0\}$ that is assigned from V to w.

b. $\perp$ is true in w, iff, V assigns $\perp$ to w.

c. a causal conditional is true in w, iff, V assigns this conditional to w.

d. $\phi^n(x_{(m)_1}, y_{(t)_2}, \ldots, z_{(s)_n})$ is true in w, iff,

$$\langle x_{(m)_1}, y_{(n)_2}, \ldots, z_{(t)_n} \rangle$$

belongs to

$$(D(w) \times^{n-1} D(w))$$

that is assigned from V to ϕ^n with respect to w.

[36] If you want numbers to be definable through other more primitive notions, take R as generet-*able* out of every $D(w)$. Observe that the one way or the other, individuals must be infinite in number. If numbers are primitive, they are trivially. If not, we need of an infinite number of individuals in order to reach the cardinality of $\mathbb{N}$. See also Appendix III.

[37] But for this see also note 8 and the controversy about whether atomic sentences without predicates are allowed by the atomistic project. Alternatively, $\{\phi^n : n = 0\}$ is empty, and atomic sentences are as in note 8.

[38] Formally: "$T(\phi^{n=0}, w)$". In general "α is true in w" is, formally, "$T(\alpha, w)$", and "α is false in w", "$F(\alpha, w)$". Notice that informal formulations use connectives of 1.22, and thereby belong to the classical metalanguage. The formal counterparts of them also belong to the metalanguage but they do not receive a classical interpretation. Cf. 5.3.

4.4.3.4. Non-atomic sentences are sentences built upon atomic sentences and primitive connectives, or any other expression introduced by definition, as -e.g.- the connective $\neg$, introduced by: $\neg a =_{\text{df.}} \alpha \rightarrow \bot$. The truth-value of non-atomic sentences can be calculated by the usual inductive definitions of the connectives. For these, the reader is referred to A. CHURCH, [1956]. Indicatively, we will deal with $\exists$ and $\forall$.

a. $\exists x(\phi^{n+1}(x_{(m)_1}, y_{(t)_2}, \ldots, z_{(s)_n}, x))$ is true in w, iff, there is a $y \in D(w)$, such that

$$(\phi^{n+1}(x_{(m)_1}, y_{(t)_2}, \ldots, z_{(s)_n}, y)) \quad \text{is true in } w.$$ [39]

b. $\forall x(\phi^{n+1}(x_{(m)_1}, y_{(t)_2}, \ldots, z_{(s)_n}, x))$ is true in w, iff, for all $y \in D(w)$,

$$(\phi^{n+1}(x_{(m)_1}, y_{(t)_2}, \ldots, z_{(s)_n}, y)) \quad \text{is true in } w.$$ [40]

4.4.3.5. Notice that up to this point there is nothing intuitionistic in our semantics. This is an all round classical machine. It is not only that$\rightarrow, \vee, \exists, \forall$ have received the standard interpretation; observe also that

$$\phi^n(x_{(m)_1}, y_{(t)_2}, \ldots, z_{(s)_n}) \quad \text{is never true in } w,$$

if some of its arguments do not belong to $D(w)$. On the other hand, neither is it ever truth-valueless. Both these things would have hurt the sensibility of some "classical" logicians and both these things would have been the case had we taken V to assign -with respect to ϕ^n and a world w- not an element of

$$(D(w) \times^{n-1} D(w)) \quad \text{but an element of} \quad (U \times^{n-1} U),$$ [41]

or had we left ϕ^n undefined for any argument not belonging to $(D(w) \times^{n-1} D(w))$. This is -up to now- a bone-to-bone classical apparatus.

4.5. The sixth element of the septuple is the so-called forcing relation: $\Vdash$.

4.5.1. The forcing relation determines inductively all value assignments within W as follows.

It is governed by the general condition:

w forces α, iff, 'α' is true in w. (Formally: $w \Vdash \alpha \leftrightarrow T(\alpha, w)$).[42]

By standard model theory $T(\alpha, w)$ is also equivalent with $\alpha \in w$, which implies that:

w forces α, iff, α belongs to w. (Formally: $w \Vdash \alpha \leftrightarrow \alpha \in w$).[43]

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[39] As for non-closed formulas, the condition is: $\exists x(\phi^{n+1}(y_1, y_2, \ldots, y_n, x))$ is true in w, relative to the assignment $\langle x(_m)_1, y(_t)_2, \ldots, z(_s)_n\rangle$ to the free variables, iff, there is an element $y \in D(w)$, such that $\phi^{n+1}(x(_m)_1, y(_t)_2, \ldots, z(_s)_n, y)$ is true in w

[40] *Mutatis mutandis*, the same as with the previous note.

[41] As we will do, in ch. 5, pp. 146 – 47.

[42] See note 38.

[43] See note 38.

4.5.1.1. We begin with the forcing relation for atomic sentences and sentences constructed upon atomic sentences and primitive connectives $\{\rightarrow, \vee, \&, \exists, \forall\}$. The apparatus for these is classical throughout. By 4.5.1, we get:

For α atomic, or for α constructed upon atomic sentences and primitive connectives, w forces α, iff, 'α' is true in w.

Since V has made the apparatus for these kinds of sentence classical, it follows that, if α is atomic, or a well-formed formula constructed upon atomic sentences and primitive connectives, then, if a world does not establish it, it establishes its negation and *vice versa*. In terms of worlds as sets of sentences we observe that for α atomic, or constructed upon atomic sentences and primitive connectives, either $\alpha \in w$, or $\neg\alpha \in w$. Worlds of the model are maximal and consistent as for these. No surprises so far. We have just added the correspondence between "being true in a world", "being forced in a world" and "belonging to a world".

4.5.1.2. We now turn our attention to some metalinguistic formulas that we shall call "indexed". These are introduced, by the following definition:

Definition 4. $a(w(\ldots(\vec{n})\ldots))^* =_{\text{df.}} w(\ldots(\vec{n})\ldots) \Vdash \alpha$

After the application of the star function, "$w(\ldots(\vec{n})\ldots)$" becomes "$(\ldots(\vec{n})\ldots)$". This is an open sentence specifying a condition for $\vec{n}$, which (condition) can vary from anything whatsoever (provided that it is extensional), to the plain $(\ldots = \vec{n})$, in which case, we simply write "$(\vec{n})$". Obviously, if the number of elements of $\vec{n}$ is 1, the condition is a condition upon a single number - e.g. (n_1) for the trivial case, or, for the non-trivial cases: $(\ldots > n_1)$, $(\ldots > n_1 < \ldots)$ etc. In case there are more elements in $\vec{n}$, the condition can also vary from the trivial recording of their sequence to any non-trivial extensional condition, such that: this is a continuation of $n_1, n_2, \ldots, n_m$, this is the continuation of $n_1, n_2, \ldots, n_m$, up to its $n + s^{th}$ element, etc.

Before presenting the truth-conditions for these we will agree to say that a sentence is, "indexed" when it is of the general form $(\alpha)(\ldots\vec{n}\ldots)$. An indexed sentence will be called a, "quantified indexed sentence", in case the entire $\vec{n}$, or some elements of it, are bound by a quantifier and, "non quantified/unquantified indexed sentence", in case none of its elements are - i.e. if the n-tuple is constituted out of constants, or out of free variables. Otherwise it will be called "non-indexed". Non-indexed sentences are -e.g.- all sentences falling under the general schema of the *definiens* of Def. 4, all sentences of 4.5.1.1, etc. Finally, we agree to say that a sentence is "index-free", if no temporal index appears in it - e.g. sentences of 4.5.1.1 are index-free.

The semantic conditions for indexed sentences are as follows:

1. w forces $\alpha(\ldots(\vec{n})\ldots)$, iff, there is a bar b for w, such that for every $w' \in b, w' \Vdash \alpha(\ldots(\vec{n})\ldots)$.
2. w forces $\alpha(\ldots(\vec{n})\ldots) \vee \beta(\ldots(\vec{m})\ldots)$, iff, there is a bar b for w, such that for every $w' \in b$, either $w' \Vdash \alpha(\ldots(\vec{n})\ldots)$, or $w' \Vdash \beta(\ldots(\vec{m})\ldots)$.
3. w forces $\alpha(\ldots(\vec{n})\ldots) \rightarrow \beta(\ldots(\vec{m})\ldots)$, iff, for all worlds belonging to $W(w)$, if $\alpha(\ldots(\vec{n})\ldots)$ is forced, so is $\beta(\ldots(\vec{m})\ldots)$ - i.e. it is never the case that $\beta(\ldots(\vec{m})\ldots)$ is not forced, when $\alpha(\ldots(\vec{n})\ldots)$ is.
4. $w \Vdash \alpha(\ldots(\vec{n})\ldots) \leftrightarrow \beta(\ldots(\vec{m})\ldots) =_{\text{df.}} w \Vdash \alpha(\ldots(\vec{n})\ldots) \rightarrow \beta(\ldots(\vec{m})\ldots)$ and $w \Vdash \beta(\ldots(\vec{m})\ldots) \rightarrow \alpha(\ldots(\vec{n})\ldots)$
5. w forces $\alpha(\ldots(\vec{n})\ldots)\&\beta(\ldots(\vec{m})\ldots)$,
$$\text{iff, } w \Vdash \alpha(\ldots(\vec{n})\ldots) \ \text{ and } \ w \Vdash \beta(\ldots(\vec{m})\ldots).$$
6. $\neg((\alpha)(\ldots(\vec{n})\ldots)) =_{\text{df.}} \alpha(\ldots(\vec{n})\ldots) \rightarrow \perp$
7. w forces $\exists(m)\alpha(\ldots(\vec{n},m)\ldots)$, iff, there is a bar b for w, such that for every $w' \in b$, there is an r, such that $w' \Vdash \alpha(\ldots(\vec{n},r)\ldots)$.
8. w forces $\forall(m)\alpha(\ldots(\vec{n},m)\ldots)$, iff, for any r, $w \Vdash \alpha(\ldots(\vec{n},r)\ldots)$.[44]

Temporal operators are introduced as follows:

9. $w \Vdash F\alpha =_{\text{df.}} w \Vdash \alpha(n > f(w))$
10. $w \Vdash P\alpha =_{\text{df.}} w \Vdash \alpha(n < f(w))$
11. $w \Vdash Fn\alpha =_{\text{df.}} w \Vdash \alpha(f(w) + n)$
12. $w \Vdash Pn\alpha =_{\text{df.}} w \Vdash \alpha(f(w) - n)$
13. $(\text{now} +/- n) \Vdash \alpha =_{\text{df.}} F/P(n)\alpha$ [45]
14. $w \Vdash G\alpha =_{\text{df.}} w \Vdash (\forall n)(Fn\alpha)$
15. $w \Vdash H\alpha =_{\text{df.}} w \Vdash (\forall n)(Pn\alpha)$

Finally, the modal cases:

16. w forces $\Box\alpha$, iff, for all w', such that wRw' or $w'Rw$, $w' \Vdash \alpha$.
17. w forces $\Diamond\alpha$, iff, for some w', such that wRw' or $w'Rw$, $w' \Vdash \alpha$.
18. $w \Vdash \triangle\alpha =_{\text{df.}} \Diamond\alpha\&\Diamond\neg\alpha$

[44](7) and (8) can be made contain free variables as well. See notes 39, 40.

[45]In the case of $n = 0$, we write simply *now* $\Vdash \alpha$. This introduction of "*now*" is purely formalistic. For its "philosophical" background, see 4.6. Obviously, "*now* $+/- 1, 2, \ldots$" aims at formalizing other tensed expressions like "the next/previous moment", etc. Notice that $+/-$ within the expressions "*now* $+/- 1, 2, \ldots$" are no arithmetical operators. The resulting expressions are not numbers but definite descriptions of the A-series.

4.5.1.3. The evaluation of compound sentences containing both indexed and non-indexed sentences is to be determined as follows. First let us enumerate three distinct categories of sentences:

1. Sentences that either are, or can be turned into -by an application of a definition or equivalence- sentences of the form:
$$\alpha(\ldots\vec{n}\ldots),\ \alpha(\ldots(\vec{n})\ldots)\vee\beta(\ldots(\vec{m})\ldots),$$
$$\alpha(\ldots(\vec{n})\ldots)\rightarrow\beta(\ldots(\vec{m})\ldots),$$
$$\alpha(\ldots(\vec{n})\ldots)\ \&\ \beta(\ldots(\vec{m})\ldots),$$
$$\exists(m)\alpha(\ldots(\vec{n},m)\ldots),\forall(m)\ \alpha(\ldots(\vec{n},m)\ldots).^{46}$$
In most cases, this can be done by an application of Def.4 and the definitions within 4.5.1.2, or the equivalence between "$T(\alpha,w)$" and "$w \Vdash \alpha$". For example:
$$\text{“}w(\ldots\vec{n}\ldots)\Vdash\alpha\text{” is to be turned into}$$
$$\text{“}\alpha(\ldots\vec{n}\ldots)\text{” (by Def.4)},$$
$$\text{“}now\Vdash\alpha\text{” is to be turned into}$$
$$\text{“}\alpha(f(now))\text{” (by 4.5.1.2.(13/11))},^{47}$$
"$T(\alpha, now)$" is also to be turned into "$\alpha(f(now))$" (by its equivalence with "$now \Vdash \alpha$"), and so on so forth.

2. Sentences of which the main connective belongs to 1.22 and are not under the immediate[48] scope of an index. Examples of such sentences are all pairs of sentences before and after the "iff" connective in 4.5.1.2.(1-3, 5, 7-8, 16-17).

3. Sentences that belong to 4.5.1.1 -i.e. either atomic sentences or sentences constructed upon atomic sentences and primitive connectives- and are not under the immediate scope of an index.

We now pose that in order to evaluate any compound sentence in which both indexed and non-indexed subsentences occur, we do the following: for sentences falling under 1, we turn them into the required form, by the appropriate definitions or equivalences. For sentences falling under 2 and 3,[49]

[46] Observe that the truth-conditions for these are stated in 4.5.1.2.(1-3, 5, 7-8).

[47] Notice that $f(now)$ is here a value of a function that has $\mathbb{N}(+/-)$ as domain of values - $f(now)$ denotes a number. This is crucial, for, if one is not careful, one can easily mistake '$f(now)$' with 'now', and face the problems of tensed sentences, as presented in note 51.

[48] "Immediate" means here: without any "interference" of any other connective. For example α in $((\alpha \text{ or } \gamma)\&\beta(n))(m)$ does not satisfy the "immediate" condition, since, although (α or γ) is within the scope of m, & interferes, since it is under the scope of m and it has (α or γ) under its own scope.

[49] *In extremis*, these two latter categories are but one, for we assume that sentences falling under 2 are truth-functionally dependent upon the atomistic calculus, but this cannot be made explicit, because of the present unaccomplished state of the atomistic analysis. (Cf. ch. 3, pp. $35-39$, and ch. 4; 5.3.) See also note 53. The intuition that (2) is reducible to (3) could also be justified as follows: The topology underlying the partial order resulting from the $\Vdash$ relation is classical. Therefore, it respects all classical topological theorems, and according to 4.4.3.2, these are all deducible by the assignments of V. Now, sentences belonging to (2) either describe the topology of the points of model (e.g. "there is a set of worlds, such that ..."), or they describe, in a classical way, the sets of sentences assigned to these points.

we proceed as follows. Let us first, agree to say that if α is a sentence of w, $\alpha(\iota(w))$ is its *indexified* form. So, for sentences falling under 2 or 3 we substitute them to their indexified forms.[50] After this, the evaluation proceeds according to 4.5.1.2.(1). For example,

$$\text{if } (w(\ldots(\vec{n})\ldots) \Vdash \alpha) \vee \beta \vee (w' \Vdash \alpha \text{ and } w' \Vdash \beta)$$

is a sentence of w, it will be evaluated according to

$$(\alpha(\ldots(\vec{n})\ldots)) \vee \beta(\iota(w)) \vee ((\alpha(\iota(w')) \text{ and } \beta(\iota(w')))(\iota(w))).$$

The transformation of the first subsentence changes nothing, but the transformation of the others changes a lot, for it allows us to use 4.5.1.2 for evaluating the entire formula.[51]

4.5.2. Here we specify the semantics for the square brackets (cf. 1.21).

As one can easily verify, V assigns to all the worlds of the model all necessary material for calculating the truth-value of all atomic sentences, causal conditionals, and of sentences that are built upon atomic sentences and primitive connectives.[52] This calculation is purely algorithmic. As for the rest of sentences, 4.5.1.2 makes it clear that some knowledge about what happens in other worlds of the model is necessary and so the procedure of evaluation stops being reducible to truth-tables; some parameters concerning worlds other than the origin step forward. At this point, an obvious question arises: which, exactly, are these other worlds? The answer is equally obvious: they are the worlds located upon paths passing through the world that the sentence to be evaluated belongs to. Now, an elementary topological calculation establishes that these are exactly the worlds that either have access to this latter or are accessible from it. Which means that they are the elements of its substructure and these elements alone. More formally: take w to be any world of the model. The worlds needed for the calculation of the truth-value of the indexed sentences of w are the elements of the set $\{w' : wRw' \text{ or } w'Rw\}$. The world needed for the calculation of the rest of sentences (i.e. sentences falling under 4.5.1.1, causal conditionals and sentences falling under 4.5.1.3.(2))[53] is w itself. Therefore, for any w, $W(w)$ suffices for the calculation of the truth-value of any sentence of w. Now, one is here faced with the following ambiguity. We have said that the

[50] This operation necessitates that we are always in position to tell which is the world they belong to. As for why this is always possible, see 4.6.

[51] The *rationale* underneath this technique is obvious: we do not want -e.g.- "It either rains, or there will be a Sea-Battle at n" to come out true, if (i) there is no Sea-Battle at n, (ii) it does not rain now, but (iii) "it rains" is true in some future bar. Sentences falling under 2 have to suffer the same transformation, for exactly the same reason that the pseudo-atomic "it rains" has. Natural language connectives receive a non-intuitionistic interpretation, and so the application of the intuitionistic bar-criterion would lead to contradictions. Once indexified, no problem remains. Cf. 4.5.3.3 and 5.3.

[52] According to the assumption of note 49, to these we have to include also all sentences falling under 4.5.1.3.(2). This might seem strange, since in these latter we encounter connectives that do not belong to the vocabulary of 4.5.1.1. The idea is that one has to treat all indexified sentences of 4.5.3.(2) as atomic. When one does, 4.5.1.3 makes it clear that for their evaluation no world is needed other than w itself.

[53] See notes 49 and 52.

worlds needed for the valuations of w are the elements of $W(w)$. This is ambiguous for the following reason: as we have seen during the exposition of R (cf. 4.3.19), substructures might be considered as changing according to the occasional perspective. Therefore, "the substructure corresponding to w" might be different from the perspective of w itself, and different from the perspective of a world w', such that wRw' or such that $w'Rw$. According to the generalized notation mentioned in 4.3.19, "$W(w_1)(w_2)\dots(w_n)(w)$" is the substructure corresponding to w, in the sub$_n$-sub$_{n-1}$ ... sub$_1$structure $W(w_1)(w_2)\dots(w_n)$. Obviously, $W(w)$ is the substructure corresponding to w, from the perspective of w itself. It is equally obvious that the maximal number of paths passing through w passes through w, when the perspective is from w itself. A straightforward consequence of this is that $W(w)$ contains (not necessarily properly)[54] all other $W(w)(w_1)(w_2)\dots(w_n)$ structures. The question generating the ambiguity is the following: let w' be a world, and assume that $w'Rw$. In order to construct the valuation of w' from the perspective of w, does one have to take $W(w')$ under account, or does one have to take $W(w')(w)$ under account? Both options have advantages and disadvantages and in order to keep the advantages of both we will assume a combined approach.
The notation "$w \Vdash (w' \Vdash \alpha)$" will mean that w forces that w' forces α, after having taken into account the *constant* valuations of the origins constituting the rest of worlds within $W(w)$ - i.e. on the occasion, it evaluates $w' \Vdash \alpha$, according to $\langle W(w'), f, R, D, V, \Vdash\rangle$, and the notation "$w \Vdash [w' \Vdash \alpha]$" will mean that w forces that w' forces α, after having taken into account the valuations of the origins constituting the rest of worlds of $W(w)$, but as these latter have possibly been narrowed down by $W(w)$ - i.e. on the occasion, it evaluates $w' \Vdash \alpha$, on the basis of $\langle W(w')(w), f, R, D, V, \Vdash\rangle$.[55]

The disadvantages of each method.
Say that

(i) there is a Sea-Battle in w

(ii) $w'Rw$

(iii) $f(w) - f(w') = n$

(iv) the Sea-Battle at n is not settled from the perspective of w'.

According to the first method, to express that w' has been succeeded by a moment involving a Sea-Battle is impossible, if it has to be expressed by a sentence belonging to w. The formal statement corresponding to the above claim is of the general form: $PnFn\alpha$. Take α to stand for the Battle. '$PnFn\alpha$' is, according to the first method, untrue in w, because, since the valuation of $W(w')$ is constant, it will never validate $Fn\alpha$. On the other hand, and according to the second method, to say, from the perspective of w, that the advent of the Sea-Battle at $f(w)$ is not settled in w' is equally impossible, since there is, in

[54] e.g. in linear Histories it does not.

[55] By an obvious application of the ι function, we obtain the symmetrical notations: $(\alpha(\iota(w')))(\iota(w))$, and $[\alpha(\iota(w'))](\iota(w))$, for $w \Vdash (w' \Vdash \alpha)$ and $w \Vdash [w' \Vdash \alpha]$ respectively. Notice, also, that "$w \Vdash [\alpha]$", for α index-free is a harmless notational superfluous, since $(w' \Vdash [w \Vdash \alpha])$, for α index-free, will be constant for all w' and is extensionally the same with $(w' \Vdash (w \Vdash \alpha))$.

$W(w')(w)$, no path, where no Sea-Battle happens at $f(w)$. As time has went by some branches have been cut off.

Each among these two methods has the following advantage: neither suffers from the specific form of expressive penury that the other does.[56] One can, nonetheless, turn this asymmetry into an overall virtue, by using each method according to what one wants to express on the occasion. In the previous example, the appropriate formal statement that belongs to w and expresses the fact that w' is followed by a world with a Sea-Battle, but that this event is not settled at $f(w')$ is:

$$w \Vdash [PnFn\alpha] \ \& \ w \Vdash \neg(PnFn\alpha)$$

4.5.3. Metalanguage.

4.5.3.1. If sentences are "true", "false", "forced", "established", "long", "short", etc., metalinguistic statements are obtained by sentences that predicate of sentences truth, falsity, shortness, etc. In standard intuitionistic models, the metalanguage is external to the model, and the above clarification suffices to determine the set of sentences belonging to the metalanguage. First level sentences belong to the worlds of the model, while higher level sentences belong to the (external) metalanguage. In S-models things are more complicated, since metalinguistic statements are themselves sentences belonging to the worlds of the model. This complicates matters for the following reason. Say that $w' \Vdash \alpha$ belongs to w', and that w sees w'. "$w' \Vdash \alpha$" is clearly metalinguistic since it is about α. Now, suppose that w' bars w. If so, w forces $w' \Vdash \alpha$,[57] and, so, $w' \Vdash \alpha$ belongs to w. The question is: does this metalinguistic statement belong to the true sentences of the metalanguage of w?

Intuitively, we pose here[58] that a sentence belongs to the true sentences of the metalanguage of a world, if, and only if, (i) it belongs to this world, and (ii) it predicates something of a sentence belonging to this world.

Consequently, "$w' \Vdash \alpha$" above does not belong to the metalanguage of w, even if it belongs to w, because it predicates something of a sentence belonging to w', but not to w.[59] This is an important element of our system. It formally illustrates what we will subsequently mean by saying that, "in an S-model worlds have access to the metalanguage of *other* worlds". By saying that w has access to the metalanguage of w', we mean that w can contain sentences of the form $w' \Vdash \alpha, T(\alpha, w'), F(\alpha, w')$, etc. This remark is important for another reason as well. Usually, and when the metalanguage is external to the model one can use a simple device to calculate the level of a metalinguistic statement. The statement is of level n, if it contains $n+1$ inter-contained metalinguistic predications. For example, $T_n(F_{n-1}(\dots(T_1 p)))\&T_1(q)$ is of the above level. In S-models the same

[56] We let the reader to test this.

[57] Cf. 4.5.1.2.(1).

[58] For the formal formulation of this restriction see *infra*.

[59] α itself might belong to w as well, but this would be accidental, it can neither be inferred nor disproved by "$w' \Vdash \alpha$". See *infra*.

method will not do because the presence of such sequences of inter-contained metalinguistic predications is not necessarily constituted out of metalinguistic statements of the same world. For example: $T(T(\alpha, w'), w)$ is a metalinguistic statement of the world w, but belongs to the second linguistic level. Remember that $T(\alpha, w')$ is not a metalinguistic statement of w, and that we never assume an overall (external) metalanguage for S-models. All metalanguages are internal and therefore they are always metalanguages of this, or that, world of W.[60]

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To begin with, let us assume that the only metalinguistic predicates are "... is true", "... is false", "... is forced". In order to exclude any irrelevant occurrences, we pose the following:[61]

As for w:

(i) If α is of level n, then, $T(\alpha, w), F(\alpha, w)$, and $w \Vdash \alpha$ are of level $n+1$.

(ii) If $\alpha \otimes \beta \bullet \gamma \oplus \ldots \copyright \delta$ is well-formed, and $\otimes, \bullet, \oplus, \copyright$ are connectives, but not $\Vdash$, then the level of the entire formula is equal to the maximal level of its sub-formulas.

(iii) A formula cannot be of a level higher than 1, unless it belongs to the schemata $T(\ \ldots, w), F(\ldots, w), w \Vdash \ldots$[62]

We will also say that a formula of w is "genuinely of level n" if and only if the formula belongs to the schemata, $T(\ldots, w), F(\ldots, w), w \Vdash \ldots$, and its level is n.[63]

In that way one can avoid all tricky cases where occurrences of metalinguistic predicates and connectives do not contribute to the elevation of the level, such as the second "$\Vdash$" within $w \Vdash (w' \Vdash \alpha)$, or even, "$T$" within $w \Vdash T(\alpha, w')$. In this last example it is quite evident why it is that "T" does not (must not) be involved in the calculation of the level. The truth-value of $T(\alpha, w')$ within $w \Vdash T(\alpha, w')$ does in no sense depend upon the truth of α in w. 'α' might be true in w, as it might also be false in w, but this has nothing to do with what "$T(\alpha, w')$" is within $w \Vdash T(\alpha, w')$.

4.5.3.2. Since there are in W as many substructures as worlds the set of valuations contained in an S-model is the set of valuations assigned to each and every world of the model *qua* the origin of the substructure of which it is the origin. Now, 4.5.1.2 makes it clear that the valuations of these substructures

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[60] See also 4.6.

[61] The same will obtain for any other metalinguistic predicate that we introduce along the way.

[62] The strength of the rules diminishes on the way down - e.g. (ii) is more potent than (iii). This means that there can be sentences of the form $\alpha \otimes \beta \bullet \gamma \oplus \ldots \copyright \delta$, which are of level$> 1$, but make no part of the schemata in (iii). It suffices that one of $\alpha, \beta, \gamma, \ldots, \delta$ does. Notice also that from these rules it follows that any formula containing neither T, nor F, nor $\Vdash$ is of level 1.

[63] e.g. "$T(\alpha, w)\&\beta$" is of level 2, but not genuinely.

depend upon one another. This feature renders the underlying logic a form of supervaluation logics. A "supervaluation logic" is a logic that makes the valuations of a certain Interpretation[64] (called "supervaluation") depend upon the valuations of a class of other Interpretations. In Van Fraaseen's supervaluation[65] (see chapter 5), this latter class is the set of classical valuations. In Thomason[66] (see chapter 5), it is the class of all linear Histories. In our system, however, it is a class of supervaluations. More precisely, the supervaluation of the origin w within the substructure $W(w)$ is based upon the supervaluations of all worlds within $W(w)$. So a distinctive feature of the present supervaluation is that it is based upon supervaluations and not upon simple (classical) valuations, as in Van Fraassen and Thomason.[67]
In more concrete terms, if $\alpha(n)$ is forced throughout a bar with respect to a certain origin w, and consequently, is forced in w as well, this is because it makes part of the supervaluations of the substructures corresponding to the worlds of the bar. Moreover, and when the rank of w is different from n, $\alpha(n)$ is forced upon the origin w exactly because this latter has access to the metalanguage of (other) worlds. For consider it this way: $\alpha(n)$ belongs to all the worlds of a certain bar with respect to a certain origin w. Now, these worlds either are of rank n or are not of rank n. If they are of rank n, $\alpha(n)$ is a sentence of their own metalanguage, and consequently, the origin w has access to the metalanguage of these worlds. If they are not, they, in turn, validate $\alpha(n)$, by virtue of some other bar, where $\alpha(n)$ is forced, and so on so forth, until one reaches worlds of rank n. Therefore, the very possibility for any sentence of the form $\alpha(n)$ to be forced in any world of the S-model with rank different from n is due to the fact that any world of the model "has access to the metalanguages" of other worlds of the model.

4.5.3.3. Unlike the rest of the supervaluation systems, this model is in a position to express the fact that a sentence is truth-valueless without needing a different semantic definition for negation in the metalanguage. The negation for indexed sentences follows the same general condition (i.e. 4.5.1.2.(6)), no matter whether the sentence belongs to the metalanguage or not. Normally, in supervaluated systems, it is impossible to express the fact that a sentence is truth-valueless without adopting a classical negation for the metalanguage. We will say more about how negation works in the "classical" supervaluation models during the following chapter, where a comparison between our supervaluation technique and that of Van Fraassen-Thomason will take place. For the time being, we will just show how, in an S-model, a unitary semantics for negation can allow one to say that a sentence is truth-valueless.

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[64] Here we use the term in the technical sense of model theory.

[65] B. VAN FRAASSEN, [1966], esp. pp. 493-495, [1968].

[66] R. THOMASON, [1970], [1984].

[67] Observe also that we define no valuation corresponding to the S-model in general. Each valuation is a valuation of a certain substructure of it. This formally corresponds to our intention not to accept any atemporal nature in things, nor any atemporal standpoint. See also 4.6, and chapter 5.

The reason is plain. We will first illustrate it by an example. Consider a submodel following the conditions:

(i) wRw', wRw'',

(ii) $w' \Vdash \alpha, w'' \Vdash \neg\alpha$,

(iii) $f(w') = f(w'')$, and

(iv) $\{w', w''\}$ is a bar for w.

If so, and according to 4.5.1.2.(6), w neither forces $w(f(w')) \Vdash \alpha$, nor does it force $w(f(w')) \Vdash \neg\alpha$. This is equivalent to saying that neither $T(\alpha, w(f(w')))$, nor $T(\neg\alpha, w(f(w')))$ belong to w. All four sentences are truth-valueless there. However, the metalinguistic statements:

$$T(T(\alpha, w(f(w')), w)) \text{ and } T(T(\neg\alpha, w(f(w')), w)) \text{ are not.}$$

When indexified, they become:

$$(\alpha(f(w')))(\iota(w)) \text{ and } ((\neg\alpha)(f(w')))(\iota(w)),$$

and, as one can readily see, their negation obtains for both of them at the origin w. None of them belongs to any world of the visual field of w,[68] and therefore

$$\text{both } (\alpha(f(w')))(\iota(w)) \to \bot \text{ and } ((\neg\alpha)(f(w')))(\iota(w)) \to \bot \text{ belong to } w.$$

This means that whereas $T(\alpha, f(w'))$ might, as Time goes by, obtain a truth-value and become true/false, $T(T(\alpha, f(w')), w)$ cannot loose or change the truth-value it has in w; it will remain false throughout. This is because even if it is possible that there is no bar for either $T(\alpha, f(w'))$ or $T(\neg\alpha, f(w'))$ with respect to w, it is not possible that there is no bar for either, "There is a bar for $T(\alpha, f(w'))$ with respect to w" or, "There is no bar for $T(\alpha, f(w'))$ with respect to w".[69] For one of them there has to be a bar which is none other than w itself. At this point it is important to observe that all sentences belonging to the metalanguage of w behave exactly like sentences belonging to 4.5.1.1; i.e. for α belonging to the metalanguage not only is it the case that $w \Vdash \alpha$, if, and only if, $T(\alpha, w)$, but, moreover, if w does not establish α, it establishes $\neg\alpha$, and *vice versa*.[70] So there can be no gaps in the metalanguage of w. More technically, one could say that the Kripke Schema obtains in our internalized metalanguage.[71]

[68] w itself sees to that because it blocks the (future) generation of any bar having only $(\alpha(f(w')))(\iota(w))$ or having only $((\neg\alpha)(f(w')))(\iota(w))$, at the level of $f(w)$.

[69] Notice that in the quoted sentences we use natural language counterparts of the existential quantifier. This is crucial; see *infra*.

[70] We remind the reader that $T(\alpha, w')$, although a metalinguistic statement, is no metalinguistic statement of w. As such it can be truth-valueless. If fact, by 4.5.1.2.(6), it follows that, if $T(\alpha, w')$ is truth-valueless for w, so would $(T_n(T_{n-1} \ldots (T_1(\alpha, w') \ldots)))$ be truth-valueless.

[71] The Kripke Schema keeps track of whether or not a propositional function has a value in a world of the model. It does so by assigning the value 1 to the function if it is established, and the value 0 otherwise. Each origin assigns one of these two values to the propositional function through a metalinguistic statement. For the Kripke Schema see -e.g.- D. VAN DALEN, [1986a], pp. 324-25; [2001], p. 251.

The above result is a formal consequence of two things: first, of our having chosen -in 4.5.2- to assume that the substructure corresponding to w (i.e. $W(w)$) remains constant throughout the entire S-model, and, second, of our having imposed a classical interpretation for the elements of 1.22.[72] Had we chosen to follow the alternative options we could not have obtained the classical behavior of the intuitionistic $\neg$ for the metalanguage. We will illustrate this in what follows.

First, assume that the substructure corresponding to w is, from the perspective of w', not $W(w)$, but $W(w)(w')$.

In other words what about $[(\alpha(f(w')))(\iota(w))]\iota(w')$?

If α obtains in w', $[(\alpha(f(w')))(\iota(w))]\iota(w')$ obtains as well, for, in $W(w)(w')$, Time's march has cut down some of the branches of $W(w)$; namely, it has cut down all branches where $\neg(\alpha(f(w'))$ obtains. More explicitly, the reason is that after the elimination of all branches having $\neg(\alpha(f(w'))$, there exists a past bar for w' at the level of $f(w)$, and upon this bar $\alpha(f(w'))$ obtains throughout. This is the singleton bar w. In general, as paths narrow down, some sets of points that do not bar the present can, after some interval, bar the (actual) present. If our semantics were to take these into account, not only, whether or not $\alpha(f(w'))$ is the case, would have been non-decidable, but so would have been, whether or not $T(\alpha(f(w')), w)$ is the case. Moreover, any attempt to formally state that $\alpha(f(w'))$ is truth-valueless in w would have been futile. The option to state it by

$$\neg(T(\alpha,(f(w'))),w) \ \&\ \neg(F(\alpha,(f(w'))),w) \text{ is condemned,}$$

because, if $\alpha(f(w'))$ is truth-valueless, both subsentences of the above formulation would have been truth-valueless, and, consequently, by 4.5.1.2.(5),

$$\neg(T(\alpha,(f(w'))),w) \ \&\ \neg(F(\alpha,(f(w'))),w)$$

would not have been true. According to our method,

$$\neg(T(\alpha,(f(w'))),w) \ \&\ \neg(F(\alpha,(f(w'))),w)$$

is true in w, exactly because, by taking $W(w)$ to be constant, we exclude any third, in-between "There is a bar for $\alpha(f(w'))$, with respect to w" and "There is no bar for $\alpha(f(w'))$, with respect to w".

Secondly, if we had chosen to provide an intuitionistic interpretation for quantification (cf. 4.5.1.2(7)), and with respect to the two sentences above, they would have been truth-valueless as well,[73] rendering, again, $\neg(T(\alpha(f(w'))),w)$ truth-valueless.

[72] These two assumptions have the same results as Van Fraassen's adopting classical negation for the metalanguage, except that, for our part, we do not impose classical negation to the metalanguage - we still interpret $\neg$, according to 4.5.1.2.(6).

[73] They would have been such, in case in some $W(w)\ldots(w')$, substructures w is bared for $\alpha(f(w'))$, while, in others, it is bared for $\neg(\alpha(f(w')))$. More precisely, the (classical) quantification describing the topology of the model is supposed to use the ordinary language connectives of 1.22. Which implies that when these quantified statements are formalized they stop behaving classically. For example, "There is a set of worlds that bars w, with respect to the sentence α" is supposed to be decidable *in situ*, and relying on the constant $W(w)$. On the other hand, its formal translation follows 4.5.1.2.(7-8), and depends upon all $W(w)\ldots(w')$ substructures of all worlds w', such that wRw' or $w'Rw$.

4.5.3.3.1. The underlying intuition motivating our decision to assign to the worlds substructures that do not depend upon the perspective, and to assume a classical quantification for the metalanguage, is the same general intuition that concerns the present's determinateness; for whether or not the conditions of a future/past state are fulfilled at present concerns the present, not the future or the past. Which implies that whereas some future/past states might be non-decidable, it has to be decidable whether they are decidable or not - i.e. the present in itself is to be taken as fully determined. In topological terms, this means that whether or not some set of points bars an origin of the model is something that is always decidable on the origin; the topology underlying the model is classical.[74]

4.6. The final element of the septuple is the higher order function "now", which assigns to one specific element of W the higher order object "*now*". It is accompanied by the condition:

$$\iota(\text{now}(W)) = f(\text{now}(W))^{75}$$

From this condition it follows that the function "now" assigns the object "*now*" to the origin of a substructure[76] (intuitively to the occasional *present moment*). Observe also that, by the *fundamental characteristic*, one straightforwardly deduces that:

$$(w \Vdash \alpha) \leftrightarrow (w \Vdash (now \ \Vdash \ \alpha))$$

From this point onwards we will also assume the abbreviation

Definition 5. $T(\alpha) =_{\text{df.}} T(a, now)$[77]

General remark about the function "now": In principle we can dispense with this seventh element, but, if so, one has to abandon any ambition to reach A-series sentences. All true sentences will be of the form $w(n, m) \Vdash \alpha$; and

[74] We do not mean by that that the underlying topology does not generate a non-classical logic. We mean that the criteria of this non-classical logic are stated in a classical way. In order to take an example from the *stricto sensu* topological models of intuitionistic logic, we mean that whereas "$\|\|\|p\| \cup \|\neg p\|\|\|$" might sometimes not be equal to the topological space, one should not evaluate "$\|\|\|\|p\| \cup \|\neg p\|\|\| =$ the topological space $\|$" by the same (intuitionistic) standards

[75] "now(W)" denotes the world the function has assigned the object "*now*" to. For a possible ambiguity of "$\iota(\text{now}(W))$" see note 29.

[76] Remember that, according to the method of indexing followed here (cf. 4.3.21.(3)), the only world that has as identifying index its own rank is the origin.

[77] This definition has to be treated with some extra caution, because, if not, it might lead to confusions such as the following. Say that the future converges at $m > n$ with respect to an event α at n=*now*. Say, also, that α obtains at present. By Def. 5, $\neg(T(\alpha(n)))$ belongs to a future bar with respect to the present. On the occasion this point is m: the point of the convergence. Here we need to use 4.5.1.3 and turn "$T(\alpha(n))$" within $\neg(T(\alpha(n)))$ into indexed, before applying the bar criterion. Namely, it has to be turned into $(T(\alpha(n)))\iota(now)$, and, then into $(\alpha(n))(\iota(now))$; otherwise we would run into the contradictory: $now \Vdash (T(\alpha(n))\ \&\ \neg T(\alpha(n)))$. The same obtains as for the point of the convergence. This point sees a past bar where "'α at n' is not truth-valueless" is true - i.e. it sees a bar, where $\neg(\neg T(\alpha(n))\ \&\ \neg F(\alpha(n)))$ obtains throughout. However, this sentence contains non-indexed subparts that need to be turned into indexed by Def. 5, so as to prevent the contradictory "'α at n' is and is not truth-valueless" at m.

this will be so even if α involves (tensed) expressions like "*now*", "Fn", "*now* $+/-n$" etc. Moreover, the index "(n, m)" will have to follow the "objective" indexing technique of 4.3.20. For without any specification of the "actual" origin, the "economic" indexing techniques are not applicable. One could only arrive at assertions of the form: "if $w(n, m) =$*now*, then, ... is the appropriate economic indexing", but without the function "now", one can never arrive at assertions like, "... *is* the correct economic indexing". Intuitively, the presence of this function corresponds to the fact that the *Ideal Scientist* is constantly in a position to identify a specific world of the model as being *now* - i.e. as being his actual present.[78] And he is able to do that because he is constantly aware of the truth-value of any atomic (tensed) sentence. Besides, this is supposed to be the reason why he is always able to indexify these sentences (cf. 4.5.1.3). He constantly knows which one among the worlds of the model is the (occasional) origin. Needless to say that this copes well with our intention to -so to speak- "manufacture" the model from within a world of the model itself. The *Ideal Scientist* is meant to assign the object *now* to a certain world of the model, on the basis of being aware of all true atomic facts, and as long as he does so he has to "inhabit" this same world.

4.7. Soundness.[79]

We will here establish the soundness of S-models. The proof is in three phases. First, we establish the soundness of sentences falling under 1 of 4.5.1.3. Then, we establish the soundness of sentences that belong to either 2 or 3 of 4.5.1.3. Finally, we establish the soundness of combinations of these two main categories.[80]

First phase: We establish the soundness of the first category by proving that if indexed sentences are taken to be atomic there exists for each S-model a corresponding Beth model. Soundness comes from the soundness of Beth models.

(Atomic sentences' variables run through the domain of indexed sentences.)

Claim 1. *To each submodel* [81] *of any S-model there corresponds a Beth model.*

Proof. We construct the corresponding Beth model as follows. (Transfinite set theory with the Axiom of Choice is needed). Assume the submodel

$$\langle W(w), f, R, D, V, \Vdash \rangle.$$

Let "forward semi-submodel" stand for

$$\langle \{w' : w' \in W(w) \& f(w') \geq f(w)\}, f, R, D, V, \Vdash \rangle$$

and "backward semi-submodel" stand for

$$\langle \{w' : w' \in W(w) \& f(w') \leq f(w)\}, f, R, D, V, \Vdash \rangle.$$

[78] See 4.4.2 and Appendix III.

[79] For the economy of the presentation proofs are not presented in full length. The same obtains as for 4.8.

[80] We let the reader test that these categories exhaust all sentences.

[81] If $\langle W(w), f, R \rangle$ is a substructure, let $\langle W(w), f, R, D, V, \Vdash \rangle$ be the corresponding "sub-model".

Let also "(forward semi-submodel)′" stand for

$$\langle\{w' : w' \in W(w) \& f(w') > f(w)\}, f, R, D, V, \Vdash\rangle$$

and "(backward semi-submodel)′" stand for

$$\langle\{w' : w' \in W(w) \& f(w') < f(w)\}, f, R, D, V, \Vdash\rangle.$$ [82]

Step 1: Make the (backward semi-submodel)′ of $\langle W(w), f, R, D, V, \Vdash\rangle$ the initial part of the (forward semi-submodel)′ of the entire submodel. To the end of each maximal path of the transposed (backward semi-submodel)′ attach the previous (forward semi-submodel)′. (See figure 5 (i) and (ii)).[83]

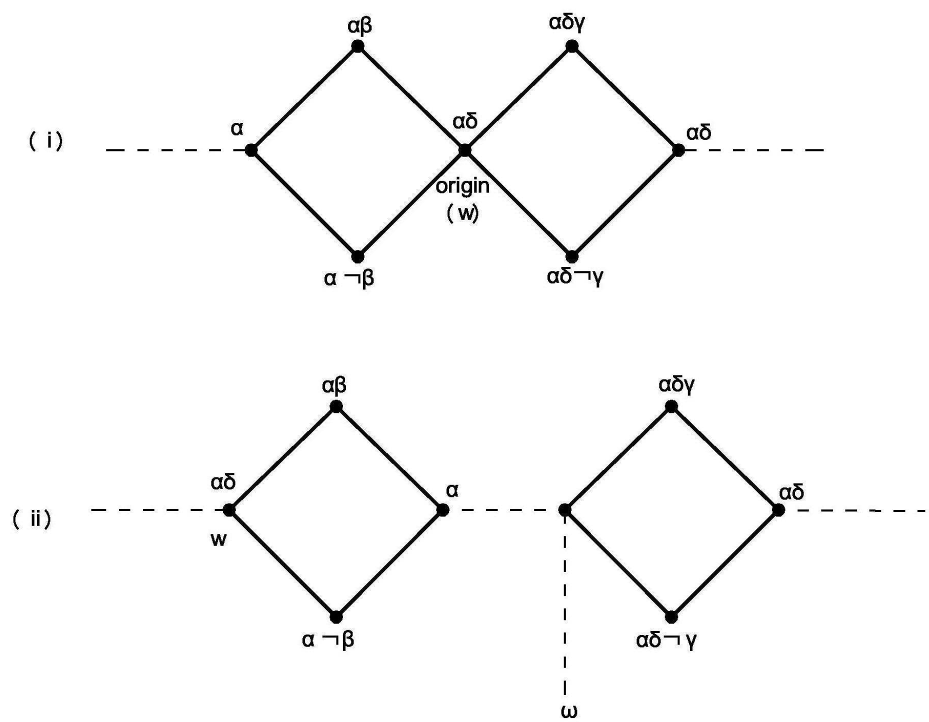

Figure 5.

Step 2: To each set of worlds $\{w_1, w_2, \ldots, w_n, w^*\}$ such that

$$w_1R^1w^*, w_2R^1w^*, \ldots, w_nR^1w^*,$$

carry out the following operation: make the (forward semi-submodel) of

$$\langle W(w^*), f, R, D, V, \Vdash\rangle$$

[82] Cf. the notion of semi-substructure in 4.3.19.

[83] Because of backward convergence there is only one such path. Cf. 4.3.10.

the (forward semi-submodel)$'$ of

$$\langle W(w_1), f, R, D, V, \Vdash\rangle, \langle W(w_2), f, R, D, V, \Vdash\rangle, \ldots, \langle W(w_n), f, R, D, V, \Vdash\rangle.$$

(See figure 6).

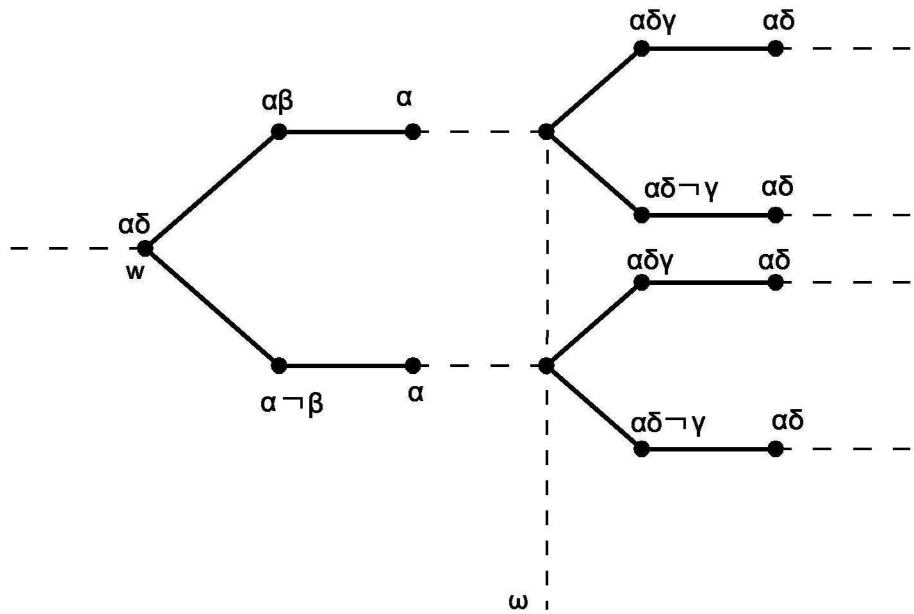

Figure 6.

Step 3: Make the model respect the condition: If w forces α and wRw',[84] then w' forces α.[85] (See figure 7).

[84]Diagrammatically, wRw' is equivalent to w' being either w or in front of w, and connected to it by a path.

[85]Notice that this is done automatically, if one, after step 2, applies the bar criterion, since, in that case, the absence of backward branching turns the model into truth-preserving.

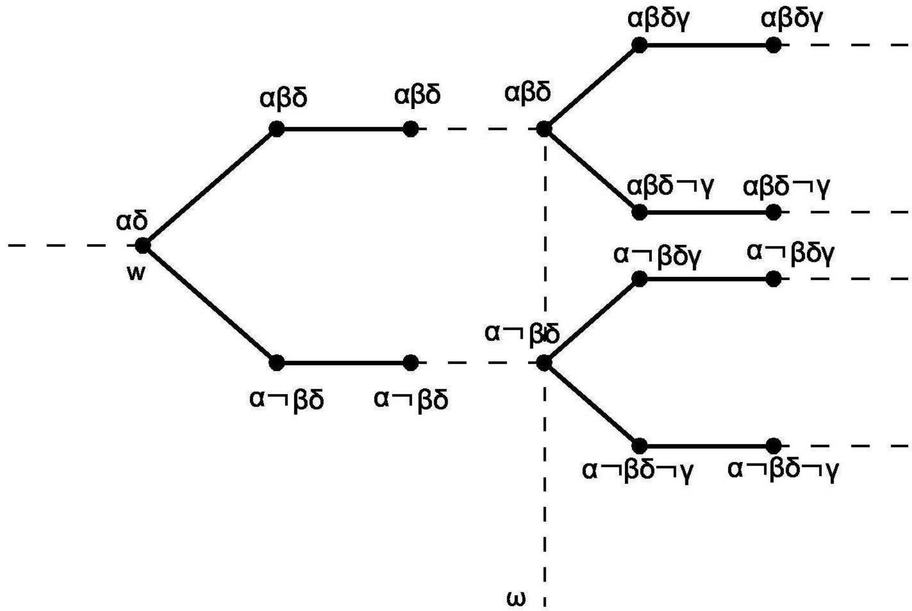

Figure 7.

The resulting model is a Beth model. Step 1 eliminates past branching with respect to the origin. Step 2 eliminates future converging, and the structure becomes a tree. Step 3 reestablishes truth-perseverance.[86] □

Claim 2. *The resulting Beth model is equivalent to the initial S-submodel.*

Since a formula α is established in a world of the Beth model, if, and only if, there is bar with respect to this world, where α obtains throughout,[87] *claim 2 is equivalent to the complex claim (i)-(iii):*

(i) The semantic conditions are the same for the two models.

(ii) For every bar A for the formula α of the S-submodel, there is an equivalent bar in the Beth model.

(iii) For every bar A for the formula α of the Beth model, there is an equivalent bar in the S-submodel.

Proof. of the parts of the equivalent claim:

(i) Compare 4.5.1.2.(1-8) with the conditions of a Beth model.[88] Since indexed sentences are taken to be atomic, the conditions are identical. Notice that 3.3.(i) of Van Dalen (op. cit., p. 249), concerning the domain

[86] Cf. D. VAN DALEN, [1986a], p. 250; 3.4.(1).
[87] Cf. D. VAN DALEN, [1986a], p. 250, 3.4.(3).
[88] Cf. D. VAN DALEN, [1986a], pp. 249-50.

of elements is satisfied in S-models, because the domain of numbers for S-models is constant - cf. 4.4.3.2. Notice also that 4.5.1.2.(8) can be stated in this form, exactly because the domain of natural numbers is constant - cf. Van Dalen (op. cit. p. 251; (5′)). Notice, finally, that transcribing 3.3.(ii).(4) of Van Dalen (op. cit., p. 249-50) into the generalized form of 4.5.1.2.(3) is a harmless superfluity for the Beth condition. In classical Beth models, the condition can be thus simplified, because of past's linearity.[89]

(ii) Step 1 removes no bars for any sentence. Step 2 replaces some new bars for old bars, but all the sentences belonging to the worlds of the old bars belong also to some of the bars that have been replaced for them. Step 3 removes no bars.

(iii) Step 1 adds no bars for any sentence. The bars added by step 2 contain no new sentences in comparison to the bars of the S-submodel. Step 3 adds some new bars with respect to the origin and for some sentences, but the bars that are added bar the origin for sentences that already have bars with respect to the origin.

□

Remarks:

1) Notice that the reason why the models are equivalent is because it is impossible to use the "⊩" connective, since indexed sentences are atomic.

2) Another way to show the equivalence of the models is the following: observe that the intersections of (i) the true sentences of all linear orders that pass through the origin of the Beth model and (ii) the true sentences of all linear orders that pass through the origin of the S-submodel are the same set. The only differences between the models are (a) the position of their worlds, and (b) the fact that in the linear orders passing through the origin of the S-submodels these (common) sentences are forced into fewer worlds. This is the result of non truth-preseverance, caused by backward branching.

Second phase: The soundness of sentences falling under 2 and 3 of 4.5.1.3 comes straightforwardly from 4.5.1.1, and the assumed classical behavior of the natural language counterparts of the formal connectives.[90]

Third phase: The soundness of the calculus of sentences that combine sentences falling under 4.5.1.3.(1) and (2-3).

Assume that $(\ldots \text{Ⓒ}\alpha\ \text{Ⓡ}\ldots)$ is a true sentence of w, in which α belongs to either 2 or 3, Ⓒ and Ⓡ are connectives, and the rest falls under 1.

[89] In Beth models, the condition for implication can be simplified into: in no world accessible from the present, can the antecedent be true and the consequent untrue. In an S-model, it cannot be thus simplified, and this is because of future convergence. For example: think of α as referring to a past contingent event. Since $\neg\alpha$ is defined as $\alpha \rightarrow \bot$, and α refers to a past contingent state, the Beth-Van Dalen definition of $\rightarrow$ will make '$\neg\alpha$' true, while it should be truth-valueless.

[90] See also note 49 and 5.3.

The claim that S-models are sound with respect to sentences that combine sentences falling under 4.5.1.3.(1) and (2-3) amounts to the claim that one cannot deduce that α has in w a truth-value other than that which the truth of $(\ldots \text{Ⓒ}\alpha \ \text{Ⓡ}\ldots)$ in w implies that α has. We will take the easiest case, which is Ⓒ =Ⓡ= & and '$(\ldots \text{Ⓒ}\alpha\text{Ⓡ} \ \ldots)$' being true in w. The rest is more laborious, though routine.

Claim 3. *One cannot deduce that α has, in w, a truth-value other than that which the truth of $((\ldots)\&(\alpha)\&(\ldots))$ in w implies that α has.*

Proof. Because of 4.5.1.3, $((\ldots)\&(\alpha)\&(\ldots))$ is to be evaluated in w according to $((\ldots)\&((\alpha)\iota(w))\&(\ldots))$. Therefore, the truth of $((\ldots)\&(\alpha)\&(\ldots))$ in w implies that α is true in w, as 4.5.1. implies that it should, if $(\alpha)\iota(w)$ is true. □

The proof of claim 3 concludes the soundness general claim.

4.8. Completeness.

Completeness is, with respect to S-models, a chimera. As you will remember, the *final scientific theory S* concerns not only mathematics but also (not to say primarily) some kind of idealized physics and history, which involves the set of causal conditionals, and which, in turn, must be assumed to express truths like "if a butterfly flies in China at n, there is a war in Europe at m", and the like. The calculus must be assumed to be at least as complicated as the calculus that the early atomists dreamed of (cf. chapter 3). In this book, it starts as a thought experiment and remains such, until the end.

However, some *facts* can be proved about it.

1. The set of theorems for sentences belonging to 4.5.1.3.(1) is a conservative extension of the Heyting calculus. This comes from claims 1 and 2 of 4.7. An alternative way to show this is to point out that Beth models are sound, and complete, with respect to the intuitionistic calculus not only in their tree-like form, but in any partial order form.[91]

2. The set of theorems for sentences belonging to 4.5.1.3.(2-3) is a conservative extension of the classical calculus. As far as 4.5.1.3.(3) is concerned, this comes from 4.5.1.1. As far as 4.5.1.3.(2) is concerned, it comes from the way the natural language counterparts of formal connectives have been defined .[92]

For the following two facts we need two definitions:

Definition 6. *An index $(\ldots\vec{n}\ldots)$ will be said to "effectively determine a world for the origin w", if, and only if, there exists one and only one world in $W(w)$, such that it is identified by it, according to the economic methods of indexing, when these are applied by w itself (cf. 4.3.21.2 and 3).*

[91] A. TROELSTRA - D. VAN DALEN, [1988], p. 679: "The notion of a Beth model can also be generalized to arbitrary, partially ordered ($K \leq$), with the role of the α played by maximal linearly ordered subsets."

[92] Cf. 3.3.

Definition 7. *An index $(\ldots\vec{n}\ldots)$ will be said to "non-effectively determine a world for the origin w", if, and only if, there is in $W(w)$ a bar b with respect to w such that*

(i) b is other than w, and

(ii) for every world $w' \in b, (\ldots\vec{n}\ldots)$ effectively identifies a world of $W(w')$.

To take an example: in figure 4, "$0, 1, 1$" effectively determines a world for the origin; it denotes the upper world with rank 2. On the other hand, "$1, 1$" determines a world, but non-effectively - consider the indexes used by the worlds of the bar constituted out of the worlds with rank 1. They both use that index so as to effectively identify the upper, and the third, from the top world with rank 3.

3. For every origin w, the set of theorems for sentences of the form $\alpha(\ldots\vec{n}\ldots)$ is a conservative extension of the classical calculus, when $(\ldots\vec{n}\ldots)$ effectively determines a world in $W(w)$. This result comes straightforwardly from (2) above; the metalanguage of $w(\ldots\vec{n}\ldots)$ is classical. Since there is, in $W(n)$, only one world with that index, this world either establishes α or does not, and, if it is not the case that is establishes α, it is the case that it does not establish α, and *vice versa*.

4. For every origin w, the set of theorems and valid arguments for sentences of the form $\alpha(\ldots\vec{n}\ldots)$ is a conservative extension of Van Fraassen's supervaluation (op. cit.),[93] when $(\ldots\vec{n}\ldots)$ determines non-effectively a world in $W(w)$. This result will be proved in chapter 5. We state it here because it explains a lot about how the Sea-Battle is resolved in an S-model; *hint*: this calculus respects the Excluded Middle, while it still admits gaps.[94]

By using induction on 1-4 above, one can answer the question concerning the underlying proof-theory of an S-valid sentence for a number of other cases. For example: think of $(\ldots\vec{n}\ldots)$ as effectively determining, in $W(w)$, the world w', and of $(\ldots\vec{m}\ldots)$ as non-effectively determining a world in $W(w')$. Assume also that wRw'.

$$w \Vdash (w(\ldots\vec{n}\ldots) \Vdash ((w(\ldots\vec{m}\ldots) \Vdash \alpha) \vee \neg(w(\ldots\vec{m}\ldots) \Vdash \alpha)))$$

has to be true, because of (4) above.

5. Before moving onto the second part and presenting how the fantasy can be successfully incorporated in an S-model, I will make some comments about the model in general. They will aim, for the most part, to probe further into the general remarks of 0. Since they do not contribute much to the presentation of the solution to the fantasy, 5 can be skipped without any essential loss.

[93] Note that for this kind of supervaluation the set of theorems does not correspond one to one with to the set of valid arguments. See the following chapter.

[94] See the lemmata of theorems 7-9 of the next part.

5.1. (It picks up 0.3). As we have seen, Beth models and S-models with atomic sentences=indexed sentences are equivalent. On the other hand, if indexed sentences are removed from the S-model, the model becomes classical. These two facts can also be stated as follows: S-models are all-round classical models, except for the formalized quantification over the domain of arguments and the domain of values of the ι function.[95] (The ι function has as its domain of arguments the domain of worlds, and as its domain of values the domain of their identifying indexes.) Formalized quantification with respect to these domains is intuitionistic.[96] Which means that by the statements (i) "There is a world with index $(\ldots \vec{n} \ldots)$, such that so and so" and (ii) " There is an index $(\ldots \vec{n} \ldots)$, specifying a world, such that so and so" are, when formalized, not to be evaluated *in situ*, but upon bars. This, in turn, means that the world with that index need not be available at present, the index might identify nothing at present; it suffices that it necessarily will, or has identified some world in the future, or during the past. (The bar accounts for "necessarily"). Notice here that our conditions 4.5.1.2.(1-6) fall under (i) above, and 4.5.1.2.(7-8) fall under (ii) above. For example: 4.5.1.2.(1) poses the condition for the formalized, "There is a world with index $(\ldots \vec{n} \ldots)$ such that α obtains in it".[97] The index $(\ldots \vec{n} \ldots)$ might identify nothing in the substructure corresponding to *now*, it suffices that it does identify something in the substructures of the worlds of a bar. Hence, the necessity of our world having access to the metalanguage of all the worlds of its visual field. In brief, S-models can be taken to be normal models of tense logic with branches, having, however,

(i) an internalized metalanguage and

(ii) these specific quantification rules for formalized quantified sentences over the arguments and values of the ι function.

As for other quantifications, they are all classical - e.g. "Someone is more than 200 years old" will not become true in an origin, if there is a future bar for that origin, where it is true throughout. If there is no one today exceeding that age, it is false. Similarly, the non-formalized, "There is a world with a certain index, such that so and so" will be false if the index specifies nothing in the substructure corresponding to the present, and even if it does so upon a bar with respect the present.

5.2. (It picks up 0.1). We are now in a much better position to formally establish the fact that S-models differ from Beth models, in as much as they do not always have an equivalent Kripke model.

The fact that there are Beth models that falsify the *local* Disjunction and Existence Properties[98] while there is no such Kripke model is due their

[95] The clause "formalized" is indispensable, because quantifications using connectives of 1.22 are always classical. Cf. 4.5.1.3 and 5.3 bellow.

[96] Most of the time, these quantifications rotate around the predicate of *forcing*, or equivalents such as "being true in a world" - e.g. "there is a world with an index, such that it forces so and so"; "there is an index following a certain condition and it identifies a world, such that it forces so and so", etc.

[97] The converse is stated in 4.5.1.2.(7).

[98] By *local* Disjunction Property -as opposed to the *non local*- I mean the property that there is no model, in which there is an origin where $\alpha \vee \beta$ is forced, while neither α nor β

respective conditions for disjunction and existential quantification.[99] However, every Kripke model has a corresponding Beth model.[100] By the internalization of the metalanguage and the fact that it is complete (cf. 4.5.3.3 and 4.8.(2)), this stops being the case for S-models. Consider the Kripke model in figure 8.

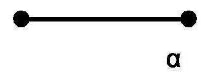

Figure 8.

The corresponding Beth model appears in figure 9.

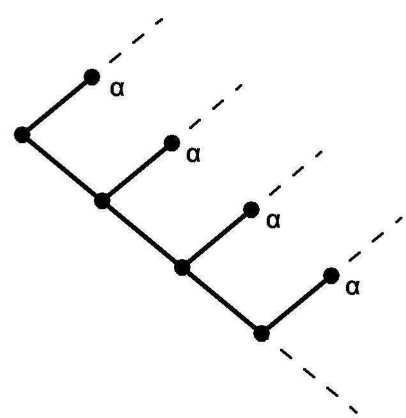

Figure 9.

Upon the infinite lower path (also called "spine"), α is never proved nor disproved. We will show that this kind of "spine" cannot be constructed for every S-model. Consider the S-model model of figure 10.

are forced. Its *non local* counterpart asserts that no $\alpha \vee \beta$ formula is valid in a certain set of models complete over some logic L, while neither α nor β are valid in L. These two do not imply each other. For example, the set of all classical Beth models falsify the *local* but verify the *non local*. The set of all Kripke models verify them both.

[99] Cf. D. VAN DALEN, [1986a], pp. 249-50; 6.5.2.

[100] Cf. D. VAN DALEN, [1986b].

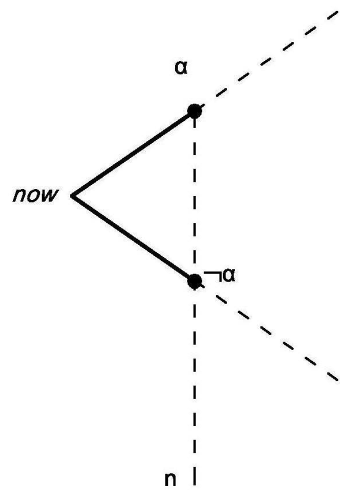

Figure 10.

Since the metalanguage makes up part of the model, the sentence

$$\text{“}(w(n) \Vdash \alpha) \vee \neg(w(n) \Vdash \alpha)\text{”belongs to the origin,}$$

while neither of the disjuncts does. The corresponding Beth model that would not have allowed such a combination would have to be like that in figure 11.

However, it falls short of the task exactly because of the completeness of the metalanguage. One among,

$$w(n) \Vdash \alpha \text{ and } \neg(w(n) \Vdash \alpha)$$

necessarily belongs to the n^{th} world of the "spine". So, although the spine can be articulated for $\alpha/\neg\alpha$, it cannot be articulated for $w(n) \Vdash \alpha/\neg(w(n) \Vdash \alpha)$.[101]

[101] This particularity of S-models can also be testified for the *Creating Subject*, when the *Subject* is allowed to think about its own activity (cf. note 5). For example: in D. VAN DALEN, [1978], p. 19, "$\alpha \Vdash\, \vdash nA$" reads: the world α forces that the *Creating Subject* has evidence for A, at stage n. Now, consider $\alpha \Vdash \neg(\vdash nA)$, and take A to assert that the *Creating Subject* has constructed a certain segment of π, and assume also that this segment of π cannot possibly be reached before stage n. Now, assume that whether or not this segment will have been constructed at that stage depends upon the activities of the *Creating Subject* and that these activities are free (i.e. indeterministic). In such a case, it is indefinite whether or not the *Subject* will have reached that segment by n, or not. On the other hand, it is not to be doubted that he will either have reached it or not. The *Subject* knows how to construct the segment; it is just that, at α, it is still indefinite whether or not he will still have his mind set on that goal until stage n. (Notice, however, that A is, on the occasion, not mathematical; it is a sentence about a particular event. It does not present the segment of π that the *Creating Subject* is supposed as being after; it asserts that this segment has, in fact, been reached by the *Subject* - i.e. it asserts an event.)

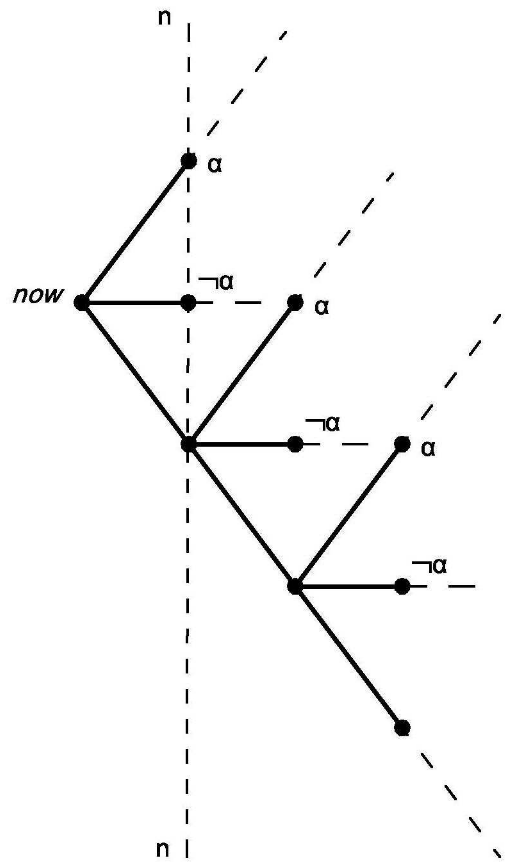

Figure 11.

We have to note here that, although the Disjunction and Existence Properties (both *local* and *non local*), are, in general, falsified in S-models, this does not necessarily harm their intuitionistic foundations.[102]

5.3. (It picks up 0.4.) We have, on several occasions, referred to the "double-life", and to how one can come to terms with it within an S-model. One should grow accustomed to the necessity of having some intuitionistic criteria for evaluating some sentences, while, at the same time, he still uses classical criteria, in order to state and test the intuitionistic criteria - cf. 4.5.3.3.1. In

[102]Cf. D. VAN DALEN, [1986a], pp. 292-93; 6.5.2: "Kreisel, however, has shown that the possession of Existence Property is neither necessary nor sufficient for constructive theories" (p. 293). The Existence Property and Disjunction Property have been proved to be independent from the calculus of Heyting.

order to facilitate this double-life, we have adopted the convention of using the natural language counterparts of the formal connectives, according to the classical interpretation. A closer look at 4.5.1.2 reveals how this is done. To the question of whether "iff", "there is", "for all", "or", "forces", etc., of our metalanguage in 4.5.1.2 are to be interpreted classically, or not, the answer comes from 1.22. Let us take the more interesting case: 4.5.1.2.(2). That "iff" must be evaluated according to the classical apparatus means that either both sentences of the biconditional are true, or both are false.[103] It also means that either they are both true, or their negations are both true.[104] What comes after "iff" is an existential statement that must be evaluated according to 4.5.1.1, because the quantifier is in ordinary English. Moreover, it means that, when the sentence asserting the existence of such a bar is not established, its negation is established. Finally, observe that "or" will not be evaluated by the standards of $\vee$, defined in the same line. To say "either α is forced, or β is forced" necessitates that one of the two belongs to the world, whereas to say that "α is forced $\vee$ β is forced" does not - i.e. the assertion might be based upon the presence of a distanced bar, where either α is forced, *or* (ordinary language) β is forced.

Therefore, we have to be highly cautious and to make certain that, even if we do live the double-life, we know exactly what part we play on each and every occasion. This is not hard to imagine, for it is exactly what the *Creating Subject* does. He first states some conditions for -say- disjunctions; then, he establishes, in a classical way, that these conditions are not met for some instances of the Excluded Middle, and, finally, he ends-up with the well-known deviant results. All this, he does, while inhabiting his very models; his theorems are the things that he either has experienced or now experiences or will experience, while located on the nodes of the model itself; the *Subject* does not contemplate his model from a distance, he makes part of the (occasional) origin. Evidence for that, is that "p is a theorem" is equivalent with "the *Subject* either has, or now does, or will experience p" (cf. 5.7), which means the *Subject* belongs to every $D(w)$.

5.4. We will briefly go back to our insistence on calling expressions of the form $w(\ldots(\vec{n})\ldots)$, "world constants depending upon free choice sequences". After 4.5.1.2, it has become clear that one could have dispensed with such vocabulary. To say -e.g.- that '$w \Vdash F\alpha$' is true amounts to saying that there is a bar for w, such that all the worlds of that bar fulfill the conditions that

(i) their rank is higher than $f(w)$, and

(ii) they force α.

A question, however, arises: why does one decide, all of a sudden, to focus attention on the fulfillment of this condition, and highlight it in such a way, so as to introduce a brand new expression into the calculus? The answer is that it is not the bare presence of all these worlds upon the bar that provokes the highlights. It is the fact, rather, that one of these worlds will belong to the

[103] Notice that it would not have been fruitful to use here the formal $\Vdash$, instead of the informal "forces", because $w \Vdash (\alpha(\ldots(\vec{n})\ldots) \ \vee \ \beta(\ldots(\vec{m})\ldots))$ can be truth-valueless, in the formal metalanguage.

[104] Their formal negations.

linear order that a free choice sequence tends to generate, as it creates its way into the future by its successive free choices. The bar, *per se*, is uninteresting; what is interesting is that one of its worlds will become a (constant) element of the linear order that is now in the becoming. The world chosen will, indeed, be a constant. Which world will this be depends upon the future "choices" of the sequence. So, if one uses only extensional criteria as for the universe of free choice sequences,[105] one had better keep the idiolect appropriate to them, instead of switching to the topological and set theoretical language of the model. Such a tactic helps, in so far as it reveals the underlying motivation.

On the other hand, since tensed and modal statements are introduced by definitions over free choice sequences (cf. 4.5.1.2.(9-18)), it is evident that one who prefers the kind of analysis that is associated with free choice sequences can dispense with tenses and modalities altogether, in favor of the sequences themselves - i.e. he can use the *definiens* and not the *definiendum*. That is (according to some) an advantage in itself. For example: "$F\alpha$" can be eliminated by "the sequence chooses, at some stage $m > f(w)$, a world w', such that "$\alpha \in w'$"".

5.5. Flirting with non-discreetness.

S-models are discrete. To make them continuous, necessitates a step no less groundbreaking than Brouwer's "second act of Intuitionism" between 1914-18. Nonetheless, since the ground has already been broken, our task is relatively easy. Instead of posing that f takes values from $\mathbb{N}(+/-)$, we can pose that it takes values out of the set of rational intervals satisfying the Cauchy sequence condition. Moreover, these rational intervals can be made to correspond to choice sequences and, consequently, the choice sequences themselves can be made to represent moments[106] of Time continuous. It is important here to notice that these choice sequences that represent the rational intervals are not the same as the choice sequences of worlds that constitute the spread of a universal (partial) History. The former are exactly what, in the discrete model, were the (static) moments of Time, and, as such, they are assigned to sets of sentences representing universal states, under exactly the same *rationale* that the static moments of the discrete model were assigned to such sets. The latter are the free choice sequences constituting the spread. This change with respect to what "a moment" is will pose no problem to the intuitionistic denial of the actual infinite, and no Zenonian paradoxes will emerge out of the general bar criterion - the bar criterion will still hold, even for bars at a level equal to - say- $\sqrt{2}$. The classical problem of the sequence that constantly converges, but never actually reaches $\sqrt{2}$ is here surmounted by the fact that, intuitionistically, $\sqrt{2}$ is not an entity apart from the sequence "converging towards" $\sqrt{2}$. The number $\sqrt{2}$ is that very sequence, and it can be effectively identified by the

[105] This is necessary, for otherwise (i.e. if we allow for intensional criteria) the topology of the model would underdetermine the free choice sequence proof theory - e.g. a single maximal path might represent many completed free choice sequences, and two sequences sharing some segment in common would be not necessarily "meeting" one other "during" that segment.

[106] These "moments" will not have the static (set theoretic) character of the moments of a "classical" Time continuous. The intuitionistic moments would be the converging sequences themselves, rather than the moments/points to which they are ("classically") converging.

constant ratio that generates the sequence. On the other hand, the proven unsplittability of the continuum (a consequence of the Continuity Theorem) guarantees not only that every bar (even if at a level represented by an irrational rank) will eventually be reached, it also guarantees that all bars will eventually be left behind.[107] In *Creating Subject* terms, this means that, although the *Subject* is not capable of any infinite act of intuition, and, therefore, the *Subject* cannot have any protension of the bar at the level of $\sqrt{2}$, he can always assert its presence, because he can apprehend the ratio generating the sequence that converges "towards $\sqrt{2}$".[108] On the other hand, because of the unsplittability of the continuum, he can dismiss of any Zenonian kind of worry.

Contrary to this generalization, and with respect to the number of worlds sharing a rank in common, we do not think that one can allow it to exceed $\aleph_0$ in cardinality - cf. 4.2 where such a condition is posed. Since these worlds form economic bars, such a move would endanger their capacity to be well-ordered, and thereby, the Bar Theorem, upon which so many of our results are founded. On the other hand, our having incorporated into the worlds of the models a higher order logic allows, theoretically, for a transfinite number of atomic states.[109] If such number of atomic states is allowed, nothing prevents the existence of immediate successors which are transfinite in number, and, thereby, of economic bars with transfinite elements. This possibility speaks in favor of the necessity for a condition which cuts down the number of atomic states to a denumerable one.

5.6. Some points about our modalities:
The first concerns "$\triangle$", the contingency operator. Some logicians believe that this modality is -if defined in this way- trivial. This objection originates from Lesniewski,[110] who has established that, if there is, within a calculus, a sentential predicate that is true of a sentence, and this predicate is also true of its negation, then, it is true of every sentence of the calculus. This theorem has often been used, in order to exhibit the "absurdity" of "being contingent". A way to defend its meaningfulness is to claim that the theorem of Lesniewski obtains only for primitive predicates. For, consider the experimental operator "$\bullet$". Suppose that it operates on predicates of individuals, and that it transforms them into set predicates, in the following way: assume $\phi(\ldots)$ to be a predicate of individuals. '$\bullet\phi(\ldots)$', is true of its argument (which now is a set), if, and only if, some members of it are ϕ, and some are non-ϕ. For example, a set of people

[107] Roughly put, what the unsplittability of the continuum guarantees with respect to the present example is that it is not the case that the "moment/point" $\sqrt{2}$ splits the continuum in such a way that $\{t : t > \sqrt{2}\} \cup \{t : t \leq \sqrt{2}\} = R$.

[108] Now it is a matter of controversy whether or not the *Subject* by so doing arrives at an infinite act of intuition - i.e. it is a matter of controversy whether or not his apprehension of the ratio generating $\sqrt{2}$ is in itself an infinite act of intuition. We think that *stricto sensu* it is not, since an infinite act of intuition is equivalent to having grasped an infinite totality of elements, which is not the same as having grasped the ratio generating the totality and converging towards $\sqrt{2}$. However, the ratio itself is the intuition is equivalent to having grasped an infinite totality of elements, which is not the same as having grasped the ratio generating the totality and converging towards $\sqrt{2}$. However, the ratio itself is the intuitionistic equivalent of the classical series converging towards that number. These two are intuitionistically the same. Therefore, the *Subject* does not apprehend less numbers than his classical "colleague".

[109] i.e. the number of functions (cf. 1.19) is transfinite.

[110] J. LUKASIEWICZ, [1957], § 45.

will be "•intelligent", if, and only if, some among them are intelligent and some are non-intelligent. Clearly, then, everything, which is •intelligent will also be •non-intelligent, but this can hardly make absolutely everything •intelligent. By a parallel argument, one can defend "contingency" in set-theoretic terms: a state is contingent, if, and only if, there is a set of worlds that are accessible from our world, or have access to our world, and in which the state obtains, and there is also another similar set, in which the state does not obtain.

A second point concerning our modalities: Necessity and possibility have been modeled according to the -so called- "Diodorian", as opposed to -so called- "Aristotelian" modalities. The above distinction is a crude one and does little justice to either of these thinkers. Roughly, it goes as follows: the Aristotelian "possible" is what either was or is or will be the case, whereas the Diodorian "possible" is what either is or will be the case.[111] (The "necessary" is defined accordingly.) We followed the interpretation that we have, because we wanted to treat of both future and past contingencies.

For switching from the "Aristotelian" to the "Diodorian" modalities, it suffices to modify 4.5.1.2.(16-17) as follows:

16. w forces $\Box\alpha$, iff, for all w', such that wRw', $w' \Vdash \alpha$.
17. w forces $\Diamond\alpha$, iff, for some w', such that wRw', $w' \Vdash \alpha$.

5.7. All formalizations of the *Creating Subject* are much closer to Beth models, than to Kripke models. This is due to the fact that the *Creating Subject* tries to capture the evolution in Time of the cognitive content of an ideal mathematician. The march of Time is inherent in these models. The corresponding formalization of the *Ideal Scientist* is identical to the formalization of the *Creating Subject*, except for perfect memory and the arbitrary iteration of the "know that" propositional attitude, due to the internalization of the metalanguage.

The three clauses for the *Scientist* are:

1) If he knows, at n, that α, then, at no subsequent moment, does he know that $\neg\alpha$.

2) At every moment, he either knows that α, or he does not know that α.

3) α is the case, if, and only if, there exists a moment, when he knows that α.[112]

(1) is stricter for the *Subject* and necessitates that he never forgets α; i.e. not only that he does not learn its negation.[113] On the other hand, by the internalization of the metalanguage, the *Scientist* is aware of (2) -it is debatable whether the *Subject* is aware of the same thing-[114] and can, thereby, deduce

[111] Cf. A. PRIOR, [1967], p. 16-17. Prior is not he who introduced the distinction.
[112] We do not consider the "intersubjective case". Cf. D. VAN DALEN, [1986a], p. 326.
[113] The reason for that can be found in note 89. See also Th. 1 in II
[114] As traditionally presented, the clauses for the *Subject* are *about the Subject*. It is unclear whether or not the *Subject* is aware of them. However, if he really thinks about his own activity (cf. note 5), he must be.

that at every moment he knows that he either knows that α, or he does not know that α.[115]

II

Before passing onto the solution of the fantasy, we will present some intermediate helpful theorems:[116]

Theorem 1. *For any $\alpha(n)$ and every w, if $w \Vdash \alpha(n)$, then, for all w', such that wRw' or $w'Rw$, $\neg(w' \Vdash \neg(\alpha(n)))$.*[117]

Auxiliary claim 1. *If w' is a random world of $W(w)$ and b a bar for w, there is at least one world of b that is within the visual field of w'.*

Proof. : think of the fact that

(i) all maximal paths pass through w,

(ii) a bar with respect to w is a bar with respect to all maximal paths passing through w,and

(iii) any world of $W(w)$ is upon at least one maximal path of $W(w)$.

□

Proof. If $w \Vdash \alpha(n)$, then, there is a bar b for w, such that, for every $w'' \in b$, $\iota(w'') = n$, and $w'' \Vdash \alpha$. By the *auxiliary claim*, it follows that all worlds w', such that wRw' or $w'Rw$ see at least one world of b. Therefore, by 4.5.1.2.(6), w' does not force $\neg((\alpha)(n))$. □

Theorem 2. *For any $\alpha(n), \beta(m)$ and any w,*

$$w \Vdash \alpha(n) \vee \beta(m), w \Vdash \alpha(n) \& \beta(m), w \Vdash a(n) \to \beta(m), w \Vdash \neg(\alpha(n)),$$

iff, there is a bar b for w, such that, if $w' \in b$, then,

$$w' \Vdash \alpha(n) \vee \beta(m), w' \Vdash \alpha(n) \& \beta(m), w' \Vdash \alpha(n) \to \beta(m), w' \Vdash \neg(\alpha(n)),$$

respectively.

Proof. By the equivalence of S-models, when indexed=atomic, with Beth models (cf. 4.7), and Lemma 3.4.(2) of D. VAN DALEN, [1986a]. □

Theorem 3. *For any $\alpha(n), \beta(m)$ and any w,*

$$\neg(w \Vdash \alpha(n) \vee \beta(m)), \neg(w \Vdash \alpha(n) \& \beta(m)), \neg(w \Vdash a(n) \to \beta(m)), \neg(w \Vdash \neg(\alpha(n))),$$

iff, there is a path p through w, such that, if $w' \in p$, then,

$$\neg(w' \Vdash \alpha(n) \vee \beta(m)), \neg(w' \Vdash \alpha(n) \& \beta(m)), \neg(w' \Vdash \alpha(n) \to \beta(m)), \neg(w' \Vdash \neg(\alpha(n))),$$

respectively.

[115] This is another instance (cf. 5.2) of the Disjunction Property failing in S-models: α is here a free propositional variable.

[116] Theorems 1-3 are adaptations of Lemma 3.4 of D. VAN DALEN, [1986a]. Notice, though, that Th.1 is less potent that 3.4.(1) of Van Dalen, and this is due (again) to backward branching. Cf. note 89, and 5.7.(1). If linear past is imposed, we obtain truth perseverance as well.

[117] In the theorems that follow we will restrict ourselves to the simple case of $(\ldots(\vec{n})\ldots) = n$. The results can be generalized.

Proof. By the equivalence of S-models, when indexed=atomic, with Beth models (cf. 4.7), and Lemma 3.4.(3) of D. VAN DALEN, [1986a]. □

Theorem 4. *For every* $w \in W$ *and every* $\alpha, (w \Vdash (w \Vdash \alpha)) \vee (w \Vdash \neg(w \Vdash \alpha))$.

Proof. By 4.8.(2);"$w \Vdash \alpha$" is metalinguistic for w. □

Other theorems relative to the Sea-Battle concern the classical behavior of some among the indexified sentences:

Theorem 5. *For every* $w \in W$*, every* n*, and every* α, β*, belonging to 4.5.1.1,*

(i) if $\alpha(n)$ *does not belong to* $w(n)$*, then,* $(\neg\alpha)(n)$ *belongs to* $w(n)$ *and vice versa;*

(ii) if $w(n) \Vdash \alpha(n)$ & $\beta(n)$*, then, both* $\alpha(n)$ *and* $\beta(n)$ *belong to* $w(n)$*;*

(iii) if $w(n) \Vdash \alpha(n) \to \beta(n)$*, then, either* $(\neg\alpha)(n)$ *belongs to* $w(n)$*, or* $\beta(n)$ *belongs to* $w(n)$*;*

(iv) if $w(n) \Vdash \alpha(n) \vee \beta(n)$*, then, either* $\alpha(n)$ *belongs to* $w(n)$ *or* $\beta(n)$ *belongs to* $w(n)$*.*

Proof. $\alpha(n), \beta(n)$ are metalinguistic with respect to w(n), and so by 4.8.(2):

(i) $'$ if $\alpha(n)$ does not belong to $w(n)$, then, $\neg((\alpha)(n))$ does belong to $w(n)$ and *vice versa*;

(ii) $'$ if $w(n) \Vdash \alpha(n)$ & $\beta(n)$, then, both $\alpha(n)$ and $\beta(n)$ belong to $w(n)$;

Since α and β belong to 4.5.1.1, if w does not force α, it forces $\neg\alpha$, and, therefore, we obtain the initial *(i) and (iii).*

(iii) if $w(n) \Vdash \alpha(n) \to \beta(n)$, then, either $\neg((\alpha)(n))$ belongs to $w(n)$, or $\beta(n)$ belongs to $w(n)$;

(iv) if $w(n) \Vdash \alpha(n) \vee \beta(n)$, then, either $\alpha(n)$ belongs to $w(n)$ or $\beta(n)$ belongs to $w(n)$.

□

Theorem 6. *The same as theorem 5, except that it starts: "For every* w*, every* n*, and every* α, β*, in which no index occurs other than* $n \ldots$*"*

Proof. The *proof* is routine. It starts with the results of Th. 5, by putting an extra "n" index in to the subparts of the proven formulas. Then, it generalizes for an arbitrary number of such indexes. □

***

We are now sufficiently equipped so as to elaborate the intuitionistic solution to the fantasy. We will present it in three ways; each one corresponds to a particular possible formulation of it. The first asserts that either there will be a Sea-Battle tomorrow, or there will be no Sea-Battle tomorrow; the second, that there either is a Sea-Battle at n, or there is no Sea-Battle at n, ("n" denoting tomorrow, by today's standards); and the last, that, "There is a Sea-Battle" is true at n, or "There is no Sea-Battle" is true at n.[118]

We will begin with the second of the above formulations:

Theorem 7. *If $\alpha(n)$ is contingent and α index-free, then,*

$$now \Vdash ((\neg(now \Vdash \alpha(n)))\&(\neg(now \Vdash (\neg\alpha)(n)))\&(now \Vdash (\alpha(n) \vee (\neg\alpha)(n)))).$$

Proof. From our earlier conclusion (4.3.16) that the set of all immediate successors of any world are bars for the same world, and the transitivity of R, we conclude that the set of all worlds having n as rank bar *now*. By the *fundamental characteristic*, it follows that all worlds of that bar identify themselves by "$w(n)$". By the fact that α is index-free and Th.5, we conclude that all worlds upon that bar either force $\alpha(n)$, or force $(\neg\alpha)(n)$. By (4.5.1.2.(2)), it follows that *now* forces $\alpha(n) \vee (\neg\alpha)(n)$. Now, suppose that $now \Vdash \alpha(n)$. By Th.1, there is no world in the substructure of *now*, such that it forces $\neg((\alpha)(n))$. But, in such a case, $\alpha(n)$ cannot be contingent as so assumed (cf. 4.5.1.2.18). Therefore, *now* forces $\neg(now \Vdash \alpha(n))$. Finally, suppose that $now \Vdash (\neg\alpha)(n)$. If so, and since the only worlds in the substructure of *now* that identify themselves by "$w(n)$" are the worlds at the level of n, all the worlds of this level form a bar, where $\neg\alpha$ obtains overall. Hence, since every world of the substructure sees at least one world of that bar,[119] there can be no bar at that level for α. Therefore, $\alpha(n)$ is forced nowhere in the substructure. This excludes that *now* sees a world that forces $\alpha(n)$. Therefore, $\alpha(n)$ is not contingent as assumed. Therefore, *now* forces $\neg(now \Vdash (\neg\alpha)(n))$. □

NOTA BENE The *fundamental characteristic* plays a key role in the demonstration of Th.7; it guarantees that, in the substructures corresponding to each and every world of the bar $\{w : f(w) = n\}$, n is the rank of the occasional origin, and of no other world of the same substructure. It is on that basis that all these worlds force either $w(n) \Vdash \alpha$ or $w(n) \Vdash \neg\alpha$. Figuratively speaking, Th.7 says that the present world will, sooner or later, discover that "it" has/came to have n as rank and, at that moment, it forces (will come to force) either α or $\neg\alpha$.

Lemma. *If $\alpha(n)$ is contingent and α index-free, then,*

$$now \Vdash ((\neg(now \Vdash \alpha(n)))\&(\neg(now \Vdash \neg(\alpha(n))))\&(now \Vdash (\alpha(n) \vee \neg(\alpha(n))))).$$

[118]The second case is -as formulated above- in accordance with the atemporal approach towards sentences - i.e. it is implied that the "there is" uses the atemporal present. This is for simplicity. What would have been obedient to the non atemporal approach is "... either ϵ(Ohere is a Sea-Battle at n), or ϵ(Ohere is no Sea-Battle at n)", cf. chapter II. In general, and since we like to think that there is no atemporal perspective "$\epsilon(\ldots)$", though longer and more tiring, should, in principle, be preferred.

[119]Cf. the auxiliary claim of Th. 1.

Proof. It comes from the proof of Th.7.[120] □

Theorem 8. *If $\alpha(n)$ is contingent, α index-free and $f(now)+m=n$, then,*

$$now \Vdash ((\neg(now \Vdash Fm\alpha))\&(\neg(now \Vdash Fm\neg\alpha))\&(now \Vdash (Fm\alpha \vee Fm\neg\alpha))).$$

Proof. By 4.5.1.2.(11), we end up with Th.7. □

NOTA BENE Notice that

$$now \Vdash ([(Fm\alpha)(n-m)](n) \vee [(Fm\neg\alpha)(n-m)](n))$$

is also a theorem, because, in the substructures corresponding to the actual *now* from the perspective of the two alternative n^{th} moments, there is no longer any alternative for the advent/non-advent of α at n.

Lemma. *If $\alpha(n)$ is contingent, α index-free and $f(now)+m=n$, then,*

$$now \Vdash ((\neg(now \Vdash Fm\alpha))\&(\neg(now \Vdash Fm\alpha))\&(now \Vdash (Fm\alpha \vee \neg Fm\alpha))).$$

Proof. See the Lemma of Th.7. □

In the final (third) formulation of the fantasy, the benevolent consequences of the introduction of a formalized metalanguage within the model are ideally exemplified.

Theorem 9. *If $\alpha(n)$ is contingent and α index-free, then,*

$$now \Vdash ((\neg(now \Vdash (T(\alpha))(n)))\&(\neg(now \Vdash ((T(\neg\alpha))(n)))) \\ \&(now \Vdash (((T(\alpha))(n)) \vee ((T(\neg\alpha))(n))))).$$

Proof. The procedure takes up from Th.7, and uses the equivalence of 4.6 for transcribing it into:

$$now \Vdash ((\neg(now \Vdash ((now \Vdash \alpha)(n))))\&(\neg(now \Vdash ((now \Vdash \neg\alpha)(n)))) \\ \&(now \Vdash (((now \Vdash \alpha)(n)) \vee ((now \Vdash \neg\alpha)(n))))).$$

Then, by using the general schema of 4.5.1, we transcribe it into:

$$now \Vdash ((\neg(now \Vdash (T(\alpha, now))(n)))\&(\neg(now \Vdash (T(\neg\alpha, now))(n))) \\ \&(now \Vdash (((T(\alpha, now))(n)) \vee ((T(\neg\alpha, now))(n))))),$$

and finally, we use Def.6 for its final form.[121] □

Lemma. *If $\alpha(n)$ is contingent and α index-free, then,*

$$now \Vdash ((\neg(now \Vdash (T(\alpha))(n)))\&(\neg(now \Vdash ((\neg T(\alpha))(n)))) \\ \&(now \Vdash (((T(\alpha))(n)) \vee ((\neg T(\alpha))(n))))).$$

Proof. the proof for Th.9 to the Lemma for Th.7. □

[120] *Hint*: if not, there would be a bar -i.e. all worlds at the level n- that would force $\neg\alpha$ and not force $\neg\alpha$

[121] Cf. An alternative proof comes from some variation of the Tarski Schema, which happens to be valid in S-models. See chapter 5.

***

We can, now, adapt Theorems 7 to 9 to the Sea-Battle, by making only the following (obvious) assumptions:

(i) "There is a Sea-Battle" is analyzable into atomic sentences and primitive connectives and, so, is index-free,

(ii) tomorrow is m moments away and $now+m = n$ and

(iii) there are *now* no sufficient conditions either for the Sea-Battle, or for its non advent.

By (iii), we conclude that the sentence representing the cause of the Battle is, for the time being, truth-valueless. Otherwise, by our rules for causal conditionals, we should be in position to deduce that there are (*now*) sufficient conditions either for the Battle or for its non advent. This stresses the fact that, in indeterministic Histories, there have to be worlds, in which some well-formed sentences are truth-valueless: the sentences describing indeterministic states. On the other hand, $\{w : f(w) = n\}$ bars the present for "There either is a Sea-Battle at n, or there is no Sea-Battle at n". From the above, it follows both that the event is contingent and that "There either is a Sea-Battle at n, or there is no Sea-Battle at n" is established, even in the present.[122]

NOTA BENE This is the appropriate moment at which to illustrate a point that we have stressed already: this particular solution to the Sea-Battle rests not only on Beth semantics, but also upon the internalization of the metalanguage and the completeness of the atomistic calculus. Without an internalized metalanguage, and without the supervaluation techniques allowing all origins to have access to the metalanguage of other worlds, expressions like "$\alpha(n)$" would have been meaningless, in all cases that n is different from *now* - e.g. in the theorems above, "n" denotes nothing in the substructure corresponding to *now*, although it denotes the origin of the substructures of the worlds belonging to the crucial bar. It has a sense, at present, because the present has access to the metalanguages of these latter. Beside that, the completeness of the atomistic calculus is also indispensable.[123]Consider figure 12.

[122]As for the past (contingent) Sea-Battle, the general argument is the mirror image of the argument concerning tomorrow's Sea-Battle. Theorems 7 and 9 stay as they are, and Th.8 is changed into its mirror image. What further needs to be noticed here is that, unlike future supervaluated disjunctions that will, sooner or later, stop being true by supervaluation, and become truth-functionally true/false, past supervaluated disjunctions will remain such forever. The reason is plain. Once the future converges, the world will never again make up a part of any of the branches that have been merged into one, after the convergence. R sees to that, for it disallows wRw', when $f(w) > f(w')$. Intuitively, this gives justice to the claim that, once all the singularizing effects of a certain state have been annihilated, no evidence of that state will ever appear again. Cf. 0.2.

[123]Cf. chapter 3, p. 64.

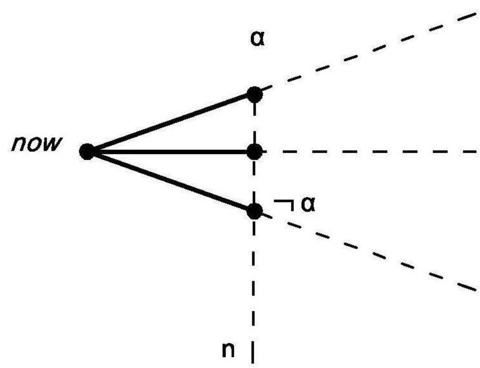

Figure 12.

It represents a model with an incomplete atomistic calculus. The claim that either α obtains at n, or $\neg\alpha$ obtains at n, is not true *now*, nor will it ever be, upon the middle path. Less formally, and with respect to the Sea-Battle, this says that the Aristotelian argument looses its vigor in contexts of vagueness. Aristotle has not raised the problem of "today's Sea-Battle"; vagueness is not his concern in *De Interpretatione* 9. Whether or not there is *now* a Sea-Battle going on has to be assumed as effectively decidable by classical methods, and, more importantly, it needs to be so, in order to load the argument solving the problem about tomorrow's Sea-Battle. For, if it isn't decidable, nothing excludes the possibility that the matter remains undecided forever, and, so, one would have no grounds to claim the actual truth of the disjunction. It is the present's (every present's) completeness that creates the bar for "$\alpha(n) \vee (\neg\alpha)(n)$".

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S-models are, with respect to their indexed sentences, intuitionistic. One of the main intuitionistic features of them is the way they falsify the Excluded Middle. The counterexamples provided are similar to the counterexamples one encounters in Beth models.

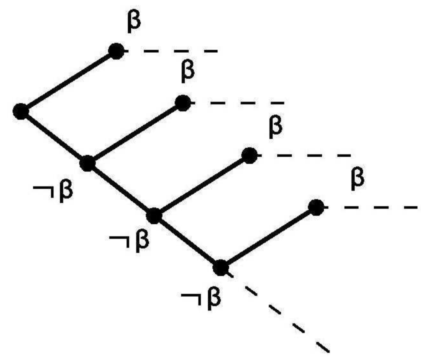

Figure 13.

Metatheorem: $\alpha \vee \neg\alpha$ is not valid in S-models.

Proof. Figure 13 depicts an appropriate counterexample. Take $F\beta$ for α, and assume β to be index-free. The lower path is a path, where β is never forced. Since β is index-free, we conclude, by 4.5.1.1, that $\neg\beta$ is forced throughout the path. On the other hand, the worlds of this path are assumed never to stop having access to worlds where β is forced. It follows, from 4.5.1.2.(6) and 4.5.1.2.(9), that neither $F\beta$, nor $\neg F\beta$ are forced upon the crucial path, and, so, that there is no bar having $F\beta \vee \neg F\beta$ throughout. By 4.5.1.2.(2), we conclude that $F\beta \vee \neg F\beta$ does not belong to the origin. □

NOTA BENE Notice that in Th.7-9 the Excluded Middle was not falsifiable but the Disjunction Property did not hold.[124] In the Metatheorem, it is the converse that happens. We may generalize this observation, by saying that wherever the Excluded Middle is not falsifiable, the Disjunction Property fails, and *vice versa*. For the Disjunction Property of intuitionistic logic holds good in that logic, because the same logic does not accept the Law of Excluded Middle, and, as a consequence, excludes the possibility that an instance of the "Law" can be taken as established, without necessitating the proof of one of its disjuncts. On the other hand, the classical logician is more "prolific" with respect to theorems, exactly because he allows an extra tool into his apparatus: the Excluded Middle, which, in turn, allows him to prove any of its substitution instances. Hence, he dismisses of the Disjunction Property. S-models, and with respect to that matter, stand midway between classical and intuitionistic logic. For as for sentences like, "There will be a Sea-Battle in the future" the Excluded Middle can be falsified, and, consequently, the Disjunction Property holds good, whereas, for some indexed sentences, like "There will be a Sea-Battle at n", it is the Law, which holds good, and the Property, which can be falsified.

[124] Since the theorem concerned a certain specific S-model, it is the *local* Disjunction Property that did not obtain in Th.7-9. However, in S-models, and unlike Beth models, the *non local* Disjunction Property fails as well. Which means that there can be within the set of all S-models valid formulas of the form $\alpha \vee \beta$, at the same time that neither α nor β are valid in the same set. The reason is presented in 5.2.

III

We proceed here in a comparison (with the claim of correspondence) of the informal properties of the ideal agent of S (here called *Ideal Scientist*) of chapter 3 with the formal properties of S-models.

As we said, at the end of chapter 3, the thought experiment involving the *Ideal Scientist* corresponds to a formal model, which, unlike the experiment itself, bears no conscious-centered connotations. In part III of chapter 3, we have presented the conditions of the experiment performed by the ideal agent of S. These conditions correspond to:[125]

1) f, and R guarantee that Time advances monotonically towards the future. No world w sees any world w', such as $f(w) > f(w')$.[126] Therefore, Time will never go backwards. The fact that $f(w)$ might be the same as $f(w')$ cannot stop the "monotonic advancement of Time towards the future", because of the bar device for the evaluation of indexed sentences. In fact the bar devise is the formal counterpart of the proviso that the *Ideal Scientist* will pass from all future moments and cannot thereby stay at the same node forever (see 2 *infra*).

2) The fact that the *Ideal Scientist* will necessarily arrive at all future moments and has, in the past, been at all past moments, is guaranteed by the bar general device. What the bar device formalizes is the intuition that the *Scientist* cannot stay in the same world forever.[127] For, if he could, the fact that some indexed sentences are true in a bar would not have implied that the same sentences are also true in the actual world. The device gives formal justice to the intuition that the *Scientist* will move on and eventually be located upon a world of the bar, where he will validate the sentence by a (classical) truth-functional reasoning. Since he already knows that, he can assert the sentence already. The fact that there is no moment that will be left "unvisited" is guaranteed by the fact that the bar condition obtains universally - i.e. it obtains for all bars. Therefore, since all sets of worlds sharing a rank in common are bars, it follows that the *Scientist* will pass from all intermediate moments between his occasional present and any future destination. (Otherwise, the possibility might be that he could "skip/jump over" some moments.) Finally, discrete Time comes from f and R (cf. 4.3.12). Notice, however, that the possibility that the *Scientist* will never reach a certain moment that lies ahead can be excluded, even if Time is continuous (cf. 5.5). Another thing that needs to be stressed is that the semantics of S-models answer a possible objection to the legitimacy of the intuitionistic solution that we raised back in chapter 3. This objection targeted the

[125] The present numbering corresponds to the numbering of the conditions of chapter 3. Conditions (A) and (B) are not taken into account, because by assuming that the *Scientist* knows all logico-mathematical truths there are, we have already postulated that he is capable of infinite acts of intuition, so (B) is no longer necessary; we can -e.g.- assume equally that he can simultaneously grasp an infinite amount of premises. (A) is covered by (iii) of page 117.
[126] Remember that f is the "clock function" assigning to each world a moment. Cf. 4, 4.2.
[127] As he can do in the "antagonistic" Kripke (intuitionistic) models.

subject (conscious-)centered framework that intuitionism is usually affiliated with. It proclaimed that the "intuitionistic solution", in fact, argues in favor of epistemic indeterminism. And it also proclaimed that, according to this solution, it is not the (future/past) state itself that is undetermined, but our cognitive shortcomings with respect to it. What the (intuitionistic) formalism of S-models has revealed is that the presentation of the solution can be brought home without any reference to either the *Ideal Scientist*, or, any other cognitive being. The model is geometrical throughout: any locution of the form "the agent, being at present at moment ..., will be/has been at moment —" has been replaced by a locution of the form "— constitutes a forward/backward bar for ...". The exposition is no longer anthropomorphic.[128]Therefore, and since this presentation can go all the way without adopting any anthropomorphic terminology, conscious beings are not indispensable for the intuitionistic solution itself or for the *Weltanschauung* this solution supports. The model will still make sense, without any thinking individual living in it. The only way for one to claim that conscious beings are indispensable for the intuitionistic solution is to argue that even the geometrical device of points and bars presupposes the notion of a series (cf. the paths within the S-model), and that this notion is intrinsically related to Time and -this is crucial- that Time concerns the way we, human beings, are condemned to approach reality; i.e. that Time does not refer to the world *per se*. This interpretation of mathematics excludes indeed any possibility that the intuitionistic solution can ever be detached from the thinking individual, but it does so for a trivial reason. The only reason that, according this interpretation, the intuitionistic solution is conscious-centered is that every solution is. For, according to the same interpretation, all solutions assume the notion of a sequence (series), which presupposes Time, and, which, in turn, presupposes the thinking individual and its activities. This radical interpretation put aside, one can abandon any presentation that takes the model to be a function from moments of Time to possible cognitive contents of the *Ideal Scientist* in favor of a presentation that takes the model to be a function from $\mathbb{N}(+/-)$ towards consistent sets of sentences.

3) Infinity in both directions is imposed by the function f being from $\mathbb{N}(+/-)$ and not from a finite subset of it. This parameter is indispensable if one wants S-models to be intuitionistic. Finite Beth models are classical: every classical theorem is valid in them, and so is the Excluded Middle.[129]

4) That the *Ideal Scientist* has a full grasp of the events of his occasional present corresponds to 4.5.1.1, which guarantees that the set of sentences containing no reference to (other) worlds is maximal. Now, if some of these sentences are also antecedents of true causal conditionals, the consequents of these latter will also be contained in the present, no matter whether they refer to the future or not. This says that 4.5.1.1 not only guarantees the present's decidability, but also allows for some portion of the future or past to be

[128] Cf. E. BETH, [1956]. In the last parts of his paper, Beth cites some remarks of Heyting, according to which the subjective element vanishes by the introduction of choice sequences and the trees of the models - p. 384: "By means of this terminology every concept, which does not involve 'the notion of time and a subjective element that do not belong (in) mathematics' ⟨citation from Heyting⟩ can be expressed."

[129] Cf. D. VAN DALEN, [1986a], p. 251.

determined as well. Limiting cases of this "allowance" are deterministic universes, where not only the present, but also the future and the past are fully determined. Converse limiting cases are universes containing only self-caused events. In these, nothing but the present is determined.[130]

5) The fact that the *Ideal Scientist* knows, at any moment, what time it is, comes from the seventh element of the septuple - i.e. the function "now" assigning the object "*now*" to a world of W. The rank of *now* can be reached by the application of the f function to this world. This answers the "what time is it?" question. It is important to notice here that, since we have preferred the non-atemporal approach towards sentences, the answer to the question about the rank of the actual world is always given during a certain moment - i.e. during the moment denoted by the rank. The whole elaboration/presentation of the model is always performed from within. This says that the occasional "universal shape" of the world provides sufficient evidence, in order to determine the actual origin,[131] and, *a fortiori*, provides the rank of the same origin as well - i.e. the occasional actual time.[132]

This concludes the juxtaposition (with the claim of correspondence) of the conscious-centered thought experiment of chapter 3 and the conscious-free presentation of this chapter.

***

As we said in chapter 3, one of the very few ways one can claim that the absence of a state from within a world does not imply the presence of its negation, without being immediately shown into the logician's asylum, is to build a time horizon into the logic, and to claim that worlds are moments in time and that matters concerning the presence/absence of states are not always decidable *in situ*. Our S-models do exactly that, by further elaborating some old

[130] Michael Frede was of the opinion (graduate seminars 2004-07, Athens University), that these latter limiting cases are the ones providing the appropriate picture for the Aristotelian sublunary universe, and, consequently, that the Sea-Battle type of events contains all events of the same universe. In fact, Professor Frede assumed something stronger still. According to his interpretation, the notions of causal conditional and causal chain are quite posterior to Aristotle's text - see also M. FREDE, [1980]. It is obvious, however, that our model speaks neither against, nor in favor of this possibility. In order to depict Frede's intuition in an S-model, just assume that the set of causal conditionals assigned by V to the elements of W describes only self-caused events - i.e. degenerated causal chains of one element. Notice that, from the extensional point of view, Histories governed by (i) a total absence of causal chains or (ii) only by degenerated chains of a unique element are exactly alike; the difference is intensional. Notice, also, that neither a total absence (senselessness) of causal conditionals, nor a "monopoly" of self-caused events invalidates Th. 7-9. The solution to the fantasy is based on present's completeness, not on the presence or absence of causal conditionals.

[131] This comes from 4.4.2. See also Appendix III.

[132] Notice, however, that the condition that the occasional "universal shape" of the world provides sufficient evidence, in order to determine the origin, is stronger than the condition that the *Scientist* always knows what time it is. For the last condition to be met, it suffices that the occasional "universal shape" determines the rank of the origin, which, in turn, allows that the scientist mistakes the real (actual) substructure with one having an origin with the same rank. In other words, for the fantasy to be countenanced, it suffices that the scientist knows that tomorrow is open both for the advent and the non-advent of the Battle. It is not necessary that he does not mistake the actual *now* with another world of the same rank.

intuitionistic devices. As a matter of fact, Beth models have been proved to be special cases of S-models (cf. 4.8), and so S-models are not only models aiming at the mathematician and the mathematical universe he works in, but, more generally, models aiming at the activities of any scientist, and our universe, in general.

Chapter 5

Comparing Intuitionistic with "Classical" Supervaluation

In this chapter, I proceed to a general comparison of the intuitionistic with what I call "classical" superevaluation, which, in fact, refers to Thomason's adaptation of Van Fraassen's supervaluation method, so as to fit it to temporal logic models. In the first part, I present a relative advantage intuitionistic supervaluation has over the "classical", with respect to unaccomplished prophecies, and in general, with respect to the way one evaluates all kinds of predictions. In the second, I juxtapose the two methods, with respect to their central points of divergence - i.e. Tarski Schema, and the metalanguage they use. In the third, I further elaborate their basic differences, by insisting on the fact that while supervaluation classes of "classical" supervaluation are built upon classical valuations, intuitionistic supervaluation classes are built upon supervaluation classes. I try to show why this characteristic enriches the expressive capabilities of S-models.

I

Throughout this treatise, I have been advertising a new solution to the Sea-Battle paradox. This solution proceeds within a variation of Beth models for intuitionistic logic. In Beth models $\neg, \vee, \rightarrow$ are not evaluated on the spot but depend on bars. In order to make Beth models competent to treat cases like the Sea-Battle, I have made some adaptations, but the core of Beth's intuition has remained the same. The key to my results is that future bars allow, 'There will be a Sea-Battle tomorrow or not' to be true, while neither among, 'There will be a Sea-Battle tomorrow' and, 'There will be no Sea-Battle tomorrow' has any truth-value. To spell it out: future bars provide supervaluation for these sentences, and thereby do justice to Aristotle's "fantasy".

An obvious question -if not criticism- concerns the novelties involved in this semantics, for it is not the first semantics to provide a superevaluative way out of the problem. Bas Van Fraassen has pointed towards,[1] and Richmond Thomason

[1] B. VAN FRAASSEN, [1966], esp. pp. 493-495, [1968]. Van Fraassen's supervaluation has, in the years that followed, gained a normative status. We will be referring to his supereval-

has elaborated[2] a logical framework to the same effect. Indeed, what Thomason's supervaluation solution advertises is a solution kindred to our own. 'There will be a Sea-Battle tomorrow' and, 'There will be no Sea-Battle tomorrow' represent truth-value gaps, whereas their disjunction is true.

Hence the question: what is new in our model? Or, to put it more aggressively: how can there be any need for S-models?

We are going to address this question step by step.

First, let me show that the logic underlying these two models is indeed different. According to Thomason's semantics a sentence is true in w, if, and only if, all maximal linear orders passing through w assign the same truth-value to the formula. Now, having in mind that these maximal orders stand for complete linear Histories, and that the logic underlying these latter is classical, consider the following situation. Several maximal Histories traverse *now*. One of these has no Sea-Battle in it, but it endlessly traverses Histories, which do have such an event upon their occasional future. In other words, although there is no such Battle in that particular History, the fact of this absence is not definitely settled for any future moment.[3] Now, Thomason's semantics give the truth-value True to the disjunctive sentence, 'Either there will be a Sea-Battle in the future, or not'. This is because since all maximal linear orders traversing *now* represent classical valuations, one of the disjuncts (perhaps not the same one) has to be true in all of them. Consequently, their disjunction is true at present.[4] The above result exemplifies a much stressed feature of "classical" supervaluation, according to which, classical bivalent logic and supervaluation share almost the same body of theorems, and exactly the same set of valid arguments.[5] The Excluded Middle is definitely not among the exceptions, and, 'Either there will be a Sea-Battle in the future or not' is an instance of it.

We now come to the first difference between these two superevaluative methods. According to our semantics, this last disjunctive sentence is, in the world just described, neither true nor false.[6] The reason is that the truth/falsity of it depends on the presence of a bar, upon which it is true/false throughout. This shows also that the foundations of our semantics are what they were meant to be - i.e. intuitionistic. Intuitionistic logic does not subscribe to the Law of the Excluded Middle. The fact that it does subscribe to it, as far as unquantified

uative technique as "classical" supervaluation. The reason for this is not only its normative status, but also that it shares with classical logic almost the same body of valid arguments and theorems. See *infra*.

[2] R. THOMASON, [1970], [1984]. Chapter 5 contains no detailed presentation of the articles of this or of the previous note. Since its topic concerns a comparison between the supervaluation method of these and of my own, the reader is invited to consult them first, before going on through this chapter.

[3] Cf. figure 13 of ch. 4.

[4] Cf. R. THOMASON, [1970], pp. 274, 277. See how the valuation of moments depends upon the *classical* valuation of linear Histories passing through these moments.

[5] This implies that there is, in "classical" supervaluation, no 1-1 correspondence between valid arguments and theorems. Cf. B. VAN FRAASSEN, [1966], pp. 484ff. The question, which are the valid arguments that have no corresponding theorem should not concern us for the moment. They occur in the Predicate Calculus with non-denoting terms, and concern also the Tarski Schema.

[6] Cf. ch. 4, p. 119; the Metatheorem.

indexed states[7] are concerned, is but the benevolent effect of the metalanguage that makes up part of the model, and the decidability of atomistic and metalinguistic calculi.[8] Nonetheless, since one cannot bar the future for all quantified indexed states, the Law fails.

Let us here pause for a moment and state this -first- major difference between the two systems. Although it is the case that each supervaluated compound sentence of our model is also supervaluated in models of "classical" supervaluation, the converse does not hold. Each time there is a bar for the compound sentence in the S-model, the translated sentence into the "classical" model -if such translation there is-[9] will acquire the same value. Nonetheless, it is not the case that, each time a compound sentence acquires a value by supervaluation in the "classical" model, there is a bar for the translation of that sentence into the S-model. Therefore, it cannot be supervaluated intuitionistically.

There is no need to stress the obviousness of the following question: should one be happy, or should one be sad about this difference?

As in many similar situations, the answer to that question cannot be laconic. It eventually comes down to one's own logical taste and credo. And, therefore, a lot of Philosophy of Logic has to be called into the discussion. Once this done, the discussion can but be lengthy, complicated, and certainly beyond the purposes of the present treatise. Nonetheless, one can at least try to specify the parties involved in it.

If one agrees with Van Fraassen and, "with respect to logic, is conservative",[10] the falsification of the Excluded Middle cannot but distress him. And, since "classical" supervaluation does not go against the Law, whoever is "conservative with respect to logic" should -in principle- prefer "classical" supervaluation. On the other hand, if one does not believe in the Law, and, moreover, if one does not so believe for the same reasons as intuitionists, one cannot but rejoice in this difference; and, in such case, one should indeed prefer the intuitionistic model over the "classical" model.

In either case, the difference *per se* can provide one with no conclusive argument in favor of any among the two supervaluation methods. That is, of course, if we have no conclusive evidence in favor either of classical or of intuitionistic logic. And the chances are that -for the present moment at least- the community of logicians does not have any such evidence.

Allow us here a parenthetical remark. By saying that the intuitionistic way out is *not equivalent* to the way out of "classical" supervaluation, we have (and to a certain degree) wronged the former. For, in the way we have presented things, one might stay with the impression that:

(i) classical logic provides a supervaluation solution to the problem,

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[7] An "unquantified indexed state" is a state represented by a non quantified indexed formula. A "quantified indexed state" is a state represented by an indexed, but quantified formula. Cf. ch.4; 4.5.1.2.

[8] Cf. ch. 4, p. 117, *Nota Bene* and 4.8.(2).

[9] As we will see later, the intuitionistic model has richer expressive resources.

[10] B. VAN FRAASSEN, [1968], p. 140.

(ii) intuitionistic logic provides a supervaluation solution to the problem,

(iii) these two solutions are not equivalent.

The above impression is true only in part, and this is because, strictly speaking, classical logic provides no supervaluation solution to the problem in the first place. That we call the Van Fraassen-Thomason supervaluation "classical" should not indulge us in thinking that there is a certain solution to the problem taking place within a framework, which is both classical and superevaluative. Supervaluation, rather, is a certain amendment of the classical framework that has been proposed in order to allow logicians to treat some old puzzles in new ways, among which is the Sea-Battle. Otherwise, there is nothing classical in "classical" supervaluation, except that it shares almost the same body of valid arguments and theorems with the classical framework. Evidence for that is the very name, with which Quine baptizes Aristotle's disputed claim - i.e. "a fantasy". Adopting the "fantasy" is already a departure from classical logic and its intuitions. On the other hand, intuitionistic supervaluation belongs to a certain logical framework, which assumes gaps -so to speak- "by nature".[11] Gaps were not added to Beth models in order to bring a solution to the Sea-Battle. They were there right from the start, and this solution to the Sea-Battle problem was a latent application of them all along.

The point of this parenthetical remark is that, although "classical" supervaluation emerges only when one has left the classical framework behind, intuitionistic supervaluation comes in more naturally to -so to speak- "intuitionistic orthodoxy". Seen thus, intuitionistic logic is more familiar an environment for supervaluation and, consequently, for this particular solution to the problem.

After this parenthesis, the time has now come to present what strikes me as a slight advantage of the intuitionistic over "classical" supervaluation.

Intuitionistic supervaluated models differ from "classical" supervaluated models with respect to the following parameter. If there is, in a no backward branching S-model,[12] a bar for an indexed sentence, the sentence comes out true in the S-model, and its translation -if any- comes out true in the "classical" model as well. This is because the presence of the bar in the S-model settles the truth-value of the sentence for all the free choice sequences constituting the spread that the model essentially is. Therefore, all maximal linear orders passing through the origin assign the same truth-value to the sentence, and, consequently, the sentence has the same truth-value in "classical" supervaluation too. However, it is not the case that all supervaluated sentences of "classical" supervaluation are supervaluated in the intuitionistic model. An instance of such a sentence is, "Either there will be a Sea-Battle in the future or not". "Classical" supervaluation assigns truth to the above disjunction, while, intuitionistically, one cannot assign any truth-value to it, when there is a future path where neither the event, nor its absence ever becomes established. The above semantical feature of intuitionistic models makes them more apt to evaluate prophecies, bets

[11] *Pace* some early intuitionistic semantics, where gaps were filled by other values, or taken to be false.

[12] This proviso is added here, because there are no backward branching models in Thomason. See *infra*.

and predictions in general.

The intuitionistic framework is better equipped for these things on two accounts.

We begin with the first.

Suppose that a very optimistic (or very opportunist) Greek politician utters today the following sentence: "Some day Greeks and Turks will be really fond of one another". Now, the chances are that History follows a path where this never happens. Nonetheless, since human beings are human beings, and their reactions are sometimes radically unpredictable, one might also claim that, even if History *does* move along that path, the possibility that Greeks and Turks will someday love each other is never annihilated. If so, what about the actual truth-value of "Some day Greeks and Turks will be really fond of one another"? Both supervaluated models will take the sentence to be truth-valueless. What differentiates them is that "classical" supervaluation assigns truth to "Either Greeks and Turks will be really fond of one another or not", while, intuitionistically, this is as truth-valueless as the disjuncts. Now, let us see how this formal difference can be made to work in favor of Intuitionism. We have assumed that, upon this same path, Greeks and Turks never do actually come to terms. To the question, "When is it that what the above politician has prophesized becomes true upon the path?", the answer is quite satisfactory: never. On the other hand, to the question, "When is it that the prophecy is falsified upon it?", one gets, in the "classical" model, a somehow frustrating answer; the answer is again: never. This does not seem to do justice to the commitment of the underlying logic with respect to the truth of "Either Greeks and Turks will be really fond of one another or not". If one takes this commitment at face value, one should expect that there is, in each future path, a moment when the above prophecy becomes either established or falsified. But, in the above-mentioned path, there is no such moment. Of course, "classical" supervaluation arrives at asserting the disjunction because it does not take this commitment at face value. It arrives at this conclusion after having allowed quantification over infinite totalities - i.e. the totality of all moments of that (or indeed of all) future paths. After having posited these totalities, and after having quantified over their elements, one can easily argue that there is either a moment at which the prophecy becomes true or, that there is no such moment. But still no person who inhabits this earth will ever be in a position to establish either of the two, as long as History follows the future represented by that particular path. On the other hand, our intuitionistic supervaluation has no such worries. This is not because it establishes falsity somewhere along that path - that would have been even more problematic; it is because the same disjunction is never true upon this path. Notice, here, that the very fact that, according to intuitionistic logic, 'Either Greeks and Turks will be really fond of one another or not' is, on the occasion, truth-valueless, is no *ad hoc* stipulation of the formalism for overcoming the problem. It is the result of (i) how negation and disjunction are to be understood intuitionistically,[13] and of (ii) the intuitionistic *veto* in quantifying

[13] "Having established the negation of p" means, proof-theoretically, that we can turn each proof of p into a proof of a formal contradiction.

over infinite totalities. In an intuitionistic model, the fact that the prophecy cannot be falsified is obedient to the general principles of the underlying logic - problems would have emerged only if the prophecy *could* be falsified. This, once more, speaks in favor of the hypothesis that intuitionistic frameworks are more a natural niche for indeterministic Histories, than classical ones. Notice, finally, that Thomason is well aware of the particular problem that "classical" supervaluation encounters:

> An objection to this[14] is that the method does not serve to falsify future tense statements, since time has no end. Most predictions we make, however, have a time-limit built into them, implicitly or explicitly. Those that do not have an oracular flavor. At any rate, I do not mean to say the method is always satisfactory in practice, only that it is the definitive method; as far as the evaluation of predictions goes, there is no appeal from its results.[15]

I will now turn to the second point with respect to which intuitionistic supervaluation strikes me as superior as far as predictions are concerned.

The following proposal concerns a formal device that will enable us to evaluate prophecies and bets in intuitionistic settings. The general problem for such linguistic acts emerges if he who wins the bet, or he who claims the accuracy of his prediction, is considered as doing so on the basis of the truth-value that what he said (prophesized) had at the moment he said it. Now, at that moment, and if the prediction referred to a future indeterministic state, it had no truth-value. Thomason writes about such occasions:

> However, suppose I make a prediction that comes out true: On Friday I say it will snow the next day, and on Saturday it snows. We then say (on Saturday) that what I said was true.[16]

Yes, the prediction comes out true on Saturday, but it is true from Saturday's perspective. The Oracle, however, claims something stronger. He cannot pride himself only with the fact that his prophecy will be true if evaluated on a snowy Saturday. This is something that even one who disbelieves in what the Oracle says, unquestionably accepts. There is no much guesswork involved in saying that what he said will be true in such a context. What the Oracle is claiming is not only that his prophecy will turn out true if evaluated in that particular context but that this will be the real context in which his prophecy will be assessed. And this is a further claim that will be evaluated in what will come to be the context of tomorrow, and will, retrospectively, acquire a truth-value, though it lacks one today, and so on and so forth. The same obtains

[14] See the following note.

[15] R. THOMASON, [1970], p. 269; n. 6. The target of Thomason's criticism is here an intermediate version of his system. In that intermediate version, the evaluation of sentences depends not only upon the moment the sentence is uttered, but also upon a subsequent moment - e.g. for future indexed prophecies, it depends upon the moment the index denotes. Nonetheless, the same problem persists in Thomason's ultimate version. More precisely, the final version of his method encounters the following variation of the same problem: Consider, again, what the Greek politician predicted. Prophecies with such an "oracular flavor" can be falsified with respect to some maximal Histories taken as a whole, but not with respect to *any* of the moments these maximal Histories are composed of. At the same time, the truth of the Excluded Middle is asserted everywhere.

[16] Ibid, p. 278.

for the sequence of predictions: "It will snow on Saturday", "'It snows' will, on Saturday, be true", etc. To cut a long story short, what the Oracle asserts is either a triviality,[17] or it asserts that what it says is true, even for today; which is impossible in "classical" supervaluation. If the prophecy refers to an indeterministic future state, what the Oracle says cannot be true on Friday. The same thing is -*prima facie*- equally impossible in intuitionistic supervaluation. But, with some adjustments, this latter can offer the following way out of the problem. By exploiting the nature of intuitionistic implication, one can take prophecies of the form "p, at n" as abbreviations of the bi-conditional "The present prophecy is true, if, and only if, p at n". The classical (i.e. material) implication employed by Thomason, in combination with the fact that his metalanguage is classical,[18] will falsify the bi-conditional, if 'p at n' is truth-valueless today. The bi-conditional belongs to the metalanguage, and since this latter is classical, it cannot be true, if 'p, at n' is truth-valueless. On the other hand, intuitionistic implication allows the sentence to be true, under the condition that it is never the case that one of the two sub-sentences is untrue while the other is true. This relaxation of what counts as a true implication suffices for removing the problem. We can now have a rigid criterion for judging whether or not a prophecy has been fulfilled. For, if the linguistic community agrees that these are the terms on which the prophecy is to be judged, the community can only accept the accuracy of what the Oracle said, if, at n, p strikes the community as having the truth-value True. And this will be because what the Oracle said was true at the time he said it; the bi-conditional into which we have agreed to analyze the prophecy, was (back then), made out of two gaps. On the other hand, the Oracle cannot but concede that his prophecy is false, once 'p at n' becomes false. *Mutatis mutandis*, the same applies for bets. The self-referential speech act: "The present bet is won, if, and only if, p at n",[19] and unlike, "p at n", gives justice to the victor of the bet, even when this last sentence is untrue, when uttered.

Of course, a similar device can be applied to "classical" supervaluation as well. It suffices to transpose this self-referential speech act from material implications to, what Thomason calls, "semantic consequences":[20] the relation holding between premises and conclusions. If he who utters the prophesy is ever to be in a position to claim its accuracy, the previous situation has to be analyzed as follows:

The arguments (p at $n \vdash$ The present prophesy is true) and (The present prophesy is true $\vdash p$ at n) are valid.

In "classical" supervaluation, "semantic consequence" does the work that plain implication was doing in the intuitionistic context. The reason for this is that "semantic consequence" corresponds to what Van Fraassen calls "ne-

[17] i.e. that his prophecy is true, if the context of evaluation is a snowy Saturday.

[18] There is no supervaluation in this metalanguage. See part II.

[19] "Won" plays here the role of the metalinguistic predicate referring to the sentence it belongs to. The whole sentence *is* the bet.

[20] R. THOMASON, [1970], p. 273: "Although I'm not entirely happy with this terminology, I'll use the term 'implication' to speak of 4.1, so that A implies B if and only if $\Vdash A \supset B$, and 'semantic consequence' to speak of 4.2, so that B is a *semantic consequence* of A (or just a *consequence* of A) if and only if $A \Vdash B$."

cessitation",[21] which, in turn, corresponds to intuitionistic implication - i.e. necessitation, semantic consequence, intuitionistic implication represent (inter-logically) the same relation. They all three demand that if the sentence(-s),[22] representing the first item of the relation, is (are) true the sentence standing for the second item must be true as well.

Having now adjusted this particular device for dealing with bets and prophecies to "classical" supervaluation, we, once more, cannot resist the temptation of suggesting the intuitive superiority of the intuitionistic framework. Our device can serve as a way to evaluate the truth of predictions about future indeterministic events in both supervaluation frameworks. The relative superiority of the intuitionistic framework consists in that intuitionistic implication allows one to state the conditions under which the prediction comes out true in a more straightforward/economic way. And, here again, it is not that we have adjusted *ad hoc* the intuitionistic implication so as to meet these standards. This is the "orthodox" intuitionistic semantic interpretation of implication. It says that in no possible world is it the case that the antecedent is true and the consequent untrue. And, moreover, the above semantics is made to correspond to an older still proof-theoretic intuition, according to which $p \to q$ is intuitionistically established, if, and only if, we posses of a construction turning each proof of the antecedent into a proof of the consequent.

The temptation is, again, *ante portas*. One might think that this application of intuitionistic implication testifies in favor of the hypothesis that intuitionistic logic is the proper domain for treating future contingents.

II

The relative advantages of intuitionistic over "classical" supervaluation that we have presented in the first part were consequences of two main features of the S-model. First, they were due to the internalization of the metalanguage, and second, to the characteristics proper of intuitionistic implication.

These two particularities of S-models will be shown, in this part, to have also a -so to speak- "conservative" formal consequence: Because of the way intuitionists understand implication, and because of the internalization of the metalanguage, S-models become much more "traditional" than the corresponding models of "classical" supervaluation when the discussion comes down to the Tarski Schema and the Law of Bivalence. This topic is quite illuminating as to what separates the two approaches, and so we will examine it in more detail.

In the following tableau we juxtapose intuitionistic and "classical" supervaluation with respect to their fundamental semantic features:

[21] B. VAN FRAASSEN, [1968], p. 138: "*A necessitates B* if and only if, whenever A is true, B is also true." See also part II.

[22] The plural has sense only in the case of semantic consequence.

Classical vs. Intutionistic Supervaluation

		"Classical" Supervaluation	Intuitionistic Supervaluation
1	Excluded Middle for the object language: $p \vee \neg p$	**Yes**	**No** (Counterexample): "Either there will be a Sea-Battle in the future or not".
2	Excluded Middle for unquantified indexed states: $p(\ldots\vec{n}\ldots) \vee \neg(p(\ldots\vec{n}\ldots))$	**Yes**	**Yes**
3	Excluded Middle for the metalanguage of any level: e.g. $T(p) \vee \neg(T(p))$	**Yes**	**Yes** since metalanguage of level 2 is bivalent.
4	Law of Bivalence: $T(p) \vee F(p)$	**No**	**No** although, it does obtain with the formulation: $(Tp)(\ldots\vec{n}\ldots) \vee (Fp)(\ldots\vec{n}\ldots) \leftrightarrow (p)(\ldots\vec{n}\ldots) \vee (\neg p)(\ldots\vec{n}\ldots)$
5	Law of Bivalence for unquantified indexed states: $T(p(\ldots\vec{n}\ldots)) \vee F(p(\ldots\vec{n}\ldots))$	**No**	**No**
6	Law of Bivalence for any level highter than 2: e.g. $T(T(p)) \vee F(T(p))$	**Yes**	**Yes** since the metalanguage of level 2 is bivalent.
7	Tarski Schema: $T(p) \leftrightarrow p$	**No** (Counterexample): Consider the distribution: truth-value gap for p, and the truth tables for material implication.	**No** although, it does obtain with the formulation: $(T(p))(\ldots\vec{n}\ldots) \leftrightarrow p(\ldots\vec{n}\ldots)$

We will begin with 7. The treatment "classical" supervaluation reserves for the Tarski Schema lies at the bottom of how this method interprets the Law of Bivalence. Since (i) "classical" implication is the material implication, and since (ii) the metalanguage of "classical" supervaluation does not allow for any supervaluation, it is straightforward that T-Schema does not hold in Thomason. The fact that, "There will be a Sea-Battle tomorrow" represents a gap, excludes that the bi-conditional is true. No matter whether we assign truth or falsity to, 'It is true that there will be a Sea-Battle tomorrow',[23] the co-implication will not obtain, if, 'There will be a Sea-Battle tomorrow' is truth-valueless. Now, the fact that T-Schema does not obtain in "classical" supervaluation has two benevolent side effects. The first is that it drags along with it the Law of Bivalence. There are at least two sound arguments that show how one can deduce bivalence if the Tarski Schema is allowed into the system.[24] We have just presented a third, using truth tables. In more detail: Assume that, (i) 'There will be a Sea-Battle tomorrow' is true, if, and only if, there will be a Sea-Battle tomorrow, and (ii) that this (metalinguistic) bi-conditional is to receive a classical evaluation.[25] From (ii), it follows that the co-implication is true, if, and only if, both sub-sentences are true, or both sub-sentences are false. This criterion, however, is never met, if one of them is to stand for a gap. Therefore, either, 'There will be a Sea-Battle tomorrow' is true, or, 'There will be a Sea-Battle tomorrow' is false. On that account, the fact that the Tarski Schema does not obtain saves the system from collapsing into bivalence. In other words, an object language with gaps is incompatible with (i) a metalanguage with two truth-values, (ii) without supervaluation *and* (iii) with the Tarski Schema. One of the above three has to go, in order to allow non-bivalence in the object language. So, by abandoning the Schema, one allows for the combination of a non-bivalent object language with a bivalent metalanguage.

The other benevolent effect of abandoning the Schema is that sentences stop -as Thomason puts it- "implying their own inevitability".[26] To put it plainly, since, in Thomason's system, "being true in w", corresponds to, "being the case in all maximal linear orders passing through w", and, since "being inevitable in w and (the random) History h" corresponds to the same thing, inevitability "expresses the property of being true".[27] Thus, the validity of T-Schema would not only imply bivalence, but it would also imply that, "There will be a Sea-Battle tomorrow" implies (*via* its supposed truth) the inevitability of tomorrow's Sea-Battle.

These formal results invite Van Fraassen and Thomason to dissociate the Law of the Excluded Middle which obtains in their systems, from the Law of Bivalence, which does not. They also -i.e. these formal results- allow Thomason to give a formal account of the intuition that it is another thing to claim that something -i.e. a state- is (under the assumption of it), inevitable, and to claim that it materially implies its own inevitability.

[23] We cannot assume it to be truth-valueless, because it belongs to the metalanguage, which is classical.

[24] B. VAN FRAASSEN, [1968], p. 145; S. HAACK, [1996], pp. 67-68.

[25] The fact implying the second proviso comes from the fact of the metalanguage being classical for both Van Fraassen and Thomason. Cf. B. VAN FRAASSEN, [1968], p. 144, where it is clarified that sentences about the truth-values of sentences are either true or false.

[26] R. THOMASON, [1970]; ch. 6.

[27] Ibid, p. 275.

Far from wanting to challenge these sound results, I want now to call attention to some features of the intuitionistic model that, although it accepts (under some interpretation), the Tarski Schema, it neither implies bivalence, nor is it forced to accept that something can imply its own inevitability.

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We begin, by elaborating the first of the above two claims: The Tarski Schema is, under some interpretation, valid in S-models.

The Schema is not valid in its classical formulation:

(1) $T(p) \leftrightarrow p$

First of all, we have to observe that (1) obtains whenever p falls under 4.5.1.3.(2-3) of chapter 4. Both the atomistic calculus and the calculus of sentences, whose main connective belongs to 1.22, are classical bivalent calculi; (1) is equally valid, when p belongs to the metalanguage; the metalanguage is classical too.[28] The problem emerges for indexed p-s that describe contingent states. By contraposition on (1) -this is an intuitionistically valid operation-[29] we obtain $\neg(T(p)) \rightarrow \neg p$. Now, assume that p represents a contingent state.[30] If so, '$\neg(T(p))$' is true, but '$\neg p$' is not; it is truth-valueless. This falsifies the Tarski Schema for S-models. The remote reason for this falsification is the same as its falsification in "classical" supervaluation. The metalanguage for both systems is bivalent. Therefore, the part of the biconditional predicating truth of p has to have a truth-value, while the sentence of the object language -i.e. p itself- can be truth-valueless. This is equally disturbing whether one interprets $\leftrightarrow$ classically or intuitionistically. On the other hand, the model-theoretic reason why the metalanguage of S-models turns out to be bivalent is that we have chosen, in 4.5.2 of chapter 4, to assign to worlds of the model a substructure $W(w)$ that does not vary according to the perspective. Had we chosen to make the substructure vary according to the perspective, the Tarski Schema would have turned out valid. The price to pay would have been our inability to express the predicate of "being untrue", for, in such case, '$\neg(T(p))$' would have been truth-valueless, whenever 'p' is.[31] So, the Schema, in its standard formulation, does not obtain in S-models. However, a certain formulation that captures a great deal of the intuition underlying the Schema is valid. For what the Schema says is that something obtains, if, and only if, the sentence representing it is true. This means that, if "sometime", or "under some circumstances", or "on some occasions", or "in some universal states", a state is the case, the sentence describing it is true and *vice versa*. Now, model-theoretically, this means that, if p is valid within a model, so is $T(p)$, and *vice versa*. In S-model terms, it says that a world forces p, if, and only if, it forces that 'p' is true. Formally:

[28] Cf. 4.8.(2) of ch.4.

[29] Cf. D. VAN DALEN, [1986a], p. 238.

[30] Say that the contingent state is $\alpha(n)$. $\neg(T(p)) \rightarrow \neg p$ becomes $\neg(T(\alpha(n)), now) \rightarrow \neg(\alpha(n))$. As for why the first part of the bi-conditional has to be analyzed thus, see 4.5.1.3.(1) and Def.6 of ch. 4.

[31] Cf. ch. 4; 4.5.3.3, and also *infra* in the present part.

(2) $(T(p))(\ldots\vec{n}\ldots) \leftrightarrow p(\ldots\vec{n}\ldots)$ [32]

We remind the reader that "$(\ldots\vec{n}\ldots)$" is a world identifying-expression (constant)[33] in the substructure $(T(p))(\ldots\vec{n}\ldots) \leftrightarrow p(\ldots\vec{n}\ldots)$ belongs to.[34] What (2) says is that, if the world identified by $(\ldots\vec{n}\ldots)$ forces p, it also forces $T(p)$, and *vice versa*. (2) turns out valid, because the first part of the bi-conditional is no longer bound to have any truth-value. The crucial cases are the following: Suppose that $p(\ldots\vec{n}\ldots)$ represents a contingent event. This will be so, if "$(\ldots\vec{n}\ldots)$" determines non-effectively[35] a world, p obtains in some worlds that identify themselves by "$(\ldots\vec{n}\ldots)$", and does not obtain in other similar words. If so, '$(T(p))(\ldots\vec{n}\ldots)$' will also be truth-valueless; $(T(p))(\ldots\vec{n}\ldots)$ is an indexed sentence that has to be evaluated by the standards of 4.5.1.2.(1) and does not belong to the metalanguage of the world that $(T(p))(\ldots\vec{n}\ldots) \leftrightarrow p(\ldots\vec{n}\ldots)$ belongs to. Therefore, '$(T(p))(\ldots\vec{n}\ldots)$' can be as truth-valueless as '$p(\ldots\vec{n}\ldots)$'. In fact, '$(T(p))(\ldots\vec{n}\ldots)$' will turn out truth-valueless, if, and only if, '$p(\ldots\vec{n}\ldots)$' is truth-valueless.[36] On the other hand, if "$(\ldots\vec{n}\ldots)$" does effectively determine a world, both $p(\ldots\vec{n}\ldots)$ and $(T(p))(\ldots\vec{n}\ldots)$ will have to have a truth-value, and, more precisely, an identical truth-value.[37] The same obtains, of course, if "$(\ldots\vec{n}\ldots)$" identifies the occasional *now*.

The Tarski Schema, if stated as in (2), is valid in S-models. Contrary to that, the same -*mutatis mutandis*-[38] formulation is still non-valid, in "classical" supervaluation models. The reason is obvious. It suffices to note that what Van Fraassen does is not to overrule the intuition underlying the Schema, but to provide a non-standard interpretation for the bi-conditional. What he proposes is the following: instead of interpreting, "$T(p)$, if, and only if, p", as, "$T(p) \leftrightarrow p$", where $\leftrightarrow$ stands for the material co-implication, he proposes to read it as ("$T(p) \vdash p$" and "$p \vdash T(p)$"), where the relation that leads from antecedent to consequent is -what he calls- "necessitation".[39] But we have seen that intuitionistic implication expresses (inter-logically) the same relation with Van Fraassen's necessitation.[40] (2) is valid in S-models, because $\leftrightarrow$ is intuition-

[32] Which is equivalent to $(T((p),(\ldots\vec{n}\ldots)))(\ldots\vec{n}\ldots) \leftrightarrow p(\ldots\vec{n}\ldots)$. See note 30.

[33] Notice that it is possible that this expression identifies no world *in situ*, since it depends upon a free choice sequence; cf. ch. 4; 3.5, 5.4.

[34] In S-models, there is no external (overall) valuation. The valuation is always the valuation of the occasional *now*.

[35] Cf. Def. 8 of ch. 4.

[36] We leave the reader to check this. *Hint*: Both $p(\ldots\vec{n}\ldots)$ and $(T(p))(\ldots\vec{n}\ldots)$ belong to the metalanguage of the worlds that identify themselves by "$(\ldots\vec{n}\ldots)$".

[37] See the previous note. Consider the case where only one world of the substructure identifies itself by "$(\ldots\vec{n}\ldots)$", in combination with the fact that any metalanguage in an S-model is bivalent.

[38] There are no $(\ldots\vec{n}\ldots)$ indexes in "classical" models. For the formulation of (2) in their terms see *infra*.

[39] B. VAN FRAASSEN, [1968], p. 145: "Therefore, we will have circumvented the problem very simply by interpreting Tarski's principle not as co-implication but as *co-necessitation*". The same relation -i.e. Van Fraassen's necessitation- is called by Thomason, "semantic consequence", see *supra*. For an alternative formulation of the Tarski Schema, which uses co-implication instead of co-necessitation, and remains valid, even for "classical" supervaluation, see Appendix II.

[40] The only formal difference is that, although "$\{p,q,\ldots,r\} \vdash s$" is well-formed, "$\{p,q,\ldots,r\} \rightarrow_{(int.)} s$" is not. This problem, though, is easily amendable. It suffices to assume a translation operator (τ) from classical to intuitionistic formulas, such that

istic.

The disadvantage of the above formulation for Van Fraassen is that it expresses something that is true *about* "classical" supervaluation models, but not something that is expressible (and valid) *in* the models themselves, such as the formula $T(p)(\ldots\vec{n}\ldots) \leftrightarrow p(\ldots\vec{n}\ldots)$, which is valid *in* S-models. What (2) says is that, if a sentence belongs to a certain supervaluation class -i.e.$(\ldots\vec{n}\ldots)$- the sentence saying that this sentence is true also belongs to the same class and *vice versa*. This formulation is possible in S-models, because all supervaluation classes constituting the model are built upon supervaluation classes; the supervaluation class of the origin is a function of the supervaluation classes corresponding to all the worlds of the substructure. Contrary to this, in "classical" supervaluation, the supervaluation classes depend upon classical valuations. What comes out true in a supervaluation class is always a function of what is true in the classical valuations it depends upon. In Thomason's terms, what is true in a node α of the model is what is true in all maximal linear histories passing through α,[41] and all these are classical. Therefore, they cannot contain any sentence about any valuation of a supervaluation class, such as $p(\ldots\vec{n}\ldots)$ of (2); in Thomason's terms, they cannot contain any sentence about the supervaluations of any node χ of the model. His clauses (see note 41) leave all such sentences undefined. Therefore, one cannot opt for introducing the Tarski Schema in models of "classical" supervaluation *via* the maximal linear orders themselves. This is because what these orders refer to are not the supervaluated nodes of the entire model, but the classical moments of their linear (classical) Histories. Consequently, although the sentence, 'for every moment t: p at t, if, and only if, 'p' is true at t' is true in all maximal linear Histories, and belongs, as such, to all supervaluated nodes, the interpretation of the bi-conditional will have to be classical; just the same as in the orders it belongs to. We will have to read it as a co-material implication, and end-up with bivalence. Moreover, by admitting $T(p)$ and, in general, sentences of the metalanguage within the linear orders themselves, we would have to use supervaluation for these too. But, in such case, the price to pay would be our inability to express the fact that a sentence is truth-valueless. As Van Fraassen puts it, if we evaluate metalinguistic statements by using the same supervaluation method, "we shall then have no way of formulating the assertion that a sentence is *not true*".[42] The structural reason why supervaluation, if adopted for the metalanguage, condemns it to such an expressive penury is as follows. Consider how "classical" supervaluated models are constructed.[43] They are built upon equivalence classes of classical valuations. Now, the general value assignment for the supervaluation class corresponding to the occasional equivalence classes of classical valuations is that a sentence is true in that class, when it is true in all members of the class, false, when false in all members of the class, and valueless otherwise. If, however, one is to expand the same technique to the metalanguage, one has to take into consideration the

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$\tau(\{p, q, \ldots, r\} \vdash s) = (p\&q\&\ldots\&r) \to s$.

[41] R. THOMASON, [1970], p. 274; (5.1). In Thomason $\alpha, \beta, \ldots$ are variables of nodes.

[42] B. VAN FRAASSEN, [1968], p. 144. It is the same problem that we would have encountered, if we had not attributed constant substructures to the origins of the S-models. See *supra*.

[43] B. VAN FRAASSEN, [1968]; §. 8-11; p. 140: "If ν is a classical valuation for the language L and A, B are sentences of L, then: (a) $\nu(A) = F$ or $\nu(A) = T$, (b) $\nu(\sim A) = F$ if and only if $\nu(A) = T$, (c) $\nu(A \vee B) = F$ if and only if $\nu(A) = \nu(B) = F$." In the original text, there is an obvious misprint, and the last "F" is printed as "T".

truth-values of metalinguistic statements within the set of classical valuations - i.e. one has to enrich all classical valuations with the metalinguistic statements concerning their own sentences. And since these valuations are classical, they have to assign either truth or falsity to all these higher-level sentences. Why this is problematic can be exemplified as follows. Isolate a maximal History passing through the present and containing a metalinguistic statement assigning truth/falsity to a statement about a future contingent event. Since the event is a future contingent, there must also be another maximal History traversing the present and assigning the other among the classical truth-values to the same metalinguistic statement. Hence, the statement will represent a gap. On the other hand, the truth-value that all maximal Histories assign to the statement saying that the lower level sentence about the event is either true or false will be (by supervaluation) true. So far, so good. The expressive penury comes from the fact that not only the metalinguistic statement assigning truth/falsity to the sentence about the future contingent event will be truth-valueless, but so will the statement assigning non-truth/non-falsity to the same sentence. This is because, in classical valuations, when a sentence is untrue/unfalse, it is also false/true. Therefore, in every maximal History passing through *now*, sentences belonging to the sets $\{p : F(p) \wedge \neg T(p)\}$ and $\{p : T(p) \wedge \neg F(p)\}$ exhaust all sentences. The above is exactly the problem. Classical valuations, by assimilating falsity to non-truth, and truth to non-falsity oblige supervaluation to reserve a gap for both the sentence of the metalanguage saying -e.g.- that that there will be a Sea-Battle tomorrow is false, and to that saying that that there will be a Sea-Battle tomorrow is not true. What Van Fraassen does, in order to overcome the problem, is to restrict supervaluation to the first level, and keep classical logic for sentences of the metalanguage (of any level). This saves the day for “classical” supervaluation, because the metalanguage now uses a quantifier running through a domain of lower level sentences, which (sentences) can be but three things: true, false and neither true nor false. Sentences that are not true, could only be false or truth-valueless, and, hence, $T(p) \vee \neg T(p)$ is a theorem, while, since a sentence could be something other than true or false, $T(p) \vee F(p)$ fails. The metalanguage cannot be allowed into the “classical” supervaluation models.

With this remark, we pass to the remaining central aspect of the comparison between the two supervaluation systems: The Law of Bivalence and how it relates to the Tarski Schema.

As we have seen in the last paragraph but one the Law of Bivalence does not obtain in models of “classical” supervaluation. Moreover, from what preceded, it is clear that the Law would have obtained in them, in cases sentences of the metalanguage were open for supervaluation. ‘$T(p)$’ could be truth-valueless, ‘$F(p)$’ could be truth-valueless, while ‘$T(p) \vee F(p)$’ is true. If so, because of the validity of the Tarski Schema, the Law of Bivalence would have obtained for any sentence that respects the Excluded Middle. Which implies that, in “classical” supervaluation, the Law of Bivalence would have obtained for *any* sentence - the Excluded Middle is a theorem there. *Mutatis mutandis*, something similar happens in S-models. Had we chosen to assign to every world of the S-model, not a constant substructure, but a substructure varying according to the perspective, the Tarski Schema, in its classical formulation, would have been a theorem. This would have meant that the Law of Bivalence would have obtained for all sentences that respect the Excluded Middle. As we have seen already, the Excluded Middle does not obtain universally in S-models, but this is not our concern here. We just point out that, when it *does* obtain, bivalence follows, if substructures are to vary according to the perspective. But, if we had chosen to make substructures non-constant, the model would have suffered from the same expressive penury that Van Fraassen avoids by adopting a bivalent metalanguage. Our metalanguage is bivalent too, and this breaks down the connection between the Excluded Middle and the Law of Bivalence, in a manner similar to the models of "classical" supervaluation.[44] The Law fails, even for cases that the Excluded Middle holds; the Sea-Battle tomorrow is the paradigm case in our models.

However, an intuitive truth that still seems to imply a certain correspondence between the Law of Bivalence and the Excluded Middle is that, if some world forces that p obtains or the same world forces that $\neg p$ obtains, then it forces that $T(p)$ obtains or it forces that $F(p)$ obtains, and *vice versa*. This truth is derivable in S-models.[45] The reason is, again, that the supervaluation classes constituting the S-models are built upon other supervaluation classes, not upon classical valuations. For example, say the index identifying the above world/supervaluation is "$(\ldots\vec{n}\ldots)$". When "$(\ldots\vec{n}\ldots)$" determines non-effectively a world, the disjunction might be true and the disjuncts truth-valueless. However, all worlds that identify themselves by "$(\ldots\vec{n}\ldots)$" and force p, also force $T(p)$, and all worlds that identify themselves by "$(\ldots\vec{n}\ldots)$" and force $\neg p$, also force $F(p)$.[46] Therefore, '$(T(p))(\ldots\vec{n}\ldots) \vee (F(p))(\ldots\vec{n}\ldots)$' becomes true by supervaluation, when '$(p)(\ldots\vec{n}\ldots) \vee (\neg p)(\ldots\vec{n}\ldots)$' is true by supervaluation. "Classical" supervaluation cannot ask for a formula expressing the same intuitive truth, because it has no means to reach any sentence of the form $(p)(\ldots\vec{n}\ldots)$. No such sentence belongs to the classical valuations "classical" supervaluation is built upon. The equivalent of $(\ldots\vec{n}\ldots)$, in Thomason's models, is an index of a supervaluated node; but no classical valuation contains any such index. They only contain indexes for their non-supervaluated moments. The intuitive truth expressed by row (4) of the tableau is something that is true about models of "classical" supervaluation, but inexpressible in them. It is the same problem "classical" valuation encountered with row (7) of the tableau.

The corresponding formulation for models of "classical" supervaluation of our $((Tp)(\ldots\vec{n}\ldots) \vee (Fp)(\ldots\vec{n}\ldots)) \leftrightarrow ((p)(\ldots\vec{n}\ldots) \vee (\neg p)(\ldots\vec{n}\ldots))$ is as follows. By the Tarski Schema for "classical" supervaluation, we get: if p obtains, then (in the sense of $\vdash$), $T(p)$ obtains, and, if $\neg p$ obtains, then (in the sense of $\vdash$), $F(p)$ obtains. Therefore:

[44] B. VAN FRAASSEN, [1966], pp. 493-94.

[45] This, of course, is not the *stricto sensu* correspondence between the Law of Bivalence and the Excluded Middle. This *stricto sensu* correspondence demands that if a world forces that either p obtains or $\neg p$ obtains, then, it forces that either $T(p)$ obtains or $F(p)$ obtains. This is false both for S-models and for models of "classical" supervaluation.

[46] This comes from (2) above.

If it is either the case that p obtains in node α, or it is the case that $\neg p$ obtains in node α, it is also the case that either $T(p)$ obtains in node α, or $F(p)$ obtains in node α, and *vice versa*.

This is compatible with the possibility that neither 'p in α' is true, nor 'p in α' is false, for it is no matter of fact that either p obtains in α, or $\neg p$ obtains in α, even if it is the case that $p \vee \neg p$ obtains in α. However, as for the same classical valuations the principle cannot be stated *in* the models; it is always *about* the models.

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We said earlier that Thomason's models have ways to dissociate truth from inevitability. If intuitionistic models cannot do the same, this could give grounds for a crippling objection against our claim that intuitionistic models have richer expressive recourses than Thomason's. Our claim will be here that although intuitionistic supervaluation endorses a certain formulation of the Schema, it is not, thereby, forced to accept that anything can imply its own inevitability.

To that end, we will now examine in detail Thomason's argument concerning how the non-validity of the Tarski Schema causes, in his system, the non-validity of:

$$PT(Fp) \vdash PL(Fp)$$ [47]

The non-validity of this formula provides evidence that the system is able to distinguish between, "having been true" and, "having been inevitable".[48] The argument comes from the general truth-value assignments that Thomason has previously given to sentences with respect to maximal Histories *and* points. On page 277, he defines[49] ['p' will be true, for the point w and History h] as [there is a point belonging to h and coming after w, for which -i.e. point- h assigns truth to p] and ['p' is inevitable, for the point w and History h] as ['p' is true for the point w, and every History passing through w]. By these definitions, it is clear that $PT(Fp)$ has not $PL(Fp)$ as its "semantic consequence", in Thomason's terms.[50] Now, Thomason claims that this result speaks against (or at least smoothes), another result of his (ibid, p. 278), according to which truth and inevitability are, indeed, equivalent. But, at this point, one could pose the following question: With respect to what is Fp to be evaluated? Is it to be evaluated with respect to maximal Histories *and* points or with respect to points alone? This is crucial, since the only occasion that the above "semantic consequence" is not valid is when Fp is evaluated with respect to maximal Histories *and* points. As written

[47] R. THOMASON, [1970]; ch. 8. "$L(\ldots)$" stands for "$\ldots$ is inevitable", see *infra*. Let us agree here that "$F(p)$" says "'p' is false", and "Fp", "It will be that p".

[48] R. THOMASON, [1970], p. 279.

[49] We paraphrase into our own idiolect.

[50] See part I. Notice that p implying $T(p)$ and, thereby, $L(p)$ is, for the intuitionistic model, no real threat, because, as we have seen, the implication is not material. Therefore, the fact that $L(p)$ is implied by p does not mean that a sentence (materially) implies its own inevitability.

down, $PT(Fp) \vdash PL(Fp)$ is ambiguous. What it says is that if one assumes that it was true that p will be the case, it follows that it was inevitable that p will be the case. Now, if we stick to valuations with respect to maximal Histories and points, this is not so. But, if we make the valuations concern points, then, $PL(Fp)$ *does* follow; on page 274 (op. cit.), where these valuations are presented, 'Fp' is taken to be true in a world w, if, and only if, there is, upon *every* maximal linear History passing through w, a world coming after w, where 'p' is true. Seen thus, $PT(Fp)$ and $PL(Fp)$ are extensionally identical.

As things stand, the above ambiguity problem does not arise in Thomason. But the reason why it does not is that Thomason defines the metalinguistic predicates of truth and inevitability only with respect to maximal Histories *and* points, and not with respect to points *simpliciter*, as he has defined -e.g.- his tense operators (ibid, pp. 273-274). Now, this is an obvious consequence of "classical" supervaluation being obliged not to use supervaluation in the metalanguage. In order to pass from truth and inevitability with respect to maximal Histories *and* points, to truth and inevitability with respect to points uniquely, one has to assume the same supervaluation techniques Thomason uses for his first level language.[51] For reasons that we will elaborate further bellow, "classical" supervaluation does not -in general- assume such techniques for the metalanguage, and Thomason's formula is not ambiguous. Nonetheless, this distinction between truth with respect to maximal Histories and points, and truth with respect to points, can be used to exemplify what we have said time and again, about "classical" supervaluation being a supervaluation over sets of classical valuations and intuitionistic supervaluation being a supervaluation over sets of supervaluations. "Truth" can be dissociated from "inevitability", only when truth and inevitability concern the assignments of classical valuations. Once one moves away from classical valuations and affronts the supervaluation classes of the nodes themselves -i.e. when one takes under consideration all maximal Histories traversing a node- truth and inevitability coincide. Therefore, the distinction Thomason claims concerns bivalent temporal models, where one can indeed distinguish between, "having been true" and, "having been inevitable".[52] In supervaluated frames, however, if one is to apply supervaluation throughout linguistic levels, "having been true" will be substituted for, "having been true in all Histories" and thus, with the classical, "having been inevitable". Thomason's, "having been inevitable" could be used no longer, to express the difference Thomason calls "difference between 'truth' and 'inevitability'". 'p' would have been true with respect to w, if, and only if, *any* History passing through w, assigns truth to p. But, if this were so, 'p' would have been inevitable (in Thomason's terms), with respect to w. Therefore, what Thomason claims obtains only for metalinguistic valuations relative to maximal Histories

[51] The obvious step is to define ['$T(p)$' is true in w], as [$T(p)$ belongs to all maximal Histories passing through w *and* (with respect to) w]. Since, on the other hand, "inevitable with respect to Histories *and* the point x" has been defined as "true in all Histories passing through x", the extension of "inevitable with respect to Histories and points" will not change, if we take the further step of defining it with respect to points uniquely. Following that step it will be identical with the extension of supervaluated truth. But, defining truth by supervaluation provokes the already mentioned expressive penury. Thomason seems not to be aware of this, since, in note 14 (op. cit.), he suggests that one could follow exactly this device - i.e. superevaluate $T(p)$, in the way of p. See also part III.(2) of this chapter.

[52] Cf. A. PRIOR, [1967], pp. 122-127. This distinction works particularly well in "Ochkamist" frames.

and points (i.e. classical linear valuations), but not for metalinguistic valuations relative to supervaluated points. Of course, these latter do not concern "classical" supervaluation; they are not defined there. But, had they been defined, "truth" and "inevitability" would indeed have been identical.

In reality, what the non-identity between $PT(Fp)$ and $PL(Fp)$ stands for is not exactly what the readings, "It was true that p will be the case" and, "It was inevitable that p will be the case" allow one, *prima facie*, to gather. *Prima facie* these two expressions seem to concern moments of a (possibly indeterministic) partial History, while, in fact, they concern moments of linear Histories. What in Thomason's models does the trick for the past is that it makes no difference what particular linear History one chooses for the evaluation of $T(Fp)$. That granted, one can safely turn one's attention to $L(Fp)$, and the rest of the linear Histories will immediately (and safely) come into play. Had, however, the choice of the linear History mattered, as it does in -e.g.- $FT(Fp)$, the non-identity between the above and $FL(Fp)$ would be a trivial and absolutely non-informative result; one could no longer afford to read $FT(Fp)$ as, "It will be true that 'p' will be the case", without needing to specify the exact linear History one is talking about. On that basis, the same person could no longer suggest that the notion involved is the notion of truth in a partial supervaluated order, and not the old classical notion of truth within a maximal linear History. On the other hand -and here we come full circle- if one tries to transpose the notion of truth from the notion of truth with respect to maximal linear Histories towards the notion of truth with respect to nodes of supervaluated partial orders, the formal identity of truth and inevitability will reemerge: $FT(Fp)$ and $FL(Fp)$ would, then, be extensionally identical. What Thomason considers to be a clear-cut distinction between "truth" and "inevitability" is a clear-cut distinction indeed, because $PT(Fp)$ concerns assignments with respect to maximal linear (bivalent) orders. The intuitive charm of his argument against the identity of truth and inevitability comes from the specific counterexample he has chosen. For one -on this particular occasion- can skip mentioning which History one has in mind, when asserting $PT(Fp)$, because, no matter which History this is, the assignment will be the same. This, however, is far from implying that the predicate of truth in the above formula concerns points, rather than linear Histories. Thus, the argument is inscribed on a classical background, where the non-identity of the two is to be taken for granted.

***

In any event, even if by the bias of the past's linearity, the intuitive difference between inevitability and truth can be expressed in Thomason's system, when the difference concerns the past. What happens in intuitionistic supervaluation? The answer is that truth and inevitability (if defined as above) are, in intuitionistic supervaluation, identical in all contexts. Nonetheless, there are still other resources for one to formulate what intuitively corresponds to the difference between:

i) By the present's evidence, it is established that a past moment has been followed by a moment that p.

and

ii) By the present's evidence it is established that a past moment would inevitably have been followed by a moment that p.

According to 4.5.2 of chapter 4, one can exploit the semantics of square brackets and express (i) by:

$now \Vdash [PnFp]$

and (ii) by:

$now \Vdash (PnFp)$

The first formula implies that, as things turned out to be, the moment *f(now)*$-n$ has been followed by a moment that p. It allows, however, for the possibility that this evolution was not inevitable then. The second formula excludes this possibility, for it takes under consideration the constant substructure(-s) assigned to the element(-s)[53] of the S-model with rank *f(now)*$-n$; it does not consider how this substructure might have been narrowed down within *W(now)*.

More generally, one can define "inevitable for w" in an S-model as the intersection of what w forces within all $W(w')$ substructures of the S-model containing w, and taking under consideration how $W(w)$ has, possibly, been narrowed down by w'- i.e. what w forces in $W(w)(w')$, and with $W(w)$ being non-constant.

Formally:

$$I(p,w) =_{\text{df.}} p \in \cap\{A : (\exists w')((p \in A) \leftrightarrow (w' \Vdash [w \Vdash p]))\}$$

Notice that the things that are inevitable for w are, under this definition, not necessarily the things that w forces from the perspective of the constant substructure $W(w)$. For example: w might be forcing "p is contingent" in this substructure, and not force the same thing within $W(w)(w')$ - i.e. $\neg(w' \Vdash [w \Vdash \triangle p])$. In the above case, "$p$ is contingent" is not inevitable for w, although it is forced in it.

Notice also that the same *rationale* with respect to "inevitability" can be applied to moments as well:

$$I(p,t) =_{\text{df.}} p \in \cap\{A : (\exists w)((p \in A) \leftrightarrow (w \Vdash [w(t) \Vdash p]))\}$$

This formula is a specification on the first definition, since it specifies the implied $(\ldots\vec{n}\ldots)$ of "w".[54]

With the aid of these devices, one can dissociate things that a world (moment) forces that another world (moment) forces at a certain point from things

[53] The plurals are for when there is a past branch at the level of *f(now)*$-n$.

[54] Even more generally, one can define inevitability, with respect to any $(\ldots\vec{n}\ldots)$ condition on $\vec{n}$, by $I(p, w(\ldots\vec{n}\ldots)) =_{\text{df.}} p \in \cap\{A : (\exists w')((p \in A) \leftrightarrow (w' \Vdash [w(\ldots\vec{n}\ldots) \Vdash p]))\}$.

that this world (moment) forces from any perspective of the S-model, and are, on that account, *inevitable* for it.[55]

The remote reason that one can, in intuitionistically supervaluated models, reintegrate the distinction between truth and inevitability is that, although the equation of truth with truth-upon-a-bar has made "true in all possible futures" insufficient for expressing the distinction between truth and inevitability, the wider notion of, "true upon a bar for w, from all possible perspectives" can be called for, in order to perform what, "true in all possible futures" has been performing in Ochkamist models. There is still room for the distinction within S-models. Moreover, one can impose any limitation one pleases upon the field of perspectives that are to be taken under consideration. For example, suppose that one wants to express the fact that something about a moment in the past was not inevitable right from the start, but it became such, after a certain turning point. One can introduce the predicate "inevitable for w, after w'", and define it as "inevitable for w not in the initial S-model, but in its substructure having w' as origin". Thus "inevitable for w, from w' on" would be equivalent to "inevitable for w not in the S-model, but in $W(w)(w')$".[56]

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Following the previous analysis, one is to understand that Thomason's claim is exact, and the non-validity of $PT(Fp) \vdash PL(Fp)$ accounts for the capacity of his system to distinguish between, "having been true" and, "having been inevitable". This, however, is because the past is linear there and the distinction between possible linear Histories trivializes itself for that segment (i.e. for the past). On the other hand, in intuitionistic supervaluation, the notion of "inevitability" is formally defined for the entire model, and accounts for supervaluated worlds and not for worlds *qua* parts of linear orders.

[55]The spectrum of what a world forces has as lower limit what is inevitable for it, and, as upper limit, what an observer occupying the outside of Time standpoint of chapter 2 establishes that the world establishes. The same -latter- function could be performed by a temporal observer too, if, after some moment, any future indeterminacy stops, no backward branching has emerged, and the observer is located upon any of the moments after that terminus. Notice here that, if he occupies the outside of Time standpoint, the observer should preferably rearrange the branching model into a $T \times W$ one.

[56]In a way similar to the definition of, "inevitable for the moment t" -see *supra*- one can also define "inevitable for w, after the moment t".

III

Allow us now some recapitulating, concluding, remarks.

This comparison between "classical" and intuitionistic supervaluation turns on the fact that the former is a supervaluation built upon classical valuations, while the latter is a supervaluation built upon supervaluations. This difference manifests itself in many ways, the more significant of which is the following. While the nodes of Thomason's models have access to the maximal linear Histories passing through them, the nodes of the S-model have access to the metalanguage of the other nodes.[57] Thomason's nodes assert that, "There either will be a Sea-Battle at n or not", on the basis of having access to some linear Histories that contain a Sea-Battle at n, some linear Histories that contain no Sea-Battle at n, and to no others. An S-model node -e.g.- $w(\ldots\vec{n}\ldots)$ asserts the same sentence, on the basis of having access to the metalinguistic statement of some worlds: "$w(\ldots\vec{n}\ldots)$ forces that there is a Sea-Battle", to the metalinguistic statement of some worlds: "$w(\ldots\vec{n}\ldots)$ forces that there is no Sea-Battle", and on the basis of the fact that these worlds form a bar as for $w(\ldots\vec{n}\ldots)$.

By being in position to dissociate the metalanguage of some world w from the metalanguage of any other world, a world/node w in an S-model uses a much more flexible apparatus. More specifically, all the relative advantages of S-models elaborated in the previous two parts are the result of these two characteristics:

(i) Worlds contain their own metalanguage.

(ii) Worlds have access to the metalanguage of other worlds.

The consequences of these are:

1. The Tarski Schema -or, better, the intuition underlying it- finds a way to be expressed by means of the model itself, and without implying bivalence. $(T(p))(\ldots\vec{n}\ldots) \leftrightarrow p(\ldots\vec{n}\ldots)$ is valid, because $(T(p))(\ldots\vec{n}\ldots)$ either belongs to the metalanguage of the origin and, if so, it is of a level higher than 2, and bivalence obtains for that level (see row 6 of the tableau), or it belongs to the metalanguage of a world other than the origin, and, consequently, can stand for a gap from the perspective of the origin, without, thereby, imposing bivalence to the metalanguage of the origin.

2. The Law of Bivalence can be abolished, without exiling the metalanguage away from the model. $T(p) \vee F(p)$ is not valid, but the $T(p), F(p), w \Vdash p$, etc., general kind of sentences still belong (can belong) to the worlds of the model. This is because, when evaluating sentences containing "T", "F", "$\Vdash$", etc., we do not refer to maximal linear Histories, but to (supervaluated) points. Contrary to that, in "classical" supervaluation, this is impossible. Once the metalanguage is allowed into the model, bivalence is restored. Now, it is not clear to me whether Thomason is fully aware of this. Thomason's models allow for the possibility that the $T(p)$, $F(p)$, etc., general kind of sentences belong to the worlds, just as p and $\neg p$. In p. 278;

[57] Cf. ch.4; 4.5.3.1.

(8.1) (op. cit.), Thomason introduces the expression $V^{h/\alpha}(TA)$,[58] which are to be understood as asserting that TA obtains with respect to history h and node α - just as the expression $V^{h/\alpha}(A)$ asserts that A obtains with respect to history h and node α. He, then, points towards (note 14, op. cit.), the introduction of the superevaluative '$V_\alpha(TA)$', which would have the meaning that TA obtains with respect to node α. He suggests that one could follow "Van Fraassen's strategy" -as he puts it- and superevaluate TA, in the way one superevaluates A. If so, $V_\alpha(TA)$ would be the case, if all maximal Histories passing through assign truth to 'TA', false, when all maximal histories passing through α assign falsity to 'TA', and undefined otherwise. But, in such case, the fact that 'TA' is not true could find no formal expression, for exactly the same reason that we mentioned in part II: not all maximal Histories assign non-truth to 'TA'. 'TA' itself is undefined, if 'A' is. Perhaps this is the reason why Thomason does not lead the suggestion of note 14 all the way. As already said, the same problem was already known to Van Fraassen.[59]

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Before closing this chapter, I will present the ways in which S-model can be turned into conservative extensions of "classical" supervaluation models. This, I take it, provides sufficient formal evidence for their superior expressive recourses.

A model of "classical" supervaluation is founded upon the complete set of valid arguments of classical logic and a subset of the set of classical theorems. This suggests that "classical" supervaluation is not -as Van Fraassen puts it- "statement complete", although it is, "argument complete".[60] In other words, there is no 1-1 correspondence between valid arguments and theorems. The exceptions that Van Fraassen puts forward are two in number: the Tarski Schema[61] and $F(a) \to (\exists x)F(x)$.[62] Now, the Tarski Schema is not valid in S-models either,[63] and as for $F(a) \to (\exists x)F(x)$, which is valid in S-models, one can invalidate it, by changing the general assignments of V as follows:

4.4.3.3.(d) of chapter 4 was: $\phi^n(x_{(m)_1}, y_{(t)_2}, \ldots, z_{(s)_n})$ is true in w, iff., $\langle x_{(m)_1}, y_{(t)_2}, \ldots, z_{(s)_n}\rangle$ belongs to $(D(w)X^{n-1}D(w))$ that is assigned from V to ϕ^n with respect to w.

It needs to be changed to:

[58] 'A', 'B',... are variables of sentences, and 'α','β',... are variables of nodes in Thomason.

[59] B. VAN FRAASSEN, [1968], p. 144.

[60] B. VAN FRAASSEN, [1966], p. 484.

[61] Ibid; VIII.

[62] Ibid, p.486.

[63] Van Fraassen also mentions, *in passim*, (ibid. p. 493) the Law of Bivalence and, in that passage, he considers it to be the semantic counterpart of the (proof-theoretic) Excluded Middle. But the Law of Bivalence does not obtain in S-models either. Besides, there is certain vagueness, about what is to count for Van Fraassen as the semantic counterpart of a theorem. For he considers $p \vdash T(p)$ to be the semantic counterpart of the (theorem?) $p \to T(p)$, on the basis -I gather- of its main connective (cf. B. VAN FRAASSEN, [1968], pp. 143ff). On the other hand, he thinks of the Law of Bivalence as belonging to the semantics of the system, on the basis of its truth predicate. But such a predicate exists in $p \to T(p)$ as well!

$\phi^n(x_{(m)_1}, y_{(t)_2}, \ldots, z_{(s)_n})$ is true in w, iff., $\langle x_{(m)_1}, y_{(t)_2}, \ldots, z_{(s)_n}\rangle$ belongs to $(UX^{n-1}U)$ that is assigned from V to ϕ^n with respect to all worlds.[64]

Once this is done, the calculus for sentences containing empty terms becomes equivalent to the same calculus of free logic. This is because the clause for quantified sentences remains the same. Thus, 'Pegasus has wings' is true in w, even if, "Pegasus" does not belong to $D(w)$, but, 'There is a winged horse' is false, because of:

4.4.3.4.(a) of chapter 4: $\exists x(\phi^{n+1}(x_{(m)_1}, y_{(t)_2}, \ldots, z_{(s)_n}, x))$ is true in w, iff., there is a $y \in D(w)$, such that $\phi^{n+1}(x_{(m)_1}, y_{(t)_2}, \ldots, z_{(s)_n}, y)$ is true in w.

This change, with respect to empty terms, alters nothing essential in the S-model.

Now, what remains to be done, in order to force S-models not to falsify any of the theorems that are true in "classical" supervaluation models, is to make them respect the rest of the classically true theorems. The theorems that are true in "classical" supervaluated models are exactly these.

For the atomistic and metalinguistic calculi this is already the case in S-models.[65] Now, indexed sentences have an underlying proof theory equivalent to the Heyting calculus.[66] With respect to these, one has a double option. He can either disallow infinite S-models, or substitute intuitionistic quantification over infinite totalities by classical quantification over infinite totalities. As proven in 4.7, if indexed sentences are to be atomic, S-models are Beth models. Now, all finite Beth models are classical,[67] and, moreover, if infinite totalities are treated just like the finite ones, the calculus concerning them will become classical as well. In fact, the very reason why infinite Beth models are not classical, while finite Beth models are, is that Intuitionism considers expressions of the form, "for all x-s belonging to A" ill-formed, if A has infinite elements. This leads to the falsification of the Excluded Middle and Double Negation, which are both restored, by the moment one allows for such expressions and considers them on a par with the same expressions, when A is finite. The classical apparatus is restored, by the moment infinite totalities are taken under consideration and, consequently, quantification over their elements is allowed. This makes the Excluded Middle and Double Negation theorems of the system, since one can now argue -e.g.- that either p obtains in none of the points of a maximal linear order or in some of them, and so, the possibility of '$Fp \vee \neg Fp$'[68] being untrue fades away, and takes along with it the possibility of '$\neg\neg Fp \rightarrow Fp$' being untrue. On the other hand, that 'p' is truth-valueless is still possible.[69]

[64] Notice that this is not combined by the clause that $D = U$ is constant for all worlds. Which means (see *infra*) that 'α is ϕ' can be true in w, even if 'α' denotes nothing there.

[65] Cf. ch. 4; 4.8(2).

[66] Cf. ch. 4; 4.8.(1).

[67] See note 129 of chapter 4.

[68] Or '$Pp \vee \neg Pp$', for past branches.

[69] The general argument of this paragraph proves 4.8.(4) of chapter 4 as well: Assume α to be index-free, cut down the S-model to the level of $(\ldots\vec{n}\ldots)$, and eliminate the semi-substructure that follows the direction converse to the direction: $\langle$origin,$(\ldots\vec{n}\ldots)\rangle$. This is a finite Beth model. Now, use the same argument for proving, by induction, that a bar at $(\ldots\vec{n}_m\ldots)$ of $\alpha(\ldots\vec{n}_{m\in\mathbb{N}^*}\ldots)\ldots(\ldots\vec{n}_2\ldots)(\ldots\vec{n}_1\ldots)$ with m finite defines a finite Beth

The above concludes the transformation of S-models into conservative extensions of models of "classical" supervaluation. No theorem of these latter is untrue in the former, and all theorems of the former that can be stated in the language of the latter were already true in the latter, before the expansion of the language.

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"Classical" supervaluation has been conceived as a combination of Prior's temporal logic on the one hand, and Van Fraassen's supervaluation techniques on the other. These latter were, in turn, conceived as a way to resolve some problems of classical logic, which are -so to speak- "static" in nature. What I mean by this is the following. Consider the example of sentences containing singular terms, about which it is true to say that the fact that they denote something, rather than nothing, is contingent. Their truth-value varies from one classical valuation to the other. As a consequence of that, classical supervaluation attributes no truth-value to -e.g.- "Aristotle is a philosopher", if "Aristotle" can be denotationless. Nonetheless, "It is the case that Aristotle is a philosopher or it is not" is still a theorem. What classical supervaluation techniques start with are some "already given" classes of classical valuations, upon which they built supervaluation classes.[70] To put it plainly, the classical valuations are the foundation of the method. And it is in this respect that "classical" supervaluation (which applies Van Fraassen supervaluation techniques to cases where the initial classical valuations are maximal linear Histories), is built upon these Histories. On the other hand, intuitionistic supervaluation by setting out not with linear orders but with ("already") supervaluated points, works recursively and stresses the fact that what the linear orders will come to be is a function of the accessibility relation between these points; not the other way around. One -if one happens to be the ideal agent of S- can construct the entire S-model just by knowing the primitive valuations of V that are occasionally true,[71] and the constant general clauses for the primitive connectives and $\Vdash$. If it is as a result of this construction that the maximal linear orders will manifest themselves. They are the various choice sequences constituting the spread that the (constructed) S-model essentially is. It is in this sense that "classical" supervaluation can look far more "static" than intuitionistic supervaluation. For, to the question, how is it that there are many possible maximal Histories, "classical" supervaluation can provide but a circular answer: It is because there are moments

model. The same argument proves also row (2) of our tableau, which can even be generalized into "Excluded Middle for unquantified or quantified sentences containing quantifications over finite sub-domains of W", and, finally, provides an alternative, more general, proof for the lemmas of Th.7-9 in chapter 4.

[70] As Restall puts it "...we have only one relation $\models$ between moments and statements, and this relation is parasitic on another relation between *histories* and statements". G. RESTALL, [2005], p. 4. On the other hand, in intuitionistic supervaluation -if we are to paraphrase Restall's characterization- we have only one $\Vdash$ relation between points and statements, and this relation generates the partial order these same points belong to. The forcing relation is no longer parasitic on the classical valuations, but directly refers to (is the same with) the assignments of $\Vdash$.

[71] i.e. the atomic sentences that are "true at present" and the constant set of true causal conditionals.

where for some sentences some classical valuations are in disagreement. And hence, the many possible classical valuations. Contrary to that, intuitionistic supervaluation by being a supervaluation over supervaluations, which in turn are generated by different sets of atomic sentences the constant set of causal conditionals, and general clauses for the primitive connectives and $\Vdash$, is in a much better shape to show how the classes of linear orders result from -rather than create- the set of sentences that come out true upon the nodes of the model.

Usually, Columbus' egg is a non-decidable matter. On our occasion, however, we think that what comes first is what is true upon the nodes of the model, and not what is true upon the maximal branches of it.

Appendix A

Yesterday's Space-Battle[1]

(An intuitionistic approach to contingent events in relativistic universes)

In part I, I present the changes of the S-model that are necessary for it to be able to represent indeterministic Histories of relativistic universes. In part II, I try to show that the Sea-Battle problem in non-relativistic and relativistic universes is essentially the same. What differs is that in relativistic universes the indeterminacy may concern some portion of the present, and of the past as well. In general, the presentation is based on the "ideal agent of S - *Ideal Scientist*" thought experiment. As with the same (subject-centered) experiment of chapter 3, this one, too, can be objectified. How this can be achieved follows straightforwardly from chapter 4. The technique is the same when adapted to the limitations of relativity.

I

An assumption that we have adopted so far has dictated that there must be some states that belong to the general categories of "cause" and "effect". We have also assumed that at least one strong causal chain is actualized at every moment. We have not, however, claimed that *all* states belong to one of these two categories. Normally, the state represented by -e.g.- "7+5=12" is not to be counted among these. In relativity, however, the cause to effect relation is so fundamental that it is sometimes hard to resist the temptation to posit that all states belong to one of these categories. (The speed of light and

[1]For combining relativity and indeterminism, I have, for the main part, followed the classical paper of Belnap (N. BELNAP, [1992]; see also the officially unpublished electronic post-print of 2003); I have taken my distance only with respect to some points of interpretation concerning the Minkowski diagrams. As for these, I rather agree with S. McCall (S. McCALL, [1994], [1995], [2000]), and G. FLEMING, [1996]). Much of this Appendix is also grateful to a couple of pages of *Past, Present and Future* (A. PRIOR, [1967], Appendix B, 5). It has to be emphasized that my ambition is not to provide here a relativistic account for indeterministic events; neither Special, nor General Relativity accounts for these. However, and although it might be the case that the differential equations of the theory do not account for indeterministic events, they do account for the way information about them travels in Space-Time. Now, what I try to do is to draw some conclusions about ontic commitment with respect to these by the relativistic apparatus concerning information.

the relation of cause to effect are among the primary constants of all frames of reference).[2] Now, how the state represented by -e.g.- "7+5=12" can be made to look either like a cause or like an effect, is a tricky question, which, surprisingly enough, does not necessarily get a trivially negative answer. In its most rigid form the claim is that any state of any *here and now*[3] is inscribed into a causal chain that is causally connected with that specific *here and now*. "Causally connected" means here that it is either upon the past segment of a weak causal chain passing through *here and now*, or upon the future segment of a strong causal chain traversing the same spatio-temporal coordinates.[4] Despite the fact that one can, in relativistic frameworks, make the above extreme claim sound less farfetched than it does *prima facie*, we will here adopt a milder approach, and pose that any state that obtains is implied (not necessarily causally) by a state that

(i) obtains and

(ii) is causally related to our occasional *here and now*.

This says that any sentence denying something that obtains leads to a (logical) contradiction, because of some causal state that obtains and, therefore, if consistency is assumed, the sentence can be shown to be false, no matter whether causal or not.[5] We will also pose that there is always a causal state that obtains, and is causally related to the occasional *here and now*. This mild version, although it does not turn any truth into a causal truth, still suggests that any truth is logically implied by a causal truth.

Following this straightening of the role of the causal apparatus we will now present all the modifications required for the transformation of the standard S-model into a model that fits Special Relativity.

First, the expression, "the (occasional) *now* of the ideal agent of S" will be replaced for, "the (occasional) *here and now* of the ideal agent of S". This is typical in relativity because the barrier imposed by the speed of light makes it impossible for events having at a certain moment some causal influence upon a certain region of the universe to have such an influence upon all regions at that same moment. In Special Relativity they will necessarily have one such influence, in the future, on *every* region, but when this is is a function of the

[2]As there is no frame of reference in which light has another speed, there are no two frames of reference, where the same causal chain presents two different orderings. Any two causally related states always precede or succeed one another in all frames of reference.

[3] Needless to add here "... of the occasional frame of reference". In relativity there is no universal *here and now*.

[4] Being upon the past segment of a strong causal chain traversing *here and now* does not suffice to causally connect the state with *here and now*. This is because of possible past forks - i.e. upon two mutually exclusive backward branches there have to be two mutually exclusive states and both of them should represent sufficient conditions for an effect obtaining *now*. On the other hand, being upon the future segment of a weak causal chain does not suffice to causally connect the state with *here and now* since the chain might break down and the effect never be actualized. Notice here that this will certainly be the case for one of the two branches coming out of any future fork since only one of the semi-paths coming out of the fork will eventually be taken.

[5]In other words the premise leading to the contradiction can always be chosen from the set of premises describing causal states.

distance separating the region from the event, and the observers velocity and direction.

A consequence of this first amendment is that we have to assume an infinite number of ideal agents of S, each of whom follows a different world-line. More precisely to each moment we assign an infinite number of agents occupying all the *loci*[6] of the universe and to each *locus* of each moment, another infinity of agents each of whom has another direction in space. Finally, to each agent of this last class we associate yet another infinity of agents each of whom has a different constant speed.[7] It is easy to see that the classes of agents that result from these consecutive assignments are equivalence classes. The first is defined by the relation "being at the same moment with", the second, by the relation "being at the same moment and following the same direction with", and *mutatis mutandis* so for the third. These multiplications of ideal agents are -as we have said already- necessary in Special Relativity, because the agents are no longer in a position to be causally influenced by an event that *now* happens, if this event does not happen at their specific *here and now*.

We will now give the general truth-value assignments for an S-relativistic model.

'p' is true *here and now*, if, and only if:

(i) p represents a state that either belongs to the past segment of a weak causal chain passing through *here and now*, or belongs to the future or present segment[8] of a strong causal chain passing through *here and now*.[9]

(ii) p is implied by a sentence satisfying (i).[10]

'p' is false *here and now*, if and only if, its negation is true.

'p' is truth-valueless *here and now*, if otherwise.

On the assumption that there is always a sentence belonging to (i), we can conclude that the causal apparatus of the world suffices to define which are the true well-formed formulas, which are the false, and which are the truth-valueless.[11]

[6] These are idealized and correspond to dimensionless spatial points.

[7] The speed has to be constant for Special Relativity.

[8] The "present segment" accounts for self-caused states.

[9] Notice that each maximal strong causal chain is a segment of a world-line, but not *vice versa*. In indeterministic universes, with future and past branching, the world-line continues to be there, even if the causal chain has stopped dead, or has not yet begun. Consider, for example, the case of convergent futures where the agent will still continue his journey in Space-Time, even if there is no weak causal chain connecting his present with this part of his future that follows the convergence, and consider also the case of future forks where the agent will also continue his world-line even if there is *now* no strong causal chain forcing him to take one branch instead of the other.

[10] Many of the sentences satisfying (ii) have to be non-causal - i.e. they are true sentences like "7+5=12". Note also that if we were to follow the rigid version and make every sentence represent a causal truth, (ii) would be superfluous.

[11] We can even introduce an existence predicate for states by posing that "a state exists", if, and only if, it is represented by a true formula. Cf. Appendix II.

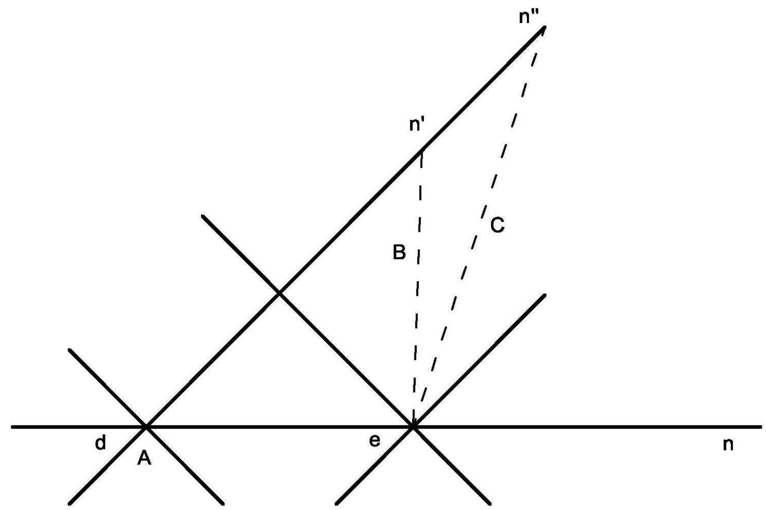

Figure 1.

With the above rearrangements of truth-value assignments we can now exemplify what we meant to say by saying that what is true during a moment is no longer a constant but a function of where one is located during that moment, of the direction one has, and of one's speed.

Figure 1 is a two-dimensional Minkowski diagram. A, B and C are three observers and d and e point events. Consider that our inertial frame of reference is A's and that *now* is n. A witnesses the self-caused event d taking place exactly where he is located. B and C are witnessing the point-event e, but B stands still, while C moves away from it with the speed suggested by his world-line. According to the above, 'd takes place at n' is true for A. B will have to wait until moment n', in order for the sentence to acquire the same value in his frame of reference. This is because it is at that moment that the event will causally interact with his spatio-temporal coordinates. Therefore, at moment n (according to the frame of reference of A),[12] the sentence will be truth-valueless for B. Finally, C will have to wait until moment n'', in order to be informed about what happened. On the other hand, the sentence 'd takes place at n, or does not take place at n' will be true for all three observers, at any time and place. This example establishes our previous claim that in relativistic universes what is true at a certain moment is not only a function of the moment but is

[12]We here adopt Einstein's convention for synchronic events - i.e. if a ray of light is emitted from our *here and now* at t, arrives at another location and witnesses some event there and then comes back to us at $t+n$, this "other event" is synchronic with the event taking place in our world-line at $t+(n/2)$. It must be noticed, however, that the way one calculates the line of simultaneity does not affect the general conclusions of this Appendix. It suffices that the priority of the cause with respect to its effect is respected. The more radical question about the very sense of defining a line of simultaneity in relativistic universes will not concern us here.

also a function of where the ideal agent is located within the universe and how is he moving (i.e. direction *plus* speed).

The above complications cause some necessary rearrangements in the figures representing branching Histories. For whereas in a non-relativistic universe each point of the figure stands for a universal state common to all observers, in the relativistic case a single point is, as we have seen, no longer sufficient. In order to create a visual representation of this complication we will use all three dimensions of standard three-dimensional space and therefore the model will depict a branching relativistic universe of one-dimensional space - i.e. of a single straight line. So, whereas standard Minkowski diagrams can, in principle, also represent Space-Time of two spatial dimensions, the present diagram has to be confined in just one.

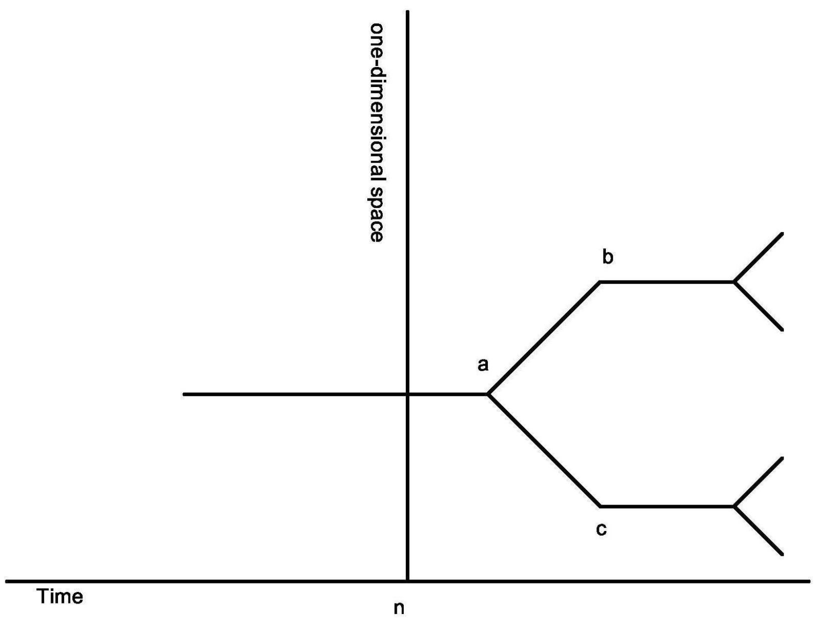

Figure 2.

We begin by enriching the branching figures we used for non-relativistic universes. Consider figure 2. The branching tree is the S-model that our *here and now* belongs to. (Let us say that now $= n$). In non-relativistic universes, the substructure corresponding to *now* would have been the same for every observer of the assumed one-dimensional space. Since this is not true for the relativistic case, we draw a vertical straight line passing through *here and now*, and standing for space. To each point of the line we associate a set of ideal agents occupying that point at moment n. Each such set is constituted out of agents moving in different directions and at constant velocities along the line. More precisely: suppose that there are m possible velocities; since the number of possible directions is two, each such set will have $2 \times m$ members. Now, to each of these members, we associate a different substructure which has the same general properties as the substructures of the non-relativistic model; the structure formed by these is the required S-model, with respect for moment n. The complete S-model is obtained when this procedure is repeated for all moments. Little remains to be done in order to convert this diagram into a Minkowski diagram. First, following the general convention concerning Minkowski diagrams, we turn it upside down, as in figure 3.

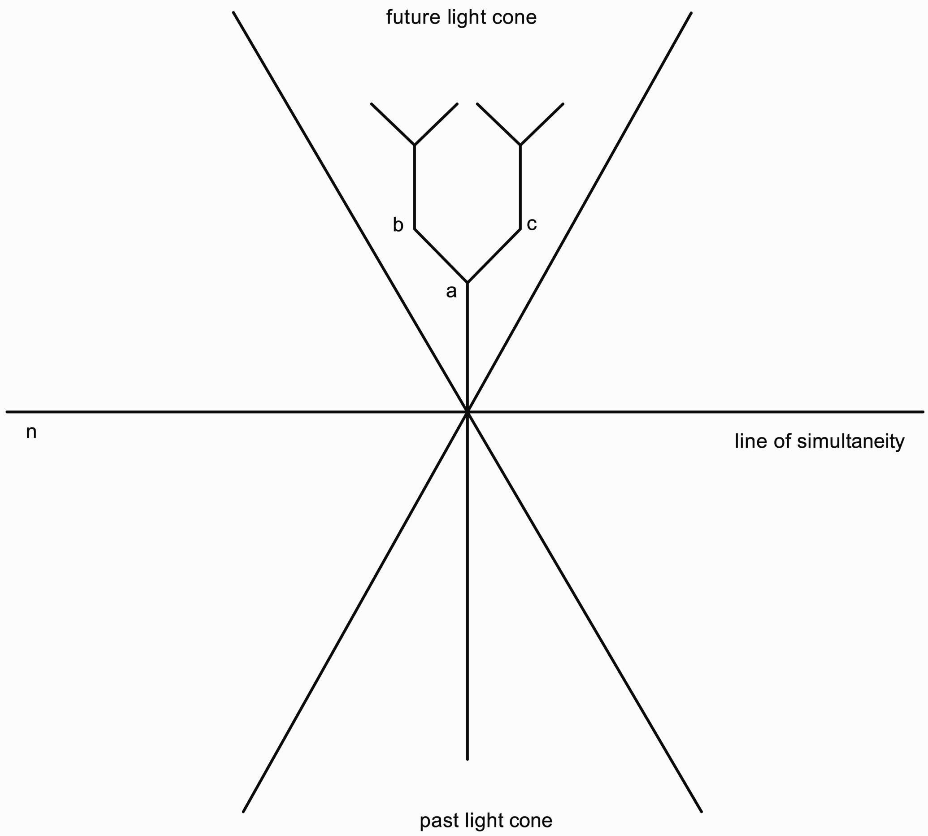

Figure 3.

Figure 3 looks like a standard Minkowski diagram for one-dimensional Space. The horizontal line is the line of simultaneity; the cone above is the future light cone, containing all the future point-events that the ideal agent can, in principle, attend, and the cone below is the past light cone, accumulating all the information the agent has received. As we have said, the future light cone defines the point-events that the agent can, in principle, attend. In reality, and for deterministic universes, all these point-events are upon a maximal linear order, represented by a non-branching line. (Everything is fully determined in these universes, as is the agent's journey - i.e. his world-line.) This leads to the next necessary transformation. Obviously, the tree that in figure 2, was the representation of the S-model can no longer receive the same interpretation, when integrated in figure 3. In figure 2, each point represented a (different) possible universal state of the world at a certain moment. Now each point stands for a point-event. These two interpretations are incompatible. For example: in figure 2, the point that follows n suggests that there is only one possible next universal state for n. The same point, in figure 3, suggests that the agent will not accelerate.[13] This change of interpretation, however, causes no insurmountable problems for the figure, since the interpretation of that point in figure 2 is the union of all substructures assigned to the line of simultaneity of figure 3, at $n + 1$.[14] The interpretation of branches in figure 2 underdetermines the branches of figure 3, but this poses no problem, since we no longer assume the same interpretation for points in general. Of course, all branches will have to be transcribed into a different form,[15] but, once this done, they can mislead no one. Time's march into the future will be depicted by the movement of the entire simultaneity line, instead of the single point of figure 2. Now, the real problem concerns points of figure 2 that are incompatible with one another. Consider, for example, b and c of the same figure. In figure 2, they stand for synchronic mutually exclusive universal states. In figure 3, however, they have to represent some synchronic *and* compatible with each other point-events. This problem gives justice to Belnap's intuition that each Minkowski plane represents a (different) maximal linear History;[16] the whole of them representing what he calls "Our World".[17] In order to overturn this last problem, we will take advantage of the, as yet, unemployed third spatial dimension by representing

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[13] Strictly speaking, the agent can never accelerate in a Minkowski diagram, since Minkowski diagrams represent world-lines, according to Special Relativity, which does not account for acceleration. We here follow the facilitating trend and represent (sometimes) such world-lines upon the plane of a Minkowski diagram, and not in a Riemannian curve. As for the exact way to transcribe a Minkowski diagram to a Riemannian diagram, see P. MITTELSTAEDT - P. A. WEINGARTNER, [2005], pp. 124-127.

[14] We have, however, to eliminate substructures corresponding to agents accelerating after n. This elimination is necessary in order to avoid contradictory orderings of the spatially separated events. (See point 2 of the next part). In any case, if it is Special Relativity we are dealing with, this can be skipped.

[15] e.g. the straight line below n will have to record the movements of the agent before *here and now*, and these might be of a nature different from the non-acceleration suggested by the line in figure 3. The line should be bending, if the agent has not remained at the same spot for his entire past. In the way it stands in figure 2, it just connects the spatiotemporal events that the agent has attended, or will possibly attend. It says nothing about his pace. The two different interpretations of the line, in figures 2 and 3, correspond to the difference between what Belnap calls "causal dispersion" and "causal branching". See N. BELNAP, [1992], p. 388.

[16] N. BELNAP, [1992], p. 399: "*A Minkowski branching space-time* is a model of **Our World** in which each history is a Minkowski space-time (in the standard sense found in the literature)." Belnap speaks of "Minkowski space-time" rather than "Minkowski plane", but the correspondence is obvious. A Minkowski Space-Time "in the standard sense found in the literature" is a deterministic chronicle of a relativistic universe, which is represent*able* by a (standard) Minkowski diagram, and *vice versa*. A standard Minkowski diagram is a single plane; it does not fork. Notice that, in the postpript of 2003, the above is called a "*rough* definition" (my emphasis) and the non-formal character of it is further accentuated in a footnote.

[17] Cf. N. BELNAP, [1992], p. 387ff.

the forks of figure 2, no longer as forks originating from a point, but as forks of the entire simultaneity line. Each maximal History will now become a maximal non-forking plane, no two different point-events of which could exclude each other. On the other hand, the union of every two different maximal planes will contain at least one such pair of mutually exclusive point-events. If we apply all this to our previous example in figure 1, we get figure 4.

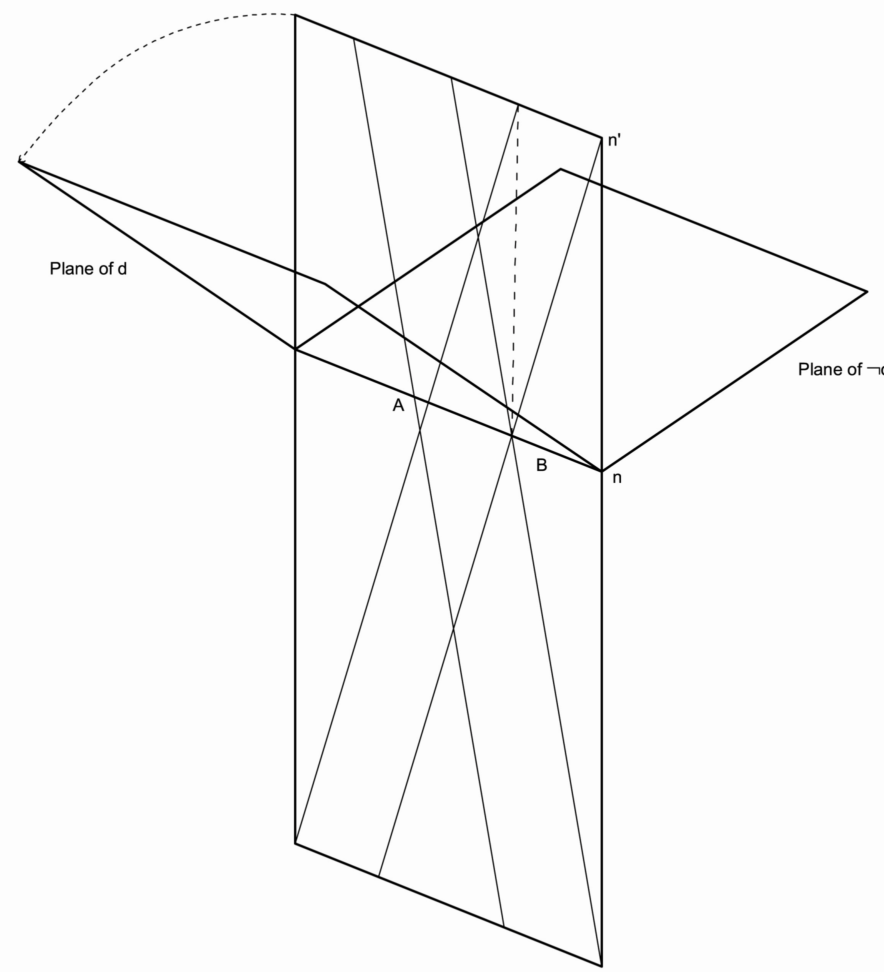

Figure 4.

"Our World", looked through the angle of A, forks at n. Looked through the angle of B, it forks at n', for it is at this point that the world "chooses" to either collapse upon the plane where d has taken place, or upon the plane, where it has not. Notice, first, that the two maximal planes corresponding to the inertial frames of reference of A and B are not identical, even though their two segments following n' are going to be such, and notice also that the information received by B at n' gives a truth-value to both the sentence concerning what this information is -i.e. decides the path- *and* to the past sentence about d itself. What the information will be at n' is a function of whether or not d has taken place, but the event of B's world-line that creates the branching at n' is the very reception of the information at n'; in other words, the reception of the information about the event is itself an event. *Mutatis mutandis*, the same applies for C. Notice, finally, that A, at n, is aware of the plane "Our World" has chosen, and he is also aware of the fact that this choice concerns B and C, as well. B and C, on the other hand, are aware of the fact that a choice has been made in the spatial coordinates A occupies at n, and that they will necessarily learn about that choice at n' and n'' respectively. By that they legitimately assert the disjunction "d occurred at n, or d did not occur at n", but, since no state represented by the disjuncts has causally interacted with their world-lines, they assert neither of the disjuncts. Therefore, the sentences "d occurred at n" and "d did not occur at n" will not acquire a value in their world-lines, before these two moments.

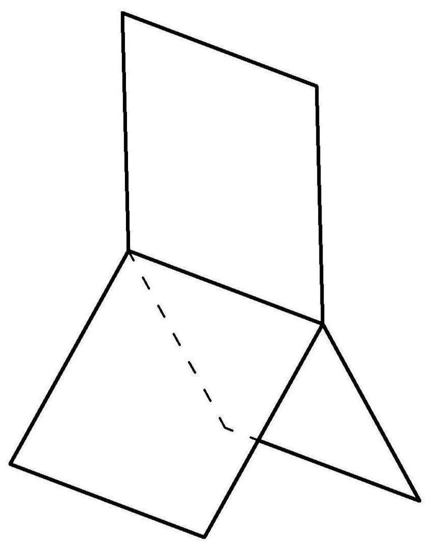

Figure 5.

It is in that sense that we claim that, in branching relativistic universes, the non yet settled states[18] do not concern the future uniquely. They concern the present and the past as well. More precisely, any self-caused event, which is

[18]Which are to be distinguished from indeterministic states in general. The latter concern also past events of which no evidence remains. The "non yet settled" imply that a certain indeterministic event has occurred (or is now occurring) somewhere in the universe and that its causal impact has not yet reached our *here and now*, but will eventually cross our world-line, and by so doing will cause a future "choice" of the plane. The others suggest that a certain past event has stopped having any singularizing effect and so one cannot tell what the maximal plane of the universe has been before the convergence. These events cannot meaningfully be said to be "non yet settled"; they have been settled during the past but following the convergence, they will remain indeterministic forever. In the figures we can distinguish between these two kinds of undetermined past events from the way the planes fork. The forking appropriate for the category of "forgotten" events suggests no distinction

simultaneous with the occasional *here and now* will remain non-yet-settled, until the information about its occurrence enters the past light cone of the world-line of the same *here and now*.[19] The same applies for the events that causally depend upon self-caused events.

See now, how intuitionistic truth-value assignments provide a different model from classical assignments. The crucial question is the following: when does the "choice" take place for the world-lines that do not cross the spatio-temporal coordinates of the indeterministic event? Belnap says that for these the choice is made as soon as the event causally interacts with them and that these world-lines participate in a "choice" which is not their own.[20] We agree with that but then fail to see why Belnap dismisses the idea that indeterministic events could be represented by branching planes.[21] He chooses instead to posit a different (Minkowski) diagram for each world-line, and he formally renounces the option of uniting all these diagrams into one shape, as we have done in figure 4. The explanation he provides is that by such a unification all the world-lines that have not suffered any causal consequence of a certain indeterministic event as yet, will be forced to prematurely respect the splitting concerning the region of the event's occurrence. This, according to Belnap, violates the principle that forks need to have a causal explanation, no matter when and where they occur. Now, before the time that the indeterministic event enters the past light cone of a world-line no such explanation can be provided for the fork. Therefore the line does not fork until the moment of the interaction. To that, we also agree, but by our having economized in an extra-dimension we are in a position to represent how the same indeterministic event causes world-lines to fork at different moments within the universe ("Our World" in Belnap's terms). Thus, we are in a position to sustain, without contradiction, that the forks are forks of the entire Minkowski plane, represented by lines of simultaneity splitting in two. This no longer condemns us to conclude that the forking of the Minkowski plane with respect to the substructure of A implies that the plane that corresponds to the substructure of B has forked as well. Perhaps the reason that prevents Belnap from considering the splitting of simultaneity lines is that he is still under the spell of classical truth-value assignments, and this spell suggests that the lines of simultaneity in Minkowski diagrams are complete - i.e. that there is no gap in these lines. If so, the line of simultaneity of the world-line of B, at n, would need to contain the event d, and consequently B would need to assert an event that has yet to have any causal impact upon his region. By our abandoning the idea that lines of simultaneity are continuous, and by *eo ipso* allowing for

between the two possibilities (see figure 5). On the other hand, the kind of forking that emerges after the arrival of the information about a distanced self-caused event forces all planes to collapse upon the planes where the event was known already. We note here that Belnap disallows backward branching (cf. N. BELNAP, [1992], pp. 388ff.), for reasons that we will not expound here. See also his postpript of 2003, on that matter.

[19]This illuminates also the title of the Appendix. In relativistic universes a spatially separated Space-Battle might have occurred yesterday while the sentence describing it might yet have to acquire a value. It is not only the future, but also the present, and the past, that might be partially underdetermined.

[20]N. BELNAP, [1992], p. 413ff., p. 422; point (ii).

[21] N. BELNAP, [1992], p. 412: "*splitting in* **Our World** *occurs at point events, not at simultaneity slices, and affects only the causal future*". Notice that, in the postprint of 2003, "necessarily" is added between "not" and "at". See also an addition in the corresponding footnote.

present indeterminacies,[22] we can distinguish the lines of simultaneity of A and B, and claim that their being distinguishable depicts the truth-value of, "d takes place/has taken place", which is true for A, valueless for B, and makes the line of B incomplete. Thus, we can consistently represent "Our World" by a unified three-dimensional shape, instead of an infinity of juxtaposed two-dimensional planes.[23] A representation that is more "eloquent", because, now, the fact that all divided planes, like the ones of figure 4, will eventually merge into one is suggested by the figure itself; the fact that all these will come to share the same History is now diagrammatically evident.[24] On the other hand, Belnap needs to single out which parts of his juxtaposed Minkowski diagrams are identical and which differ, by adopting extra notational facilities.[25]

To say that the simultaneity line of A's world-line splits in two at moment n, and that that splitting concerns also the world-lines of B and C is exact; it suffices that one adds: *from the perspective of A*. For otherwise -i.e. from the perspectives of B and C- the simultaneity line splits in two at n' and n'', and the event causing that splitting is the causal impact d has, during these

[22]In the diagram, this says that the simultaneity line of B at n, contains at least one blind spot: $d/\neg d$, at n. In classical logic, this concerns only B's *episteme*. In the intuitionistic model we have presented it concerns d itself. The line itself is not continuous; it has a gap at the point of the event $d/\neg d$.

[23] In that way we combine Belnap's answer to what he calls "the problem of the wings" (cf. N. BELNAP, [1992], ch. 8), which (answer) dictates that there can be no forking without some causal explanation, with our own adoption of a unified three-dimensional Minkowski diagram for the representation of the entire "Our World". A further reason that Belnap renounces considering the possibility of worlds splitting along simultaneity lines and not points is that he finds this splitting incompatible with the Einstein-Podolsky-Rosen phenomenon. What he says is that "you will be permanently perplexed if you try to analyze EPR in terms of a simultaneity slice" (ibid, p. 419). There is no doubt that any analysis of the phenomenon upon the basis of a diagram where the two distant correlated events are linked by a simultaneity line presents serious problems: no one among the two spatially separated indeterministic events has any causal relation to the region of the other, and yet the inhabitants of this latter attribute a truth-value to the sentence describing it. Nonetheless, the reason that the use of a simultaneity line seems to be problematic is that *any* representation of this *sui generis* correlation (see N. BELNAP, [2003]), becomes such by the moment one subscribes to the view that any splitting has to be accountable by a causal chain passing though *here and now*. See, for example, one of Belnap's proposals for overturning the paradox. He focuses attention upon the fact that the two correlated events have to share a common point in their pasts, during which the preparation of the correlation has taken place. Now, this guarantees that they belong to the same Minkowski plane, and, although this past point provides no causal explanation in the strong sense (e.g. Reichebach's) for any of them, it *does* provide a reason for excluding any combination that is not allowed - i.e. we can say that the non-allowed combinations share no common (preparatory) point, and, therefore, belong to different Minkowski diagrams. This is exact, but one can claim the same in our model too - i.e. for cases of two spatially separated events that are upon a simultaneity line and are correlated in an EPR way, our multi-plane will fork in two instead of four branches. In our representations too, each member of that pair will share a common past moment, during which the phenomenon has been prepared. This, of course, is far from being a satisfactory (final) explanation of the correlation (Belnap himself, time and again, accepts the tentative nature of his explanation), and, moreover, it is not closer to the mark, when attempted by Belnap's method of juxtaposing diagrams (cf. N. BELNAP, [1992], figure 10). The problem remains essentially the same: an observer is supposed to know about an event that seems to have no possible causal relation to his region!

[24]This phenomenon is called, by McCall, "branch attrition" (see S. McCALL, [1994]), and is parallel to a similar phenomenon of non-relativistic, but indeterministic, Histories, elaborated in chapter 3. Back then, it was the branching lines that tended to generate a final unique line, now it is the branching planes that tend to generate a final unique plane.

[25]Cf. figures 4 to 10 of N. BELNAP, [1992].

moments, in the regions of B and C. Afterwards, once the splitting is effectuated for the world-lines of these observers too, all three observers will be sharing a common past, as is suggested in the figure, by the ulterior unification of the planes. The same thing can be said as follows: at moments n' and n'', a causal chain that has been inaugurated by the self-caused event d interacts with the world-lines of B and C respectively. This interaction defines the moment of the splitting of their simultaneity lines. Which also says that whereas in non-relativistic universes, the splitting of the line was common to the entire universe, and occurred at the moment that the new strong causal chain initiated itself, in relativistic universes, it is postponed until some ulterior moment, for all points that are spatially separated from the event. So, the splitting in two, in relativistic universes, can be made either from a self-caused event (if this event happens to happen upon the world-line of the agent), or from an event that has the self-caused event as a necessary condition and happens upon a world-line that does not cross the latter event, but still is the world-line of the agent.

II

In the main corpus of this book, the notions of the "final scientific theory S" and its "ideal agent - *Ideal Scientist*" have been adapted to non-relativistic universes. Here we are going to show that to switch from non-relativistic to relativistic universes is a move that does not touch the degree of plausibility the intuitionistic solution might have, or might not have. It complicates the model, but the solution remains essentially the same. We are going to advance that claim in five steps. Firstly, we present what would have happened to a Minkowski diagram, if tuned to a Newtonian universal clock. Secondly, we present the case of a relativistic[26] but deterministic universe. Thirdly, we elaborate the consequences of making the above universe indeterministic. The forth point concerns the indeterministic events in General Relativity, and, finally (fifth point), we elaborate an argument in favor of the indeterministic truth-value assignments presented at the beginning of the Appendix, and against the classical ones.

1. In Newtonian universes, there is no limitation as for the velocity of the information. Therefore, there is no contradiction in assuming that every ideal agent of S becomes aware of every event by the time it happens, at most. If the event is not indeterministic, he knows about it by the time there are sufficient conditions for it. Which is tantamount to saying that in deterministic universes, he diachronically knows all events. The interesting question concerns what happens when a self-caused event occurs in a region that does not belong to his world-line. In order for us to stay obedient to the assumption that he knows everything that happens, by the moment it happens, at most, we must assume action at a distance; we have to assume that the information about the event reaches him in no time - this is not excluded by Newtonian mechanics. The Minkowski diagram would, then, look like figure 6.

[26] Of Special Relativity.

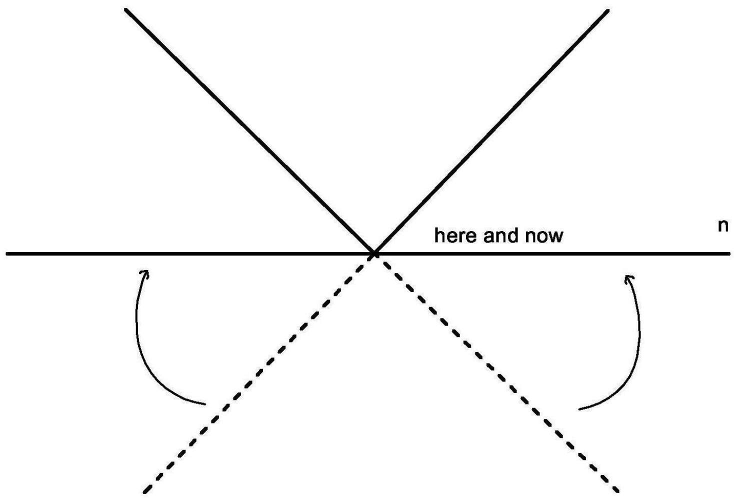

Figure 6.

The past light cone has degenerated into the semi-plane defined by the area below the line of simultaneity. If no convergence has emerged up to n, the agent knows every present *and* past event. The future light cone can remain as in the standard Minkowski diagrams, but, since the speed of light has stopped being a universal barrier, there should be no *a priori* reason imposing the limitations that the same cone suggests. The agent could (theoretically) move outside the cone without making his clock run backwards. All this translates (is equivalent to) the pre-Einsteinian assumption that there is a unique frame of reference, all travelers of which are tuned to a common clock.[27] Let us now go back to the infinity of ideal agents that were posited in part I of the Appendix. It is clear that any two agents would share, at identical moments, identical amounts of knowledge about the world, even if they inhabit indeterministic universes. Some agents would happen to be more distanced from a certain indeterministic event, than others. This distance, however, would not prevent these latter from "learning about it", simultaneously with those that, literally, attend it. If so, and since all agents share, at all moments, identical bodies of knowledge, one can simplify the diagram by falling back to the treelike two-dimensional figures of previous chapters.

2. The situation becomes more complicated, when one considers what happens in relativistic, though not indeterministic, universes. The tricky part concerns the question, whether one needs to postulate a third dimension in order to accurately represent their Histories? We will see that one does not. Belnap's "Our World" is, for these universes, a single Minkowski diagram. Of course, it is still the case that the light signal emitted by an event will necessarily arrive at the agents who are closer to it sooner than it will arrive at the agents who are more distanced; but this does not mean that the former will "know about

[27] This "Minkowski" diagram, adapted to pre-Einsteinian physics is, of course, contradictory. The very reasons that relativity prevails over classical mechanics see to that.

it" sooner than the latter. In such universes, any event of the future is always causally predetermined and, therefore, a strong causal chain leading to it propagates endlessly back into the past. Now, as it is obvious from the diagrams, all past light cones tend to have the same (infinite) basis. Which means that all ideal agents, no matter where they happen to be located at present, contain, in their past light cones, some part of the maximal strong causal chains leading to all future events. In other worlds each maximal strong causal chain propagates endlessly back into the past, and consequently, intersects with the common to all agents infinite basis of the cone. And, since any link of any strong causal chain implies all the links that follow, the agent knows all future events, no matter whether the information about the advent of the event has crossed his world-line or not.[28] Seen from that angle, no depth in the representation of deterministic Histories of relativistic universes is needed. On the other hand, some among the states known by the agents are -in such universes- functions of agent's direction and speed. These states concern the ordering of the spatially separated events. This is because the temporal ordering of events may vary, according to the frame of reference of the agent; the frame influences the ordering. If, for example, a and b are two events of the Minkowski plane, any two agents -at any *here and now*- will agree that both a and b obtain. On the other hand, and if a and b are spatially separated, they might disagree about the answer to the question concerning which one comes first. However, even if this complication is taken under consideration, the resulting graphic representation is still less complicated than the three-dimensional figure, appropriate for universes that are both relativistic *and* indeterministic. Its relative simplicity is due to the fact that, for any two agents that have two parallel simultaneity lines, the ordering of spatially separated events is identical, and, thus, one can randomly select a maximal world-line,[29] and associate to each one of its points an infinity of agents. Every one of these will be assigned with a different simultaneity line passing through the point, and their totality will represent the set of all simultaneity lines through the same point (see figure 7).

[28]The same argument can be advanced as follows: Consider the fact that the content of the future light cone increases monotonically, as one moves deeper and deeper into the past. This says that for an idealized observer who is infinitely many moments away into the past the future cone contains every event there is. Now, if every event is deterministic, the observer can, at that moment, and by his perfect mastering of S, deduce all events, since all events are located upon world-lines that could, in principle, be the ones corresponding to his own travel. But such an observer is any ideal agent of the model, when infinitely younger. Which, in turn, implies that, provided he hasn't forgotten anything in the meantime -and he cannot forget anything in deterministic universes- he always knows the same totality of things; i.e. every single event of the plane.

[29]The simpler choice is a world-line of an agent that never accelerates. This is a straight line.

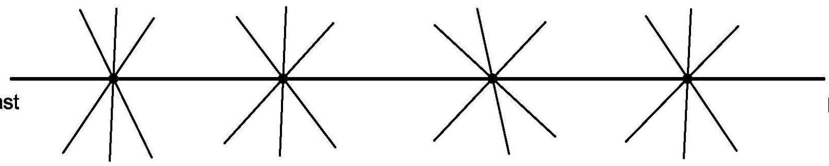

Figure 7.

Now, since in the universes under discussion all agents with the same directions and speeds (i.e. having parallel world-lines) order events by identical orderings, we can use the randomly selected world-line as the representative of all the rest. This says that any agent occupying any point upon the simultaneity line of an agent of figure 7 will, if his simultaneity line is that of the agent in the figure, necessarily assert the same set of sentences as the agent in the figure. One world-line suffices to represent the entire set of possible universal states. We have only to associate to each point of the line an infinity of agents moving according to all the allowed directions and velocities, and graphically represent them by simultaneity lines. Depth is still not needed.

3. Universes that are both relativistic (according to Special Relativity), and indeterministic, are representable by the Minkowski three-dimensional diagrams of the first part. In such universes, it is indeed the case that the indexical "here" in *here and now* cannot be eliminated. This is because the indeterministic events of these universes make the moment at which the lines of simultaneity fork a function of how distanced from the event is the agent occupying the line. It is because of this that depth is necessary for their graphic representation. However, what has to be noticed is that, in Special Relativity, although two agents occupying different points of a certain line of simultaneity or occupying the same point, but having different speeds and directions, might disagree about whether a sentence representing an indeterministic event is true, they will not do so forever. This is due to the null geodesics of Special Relativity and to the consequent fact that all past light cones tend to become identical, as time goes by. Since the future light cones of all travelers tend to enclose the same totality of events, all ideal travelers will tend to acquire identical amounts of knowledge.[30] In other words, although it is the case that information about the indeterministic event will enter the past light cone of the agent that is closer to it earlier than that of the agent who is more distanced, it is impossible that it will never enter the past light cone of the latter. This, in figure 4, is depicted by the fact that all possible planes tend to become identical by gradually collapsing upon the plane of A - e.g. by the time the information reaches all three observers, they all share the same plane. The above phenomenon of indeterministic Special Relativity is sometimes expressed by the so-called Cocchiarella's formula: $p \vee Pp \vee Fp \vee FPp$.[31]

[30] Which, nonetheless, will never actually be such, as long as there are still indeterministic events to come.

[31] N. B. COCCHIARELLA, [1984], p. 347ff.

In order to see what this formula says, let us, for a while, assume a bivalent context and suppose that one can contemplate the model from an atemporal standpoint, and that, from that standpoint, it is atemporally decided which are all the point-events of Space-Time. The formula says that, if something is such an event, then, for all observers, it is either happening now, or has happened, or will happen, or it will be the case that it has happened. This is an obvious weakening of the classical tripartition $p \vee Pp \vee Fp$. The newcomer FPp makes it that some events might be neither present, nor past, nor future. They will, in the future, become past, but, during a certain period, they are none of the above. An instance of such an event is the event d of figure 1, with respect to the Histories represented by the world-lines of B and C. For these agents, d will enter their past light cone as an (already) past event. So, at n, the only thing that obtains for the world-lines of observers B and C is that they will, in the future, assert that d is a past event. Now, the fact that it will eventually become a past event for all three agents, and, in general, for all observers, exemplifies the above-mentioned characteristic of Special Relativity. All events become known to all idealized observers as time goes by, and no matter how distanced might they be, or how fast might they be traveling away from them; they cannot travel faster than light, and in the future, information about the event will, unmistakably, reach them.[32] The classical $p \vee Pp \vee Fp$ satisfies self-caused events, only with respect to travelers attending them. For every other traveler, it is FPp that obtains, because all other travelers attend, at the time that p occurs, events that are spatially separated with respect to p.[33]

A consequence of the validity of Cocchiarella's formula is that it still does justice to the basic intuition of chapter 3, according to which all origins of an S-model are connectible with one another. Which, in the tensed idiolect, says that all states that diachronically obtain can now (or will in the future) be presented in the form of present tensed (atomic) formulas preceded by a number of tense operators. This also implies that, in an S-model, there are no two isolated substructures - i.e. no substructures the points (worlds) of which are (diachronically) not interconnected.[34] For any two origins, there is a path connecting them, and, so, there is no need for the observer inhabiting the one to use a detensed language, in order to describe what the other witnesses.

4. Things are different in General Relativity, where the bending of light forces geodesics to become non-null. In such universes, there might indeed be pairs of world-lines that will never exchange information. Consider, for example, figure 8.[35]

[32]Notice that Cocchiarella's formula obtains only if we assume the atemporal perspective for the metatheory and have as object languages the languages of the inertial observers. For, in the metalanguage of -e.g.- B, 'FPp' is not true at n. Notice also that $FPd \vee \neg FPd$ obtains diachronically for B, because of the future bar at n'.

[33]It is instructive to notice here that, according to Cocchiarella, the formula $p \vee Pp \vee Fp$ is not valid in relativistic universes, but this is so only with respect to what he calls "causal tense operators" - i.e. causal "P" and causal "F". In my system, however, it is invalid *simpliciter*, because, by my general truth-value assignments, I have turned Cocchiarella's "causal tense operators", depicting what Prior calls "Cocchiarella's causal relativistic time", into the standard tense operators that measure standard relativistic time.

[34]Cf. the "separated frames" of chapter 4.

[35]Figure 8 follows -suggests that it follows- Riemannian curved space.

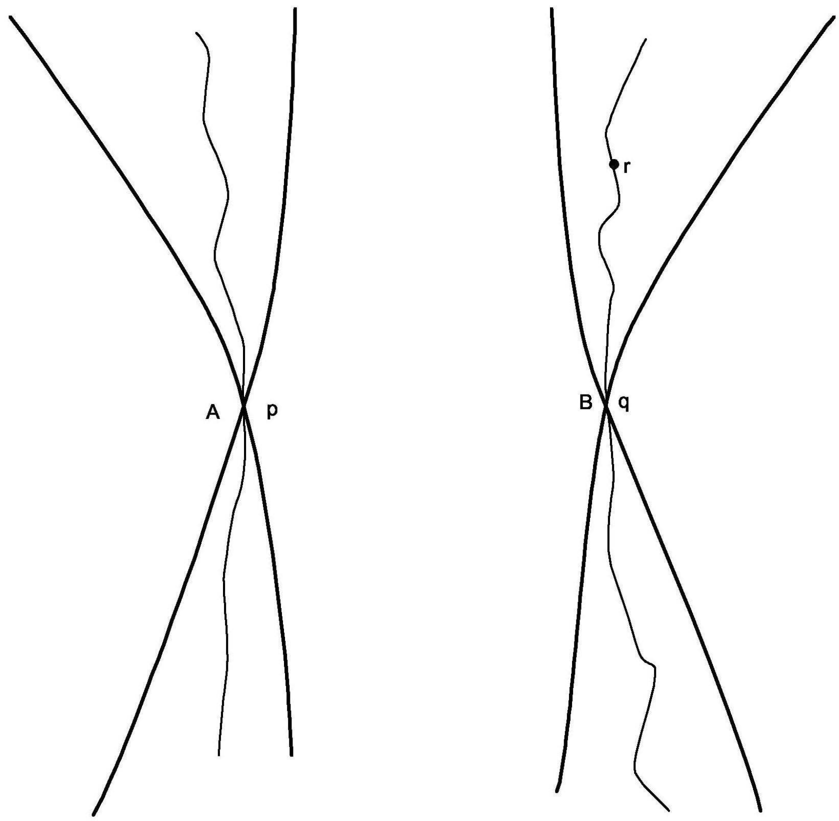

Figure 8.

Observers A and B will never exchange (can never exchange) any information. Despite the fact that p obtains at n upon the world-line of A, and q obtains at m upon the world-line of B, p and q will never be asserted (by B and A respectively). This is because these two observers could not possibly occupy the *here and now-s* of one another, which is tantamount to saying that there is no path connecting their spatio-temporal coordinates, and, so, not one of them could attend the events occurring upon the world-line of the other. Moreover, each one of these is obliged, when making a hypothesis about what the other witnesses, to index p and q by an index denoting the substructure of the agent that attends the event. This indexing is necessary, because the advent of this particular p and the advent of that particular q could never be an event obtaining at the *here and now* of the agent that is not witnessing it. The indexed sentence that this latter uses in order to state what obtains in the world-line of his "colleague" will have to be a B-series sentence, since it describes an event that is not relatable to any event of his world-line by any among the "past to ...", "simultaneous with ...", "future to ..." relations; the entire world-line of his "colleague" constitutes a B-series sequence that is not translatable into an

A-series sequence of his own.[36] A can only say that q precedes r in the substructure of B; he can never order q with respect to any event of *his* world-line, and therefore, not only the non-relativistic $p \vee Pp \vee Fp$, but also Cocchiarella's $p \vee Pp \vee Fp \vee FPp$ fails in General Relativity. Because of that, the entire S-model that is constituted out of all substructures loses the property of having no isolated substructures. The unified network metaphor, used in chapter 3, ceases to be the appropriate metaphor for the model. The appropriate model is still not a $T \times W$ one, but, on the other hand, not every point of it is reachable from all others. In a sense, the agent inhabiting one isolated region of the S-model approaches things that happen in other isolated regions in much the same way that beings that occupy the atemporal standpoint do. Suppose, for example, that, upon an origin of a substructure belonging to a region, to which a certain observer has no access, a Sea-Battle takes place, and that this is an indeterministic event. As for the S-model, this necessitates that there is also a substructure, at the origin of which no Sea-Battle occurs. Now, the observer inhabiting the region that is precluded from the event will, at no point, call one of these two substructures "the real one". The problem concerning what *really* happened in this far-away (with respect to him) region is not even raised. He positively knows that, with respect to the agents located at points that are sufficiently close to the event, the matter becomes, at some moment, decided, but, as far as his perspective is concerned, all mutually exclusive substructures of the far-away region are diachronically contemplated with an equal mind. No one among them was/is/will be the *real* one. The indexed sentence that represents for him the event in the far-away region remains valueless forever, just as its negation does. On the other hand, the disjunction is omnitemporally asserted, because of our standard intuitionistic supervaluation. Notice here that the understanding the agent has of such far-away point-events is perhaps better than the understanding an atemporal God might have had of them. This is because, for the temporal observer, the tensed sentences, "There is/was/will be a Sea-Battle at this point and at that time" are perfectly transparent, no matter how distanced might he be from the region of the event. He understands perfectly well what they mean, and he also understands that the observers who are sufficiently close to the event do, at some moment, either affirm or deny them.[37] On the other hand, it is debatable whether an atemporal being -like an atemporal God- could, under any circumstances, understand what these sentences mean. These sentences contain tenses, and he does not understand tenses.[38] The reason for this possible asymmetry between a temporal being that meditates on

[36] Notice that the line of simultaneity of the one observer never includes any of the events of the world-line of the other. The events are not defined with respect to it.

[37] Here we have in mind world-lines that are diachronically alienated from one another. For cases of world-lines that have a common origin, but have been mutually isolated at an ulterior moment, the situation is more complicated. In such combined structures, there is always a path connecting any two of their point-events, and the appropriate formulation is to say that the observer occupying what has become an isolated region will not only understand what, "There is/was/will be a Sea-Battle at this point and at that time" means, but also be aware of the fact that, even in his own past, "There will be a Sea-Battle at this point and at that time" was a sentence about a possible point-event of *his* substructure that would, eventually, acquire a determined value. It is only that his world-line has lead him to a region from where it is no longer possible to seek for the real/determined value of, "There is/was/will be a Sea-Battle at this point and at that time".

[38] He might, however, not need to understand tensed sentences in the first place, if he does not use, but just mentions them. For that point, see chapter 3.

what happens in a far-away region and an atemporal being is that any observer of Space-Time inhabits a unified field, where tenses make sense throughout, and he can, thereby, understand what tensed sentences about far-away events mean, by analogy. He can mentally substitute himself for the observer inhabiting the far-away region.

5. Before closing this Appendix, I will try to challenge an argument often used to show that relativistic universes are incompatible with indeterminism. Meeting this deterministic argument is possible, because of my intuitionistic general value-assignments. In other words, I claim that the argument in favor of determinism assumes a two-valued context, and can sound convincing only in that context. The argument tries to exploit the presence of an infinite number of different frames of reference and the absence of a common temporal ordering for the events of the entire Space-Time. More precisely, in Special Relativity, for any two spatially separated events, there is one inertial frame of reference, in which they are simultaneous, an infinite number of such frames where the one occurs earlier than the other, and an infinite number of frames where the converse obtains. Therefore -some people argue- there can be no indeterminacy in relativistic universes, for, even if there is a future indeterministic event in -say- our frame of reference, there has to be an infinite number of other frames, where the event (or its non-occurrence) is past already, and has, thereby, become necessary. Moreover, there are frames, in which the event has passed into history at moments simultaneous to, or prior to, what is now the present for us! The argument can become even more appealing when one considers that the world-lines of these frames are crossing our own.

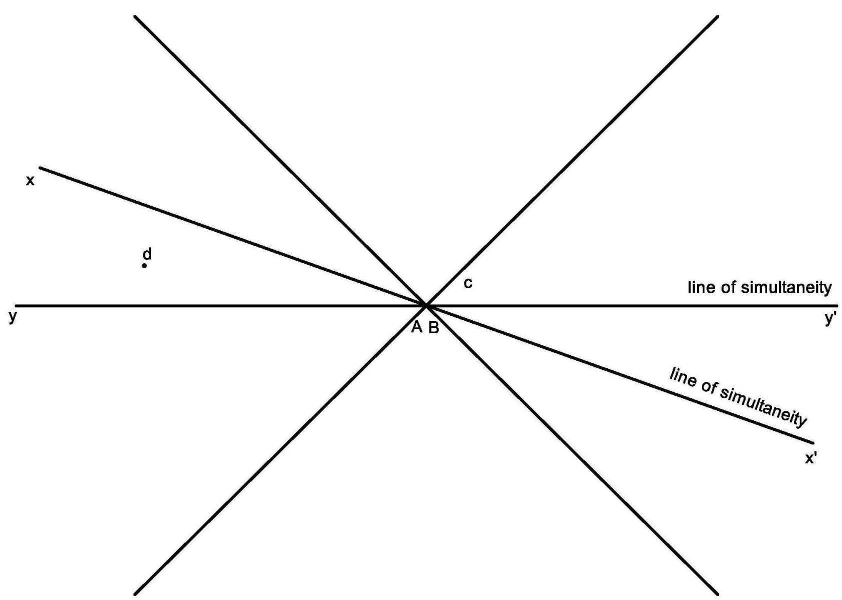

Figure 9.

Figure 9 is revealing for that. Ideal agents A and B are, at present, co-witnessing the event c. Consider that our substructure is A's, and that an event d (or any other event within the angle $< x, c, y >$), is indeterministic. As such, and since it is future with respect to our frame (A's frame), our substructure should assign no value to the sentence describing it. But, at the same time, it is a past event in B's frame and, so, one of the two possibilities is already eliminated there. The sentence has a value. The question now arises about the sense in which A might be justified in saying that the event is not yet settled, while, for B things are, by *historical necessity*, settled already. The deterministic conclusion is that there is no such sense justifying the claim. One might even imagine these two agents talking to one another[39] and disagreeing with respect to the truth-value of the sentence "¬d is still possible". How can this be true for A, when d has already happened for B?

According to our analysis, the question is misleading on two accounts. In the first place, and as for our general truth-value assignments, the above sentence is neither true nor false for any of the two, since d is indeterministic and the information about it has yet to reach B. Had it done so, it would have reached A as well. Nothing can travel faster than light, and, so, B could not possibly have transmitted the information about d to A before light itself. Now, within the entire S-model there have to be agents for whom it is both the case that the event is history, and who (at the same time), have already received information about its occurrence. But these latter cannot possibly spatially coincide with A, at c, for the reasons we mentioned in the last sentence but one. The problem arises only, when one assumes a -so to speak- "trans-structural" bivalent assignment for the events of the entire S-model, and so eliminates one of the basic postulates of chapter 4, according to which there is no overall assignment ranging over all the substructures of an S-model.[40]

Moreover, and from a less formal point of view, even if the event is now past for B and still to come for A, this does not make d more unpreventable from A's perspective, than it would have been if it did not belong already to B's past. The event is spatially separated from A and so, the way the world will settle, as far as d is concerned, is not (cannot be) a matter of his. The question about whether d or ¬d will be the case is no less outside A's reach, than it would have been, if d had settled already in A's frame of reference as well.

By the above transposition, we have granted to the determinist a premise even stronger than his own - i.e. we have granted him that any question concerning any event falling outside the occasional future light cone of any observer is "doomed already" for the same observer: the matter of d is already closed, even for A. We take our distance, however, from the deterministic argument, by suggesting that this is not because there are always sufficient reasons for d, but because A can, while attending c, do nothing about d (or ¬d). The determinist would have a genuine case within his hands, only if A could still undertake some action against d, while B could not. But this would have implied that d is temporally, and not spatially, separated from c, and, so, it would have implied that B also could undertake some action to the same effect, which is impossible.

[39] This is possible, if their "meeting point" is sufficiently far away from the event, and so, B does not have to have a very high speed.

[40] Cf. also the hybrid language we have used in the presentation of Cocchiarella's formula in 3 above.

Appendix B

Tarski Schema with Material Implication for "Classical" Supervaluation

In chapter 5, we examined the sense in which the Tarski Schema could be made valid in S-models: it was partly because implication is not interpreted as material implication, and partly because the atomistic calculus is complete. We also said that the reason why the same Schema fails in Van Fraassen and Thomason is that these authors interpret implication materially, and disallow gaps in the metalanguage.

We are now going to show that even if one keeps the standard (material) interpretation of the connective, and also assumes a bivalent metalanguage, Tarski Schema can remain valid in models of "classical" supervaluation, provided that

(i) a further domain is added: the domain of states,

(ii) the quantifier running through this domain behaves classically,

(iii) the state predicate "...obtains" is functionally complete.

The problem with Tarski Schema in "classical" supervaluation is that the untruth of a metalinguistic statement predicating truth of a sentence of the object language does not imply the negation of that latter. If the metalanguage is bivalent and the object language has gaps, $\neg Tp$ does not imply $\neg p$ and, because of that, the Schema fails.

Here is how this problem can be surmounted.

In order to postulate the additional domain of states, we need to find a way to name not only sentences but also the states they represent. For naming sentences, we will assume the standard inverted commas. For naming the state corresponding to what a declarative sentence represents (describes) we will use the operator *.[1] This operator, once applied to a sentence, will transform it

[1] Not to be confused with the star function of 1.19.1 in ch. 4.

into the name of the state it represents - e.g. one obtains the name of p, by 'p', and one obtains the name of the state it represents, by $^*p$. We further postulate that a state $^*p$ obtains in w, if, and only if, that state belongs to the set

$$\{^*p : \text{`}p\text{' is true in } w\},$$

and we also agree to consider the expressions '$^*p$ exists' and '$^*p$ obtains' as equivalent.[2] (See note 11 of Appendix I.)

If these are assumed, the Tarski Schema could be formulated as follows:

(i) $T(\text{`}p\text{'}) \leftrightarrow (^*p)$ obtains

or, alternatively:

(ii) $T(\text{`}p\text{'}) \leftrightarrow (\exists q)\ q = {}^*p$

Out of the above and of the all round classical $T(\text{`}\neg p\text{'}) \leftrightarrow F(\text{`}p\text{'})$, we immediately deduce that:

(i) $F(\text{`}p\text{'}) \leftrightarrow (^*\neg p)$ obtains

or, alternatively:

(ii) $F(\text{`}p\text{'}) \leftrightarrow (\exists q) q = {}^*\neg p$

If the Tarski Schema is stated as above, and even when the implication is material and $T(\text{`}p\text{'})/F(\text{`}p\text{'})$ are gapless, the whole system does not *eo ipso* collapse into bivalence. For, when 'p' is truth-valueless, and, thereby, '$T(\text{`}p\text{'})$' and '$F(\text{`}p\text{'})$' are both false, the possibility remains that '$(^*p)$ obtains' and '$(^*\neg p)$ obtains' are both false as well, or, which is the same, that neither of them is a state of the world.[3] This establishes that -with certain alterations in its formulation- one can allow the Tarski Schema in "classical" supervaluation systems, without being thereby committed to interpret implication as necessitation.

In fact, what does the trick for the above formulation of the Schema is some extra space in the second part of the equivalence - i.e. in its classical formulation, the second part of the Schema is indivisible and the negation connective applies either to (all) of it, or to no part of it. Its scope cannot vary. In our formulations, and thanks to the extra space created by "obtains" (or by the quantifier) and *, one can make both '$(^*p)$ is a state/obtains' and '$(^*\neg p)$ is a state/obtains' false. Contrary to that, and by the standards of the classical formulation, the second part, once negated, commits one to the existence of the

[2] These specifications can be continued even further: e.g. one could say that a state exists in w, if, and only if, the state is upon either the past segment of a weak causal chain passing through w, or upon the future segment of a strong causal chain passing through w, or is a non causal state that does not contradict the former two classes. One could also postulate that the existence or not of a state is also a state, and so on so forth.

[3] The validity of these two formulas does not commit the system to maximality of worlds. There can be instances of contradictory pairs of sentences, neither of which belongs to the world. The world becomes maximal only after the second level - i.e. for sentences, which attribute qualities to other sentences, and sentences asserting the existence of states: e.g. in $T(T(\text{`}p\text{'}))\leftrightarrow (\exists r) r = {}^*((\exists q) q = {}^*p)$, '$(\exists q) q = {}^*p$' cannot be truth-valueless, but 'p' can. (For the iteration of the * operator see note 2).

-according to our formulation- $({}^{*}\neg p)$ state.

Of course one has to make it explicit that these formulations of the Schema turn out valid for "classical" supervaluation only if the state predicate "$^{*}(\ldots)$ obtains" is functionally complete, and quantifications of the form $(\exists q)q = {}^{*}(\ldots)$ are classical. For, if one applies to either of these two the same supervaluation criteria one applies for the object language of "classical" supervaluation, one will encounter the same problem one encounters with $T(`p') \leftrightarrow p$ - i.e. the left-hand side of the equivalence could be false, while the right hand side, truth-valueless. In other words, one has to evaluate the right-hand side of the equivalence by the same criteria with which he evaluates the left-hand side; he has to interpret it as belonging to a language as classical as the metalanguage of "classical" supervaluation.

Appendix C

About the number of atomic sentences and circular Time

A question that we have left untouched concerns the total number of atomic sentences. According to some traditional atomistic conception, they must be finite in number. This is not welcome here for the following reasons.

A first consequence of assuming a finite number of atomic sentences is that the underlying logic stops being intuitionistic. In order to construct an intuitionistic apparatus, by the use of Beth semantics, we need infinite paths - finite Beth models are classical.[1] Now, one can never reach sets of any infinite cardinality, if one assumes only a finite number of atomic facts; and thereby a finite number of individuals. And without an infinite number of individuals, one cannot assign $\mathbb{N}(+/-)$ to the worlds of W (cf. ch. 4; 4.2). Infinite paths are unreachable.

Another consequence of assuming a finite number of atomic facts/sentences is that History becomes circular.[2]

Settings:

Suppose there are n (n finite), members of AS. The number of elements belonging to the power set of AS is again finite - say it is m. Therefore, there are m possible worlds. Assume that the ideal agent of S lives now in $w(n)$.

To be proved:

History moves in circles.

[1] See chapter 4.

[2] It is important to remind the reader that by "History" we mean a partial, not a linear order.

Proof. Take a random $m+1$ successive vertical bars[3] of the model that constitutes the History of the ideal agent. In every maximal linear order passing through these bars there has to be at least one subset of AS that repeats itself at least once. Isolate a pair of these, on a randomly selected such linear order. Situate the agent first upon the one and then upon the other member of the pair. By omniscience, and by the fact that, on both occasions, he deduces everything he knows out of two identical sets of atomic sentences and a common set of causal conditionals, we conclude that, on both occasions, he draws for himself identical pasts and futures. Therefore $w(n)$ repeats itself *ad indefinitum*, upon identical finite partial orderings, in both directions.

□

Notes:

1. The ideal agent, when upon each of these repeated occurrences, is in position to assert that the whole partial order repeats itself *ad indefinitum* towards both future and past. He can also effectuate a numbering of these successive occurrences taking as his "contingent constant"[4] the occasional present moment; i.e. the occasional "actual occurrence". He cannot undertake any absolute numbering of them.[5]

2. No matter how Nietzsche would have rejoiced in the perspective of this finite substructure repeating itself *ad indefinitum*, we have to say that this repetition does not touch the fundamental freedom of the agent, because it does not diminish the fact that he can never anticipate what the choice of the free choice sequence will be, with respect to any future fork. Moreover, if some of these forks stand for future indeterministic decisions of his, he will always be "condemned" to trace a radically unpredictable future, when upon these, and no matter how many times during the past he has been forced to do the same. Note here that nothing excludes that some of the worlds of the substructure are never visited by the agent.[6]

[3]They stand for the moments.

[4]For this term see C. A. MEREDITH - A. N. PRIOR, [1965].

[5]e.g. he can use expressions like: "the n^{th} future/past occurrence of $w(n)$, from where I am now"; he cannot use no expressions of the form "the n^{th} occurrence of $w(n)$".

[6]e.g. it is not impossible that the agent endlessly follows the upper path of figure 1. More precisely, what happens is that each time the agent is upon a $w(n)$ world, he cannot account for his previous choices, unless they took place, within the last circle. Consequently, he cannot answer the question about whether or not he was following the same itinerary over and over again.

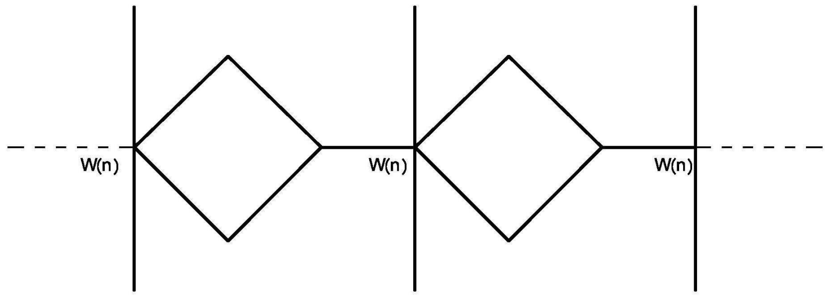

Figure 1.

3. An infinite stock of atomic sentences is a necessary but not a sufficient condition for non-circular Time, as one can see in figure 2.

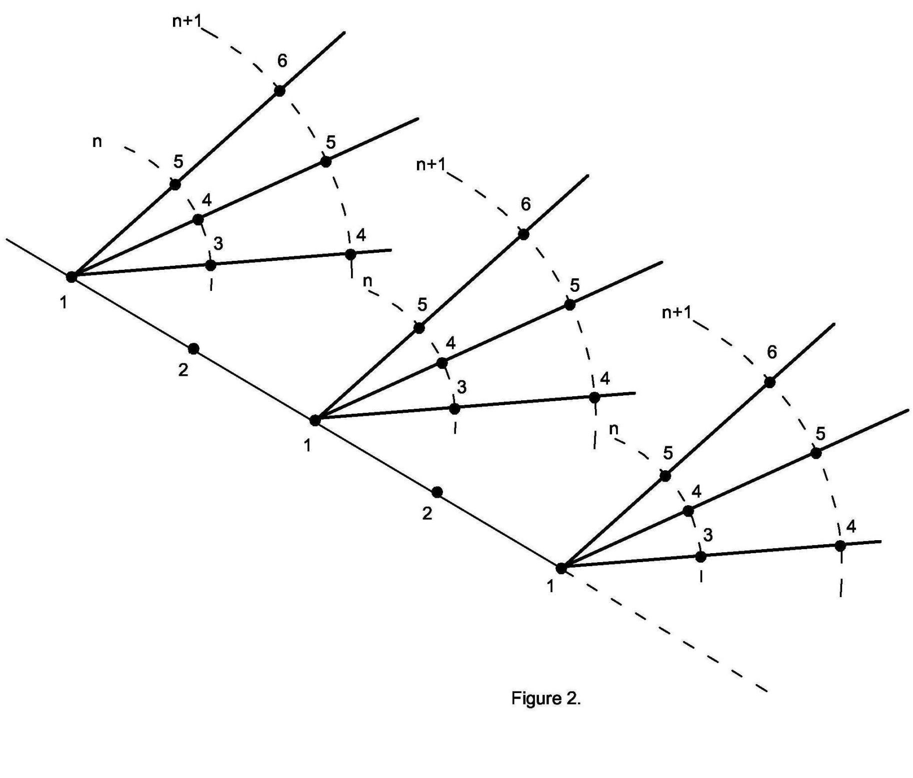

Figure 2.

Assume a canonical ordering of the power set of an infinite AS, and take the numbers of the ordering to index the worlds of the model. Time is circular in this model too. In each two distinct occurrences of the same subset of AS, the agent knows exactly the same things. Observe that all elements of the infinite power set of AS are actualized somewhere in the model, so there is no problem with the atomistic proviso that all combinations of atomic sentences are possible. These kind of situations explain the necessity for the restriction that the correspondence between subsets of AS and W is one to one (cf. ch. 4; 4.4.2).

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